mated by 2. One obtains then for δ

$$\delta = -0.76\tau_0/\tau_a \quad \text{(low fields)}. \tag{A3b}$$

This shows that the correction δ depends only weakly on field and temperature. We have therefore used

$$\delta = -0.8\tau_0/\tau_a \tag{A3c}$$

for all values of field and temperature.

The magnetoresistance calculated by Eq. (3a), $(\Delta\rho_m/\rho)$ Eq. (3a), and the experimentally determined magnetoresistance $(\Delta\rho_m/\rho_{\text{tot}})_{\text{expt}}$, of Figs. 1 and 2 are then related by

$$(\Delta\rho_m/\rho)_{\text{Eq. (3a)}} = (\Delta\rho_m/\rho_{\text{tot}})_{\text{expt}}(1+1.8\tau_0/\tau_a) \tag{A4}$$

as long as $\tau_0/\tau_a \ll 1$.

A second, similar correction factor is due to $\Delta\rho_n$ and $\Delta\rho_m$ not being additive. That Mathiessen's rule does not hold in this case either can immediately be seen if $\Delta\rho_n$ is described by an additional scattering rate $1/\tau_n$, which is field-dependent but equal for the conduction electrons of either spin direction. The deviation δ_n from Mathiessen's rule is obtained by replacing τ_a by τ_n. With the same values for f, g, and τ_0 one then obtains

$$\delta_n \approx -0.7\Delta\rho_n/\rho \tag{A5}$$

provided $\Delta\rho_n/\rho \ll 1$. $\Delta\rho_m$ and $\Delta\rho$ are then related by

$$\Delta\rho_m \approx (\Delta\rho - \Delta\rho_n)(1+0.7\Delta\rho_n/\rho). \tag{A4$'$}$$

It is interesting to consider the case $1/\tau_a \gg 1/\tau_0$, which might occur in ternary alloys or in the case of very large positive magnetoresistance. One finds from Eq. (A1) that the interference terms no longer contribute to the negative magnetoresistance and one has $\Delta\rho_m = -f/2$.

PHYSICAL REVIEW VOLUME 174, NUMBER 2 10 OCTOBER 1968

Electronic Spin Polarization around a Magnetic Impurity Using Perturbation Theory*

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(Received 9 April 1968)

The electronic spin polarization $p(r)$ arising from the s-d exchange interaction is computed using iterated solutions of Nagaoka's truncated equations and direct perturbation theory. Most of our detailed calculations are only to order J^2 (where J is the exchange constant). However, arguments are given to suggest that certain qualitative features [such as the oscillatory behavior of $p(r)$] will remain even if we work to all orders in J. We critically discuss the work of Falk and Fullenbaum as well as Suhl, whose results agree with ours apart from the important difference that our Ruderman-Kittel-Kasuya-Yosida term is proportional to the average of an effective spin rather than the bare impurity spin. This effective spin also enters the static susceptibility χ. In one of the Appendices, we briefly consider the effect of potential scattering on the spin polarization.

I. INTRODUCTION

It seems natural to expect that one of the most important manifestations of the Kondo effect[1] will be in the conduction-electron spin polarization $p(r)$ around a magnetic impurity. Historically, the first such study was made by Nagaoka[2] using a self-consistent solution of the decoupled equations of motion for retarded double-time Green's functions. He found that in contrast to the Born approximation for the polarization (associated with the names Ruderman-Kittel-Kasuya-Yosida[3] and thus referred to as RKKY) his result had a much longer range and moreover the electron spins were favored to align antiparallel to the impurity atom spin. This result led Nagaoka to the conclusion that below a critical temperature T_K there exists some sort of quasibound state between the electron spins and the impurity spin. The coherence length of this quasibound state was estimated to be 10^{-4} cm. It has since been realized that Nagaoka's original solution of his equations was incorrect and that his equations do not give rise to any bound state. As a consequence the result for the electron spin polarization given in Ref. 2 should be disregarded. In contrast, Suhl[4,5] made use of his alterna-

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* This research was sponsored by research grants from the National Research Council of Canada.

† One of the authors (H.U.E.) gratefully acknowledges a travel grant from the Deutsche Forschungsgemeinschaft.

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[4] H. Suhl, in *Proceedings of the International School of Physics, "Enrico Fermi" 1966*, edited by W. Marshall (Academic Press Ltd., London, 1967), pp. 116–205.

[5] H. Suhl, Solid State Commun. **4**, 487 (1966).

tive S-matrix theory and deduced that the Kondo effect would have little influence on the conduction-electron spin polarization at distances $r>r_0$ from the impurity atom, where r_0 is of the order $1/k_F$. We shall discuss Suhl's work later on in this paper. For now, we simply call attention to the fact that both Suhl and Nagaoka discussed the spin polarization using results which, in a certain sense, were correct to all orders in the exchange interaction J. More recently Falk and Fullenbaum[6] have calculated $p(r)$ by extending the Nagaoka equation-of-motion method to the case where rotational symmetry is destroyed by a magnetic field. A perturbational solution of the truncated set of equations of motion led them to an expression for $p(r)$ which involves a term $\propto J \ln k_F r$. This was interpreted as a manifestation of the Kondo effect in $p(r)$.

In this paper we reinvestigate $p(r)$ and related functions such as the susceptibility on the basis of Nagaoka's decoupling procedure. In Sec. II we compute $p(r)$, but instead of including the magnetic field in the equations of motion we shall start from a linear response formula for $p(r)$. Our result consists of three parts, $p_0(r)$, $p_1(r)$, and $p_2(r)$, where $p_0(r)$ has the same r dependence as the RKKY polarization. However, instead of being proportional to $\langle S_Z\rangle$ (the thermal average of the impurity spin), it contains a more complicated function of the temperature. However, this function may be interpreted as the average of an *effective* spin, the magnitude of which varies with temperature. In this aspect our result differs from that of Falk and Fullenbaum,[6] who obtained the RKKY polarization instead of $p_0(r)$. The two other contributions to $p(r)$, $p_1(r)$ and $p_2(r)$, agree exactly with the corresponding expressions of Ref. 6. $p_1(r)$ contains the term $\propto J \ln k_F r$ that we mentioned before. In Appendix C, we compare our work with that of Suhl, as well as Falk and Fullenbaum, showing where the differences come from.

In Sec. III we discuss the relation between $p_0(r)$ and the static susceptibility χ. The most important new result is that both functions have the same temperature dependence. Furthermore we show how the logarithmic singularity, which occurs in the second-order expression for the susceptibility, comes from a certain part of $\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$, the electron spin-impurity spin correlation function. The latter function was erroneously identified with the conduction-electron spin polarization in Ref. 2. A direct second-order perturbation calculation for $\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$ is presented in Appendix D. Finally, in Sec. IV we briefly compare our result with some recent experimental work. In particular, the fact that χ and $p_0(r)$ have the same temperature dependence seems to be well established experimentally. In Appendix E we discuss the charge oscillations induced by the impurity ion.

II. EVALUATION OF $\langle \mathbf{S}^{\mathrm{el}}(r)\cdot(\mathbf{S}^{\mathrm{el}}+\mathbf{S}^{\mathrm{imp}})\rangle$ USING NAGAOKA'S TRUNCATION

The interaction of conduction electrons with the localized magnetic moment is assumed to be described by the s-d exchange interaction Hamiltonian:

$$H_{sd} = -(J/2N)\sum_{\mathbf{k},\mathbf{k}'} C_{\mathbf{k}\alpha}^{\dagger}\boldsymbol{\sigma}_{\alpha\beta}\cdot\mathbf{S}^{\mathrm{imp}}C_{\mathbf{k}'\beta}. \tag{2.1}$$

We introduce the convention that repeated spin indices are summed. The operator $C_{\mathbf{k}\alpha}^{\dagger}$ creates a conduction electron in the state $|\,\mathbf{k}\alpha\rangle$, where $\mathbf{k}$ denotes the wave vector and α the spin quantum number; σ are the Pauli matrices, and the operator $\mathbf{S}^{\mathrm{imp}}$ represents the localized magnetic moment. The exchange coupling constant J is positive for ferromagnetic interaction and negative for antiferromagnetic interaction. N is the number of atoms in the lattice.

In order to obtain a finite spin polarization of the conduction electrons, we have to apply a magnetic field which polarizes the impurity spin. The conduction-electron spin polarization [henceforth referred to as $p(r)$] is then given by

$$\begin{aligned} p(r) &\equiv \langle S_z^{\mathrm{el}}(r)\rangle_h \\ &= (2\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'} \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \\ &\quad\times \frac{\mathrm{Tr}\{\exp[-\beta(H_{\mathrm{kin}}+H_{sd}+H_h)]C_{\mathbf{k}\alpha}^{\dagger}\sigma_{z\alpha\beta}C_{\mathbf{k}'\beta}\}}{\mathrm{Tr}\exp[-\beta(H_{\mathrm{kin}}+H_{sd}+H_h)]}. \end{aligned} \tag{2.2}$$

Here Ω is the volume of the system, $\beta=(k_BT)^{-1}$, Boltzmann's constant being generally set to unity. H_{kin} is the kinetic energy of the conduction electrons; H_h is the magnetic energy of the system in an external magnetic field h which is assumed to be in z direction. That is,

$$H_h = -\mu_B g h(S_z^{\mathrm{el}}+S_z^{\mathrm{imp}}), \tag{2.3}$$

where μ_B is the Bohr magneton; the Landé g factor is assumed to be the same for both conduction electron and impurity spin. We assume that the magnetic field is sufficiently weak so that the magnetic energy is small compared to the thermal energy. This allows us to expand $p(r)$ in terms of the magnetic field h and to retain only the term linear in h, with the result

$$\begin{aligned} p(r) &= -(3T)^{-1}g\mu_B h \\ &\quad\times\{\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{el}}\rangle_{h=0}+\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle_{h=0}\}. \end{aligned} \tag{2.4}$$

We have made use of the rotational invariance of thermal averages in the absence of a magnetic field. From now on, $\langle\cdots\rangle_{h=0}\equiv\langle\cdots\rangle$. Thermal averages over two equal-time operators may be conveniently rewritten as

$$\langle AB\rangle = \pi^{-1}\int d\omega\, f(\omega)\,\mathrm{Im}\langle B\,|\,A\rangle_\omega, \tag{2.5}$$

where $\langle B\,|\,A\rangle_\omega$ denotes the Fourier transform of a

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[6] M. S. Fullenbaum and D. S. Falk, Phys. Rev. 157, 452 (1967).

double-time Green's function.[7] We can express the correlation functions in Eq. (2.4) by means of appropriate Green's functions:

$$\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle = -(\pi\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \times\int d\omega\, f(\omega)\ \mathrm{Im}\Gamma_{\mathbf{k}\mathbf{k}'}(\omega), \tag{2.6}$$

$$\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{el}}\rangle = -(\pi\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \times\int d\omega\, f(\omega)\ \mathrm{Im}\theta_{\mathbf{k}\mathbf{k}'}(\omega), \tag{2.7}$$

where

$$\Gamma_{\mathbf{k}\mathbf{k}'}(\omega) \equiv \tfrac{1}{2}\langle \mathbf{S}^{\mathrm{imp}}\cdot\boldsymbol{\sigma}_{\alpha\beta}C_{\mathbf{k}'\alpha}\mid C_{\mathbf{k}\beta}{}^{\dagger}\rangle, \tag{2.8}$$

$$\theta_{\mathbf{k}\mathbf{k}'}(\omega) \equiv \tfrac{1}{4}\sum_{\mathbf{l},\mathbf{l}'}\langle \boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\alpha'\beta'}C_{\mathbf{k}\beta}C_{\mathbf{l}\alpha'}{}^{\dagger}C_{\mathbf{l}'\beta'}\mid C_{\mathbf{k}'\alpha}{}^{\dagger}\rangle. \tag{2.9}$$

It so happens that $\Gamma_{\mathbf{k}\mathbf{k}'}(\omega)$ is one of the natural functions occurring in Nagaoka's truncated equations of motion, and thus his theory implies a specific approximation for $\Gamma_{\mathbf{k}\mathbf{k}'}(\omega)$. In order to evaluate the Green's function $\theta_{\mathbf{k}\mathbf{k}'}(\omega)$ in terms of the Green's functions

$$G_{\mathbf{k}\mathbf{k}'}(\omega) \equiv \tfrac{1}{2}\langle C_{\mathbf{k}\alpha}\mid C_{\mathbf{k}\alpha}{}^{\dagger}\rangle \tag{2.10}$$

and $\Gamma_{\mathbf{k}\mathbf{k}'}(\omega)$, we follow Nagaoka's procedure.[2]

The *exact* equation of motion for $\theta_{\mathbf{k}\mathbf{k}'}(\omega)$ is given by

$$\begin{aligned}4(\omega-\epsilon_{\mathbf{k}})\theta_{\mathbf{k}\mathbf{k}'}(\omega) &= 3\{2\delta_{\mathbf{k}\mathbf{k}'}-\langle C_{\mathbf{k}'\alpha}{}^{\dagger}C_{\mathbf{k}\alpha}\rangle\}-3(J/N)\sum_{\mathbf{l}}\Gamma_{\mathbf{k}'\mathbf{l}}(\omega)\\ &+\tfrac{1}{2}(J/N)\sum_{\mathbf{l},\mathbf{l}'}\langle[\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}\cdot\mathbf{S}^{\mathrm{imp}})]_{\alpha\beta}\boldsymbol{\sigma}_{\alpha'\beta'}C_{\mathbf{l}\alpha'}{}^{\dagger}C_{\mathbf{l}'\beta}C_{\mathbf{l}\beta'}\mid C_{\mathbf{k}'\alpha}{}^{\dagger}\rangle-(J/2N)\sum_{\mathbf{l},\mathbf{l}'}\langle\sigma_{\alpha\beta}\cdot(i\boldsymbol{\sigma}\times\mathbf{S}^{\mathrm{imp}})_{\alpha'\beta'}C_{\mathbf{l}\alpha'}{}^{\dagger}C_{\mathbf{k}\beta}C_{\mathbf{l}'\beta}\mid C_{\mathbf{k}'\alpha}{}^{\dagger}\rangle.\end{aligned} \tag{2.11}$$

Subjected to Nagaoka's type of decoupling approximation, Eq. (2.11) takes the form

$$\begin{aligned}\theta_{\mathbf{k}\mathbf{k}'}(\omega) &= [4(\omega-\epsilon_{\mathbf{k}})]^{-1}[3(2\delta_{\mathbf{k}\mathbf{k}'}-\langle C_{\mathbf{k}'\alpha}{}^{\dagger}C_{\mathbf{k}\alpha}\rangle)-(J/N)\sum_{\mathbf{l}}n_{\mathbf{l}}\Gamma_{\mathbf{k}'\mathbf{l}}(\omega)+\tfrac{3}{2}(J/N)\sum_{\mathbf{l}}m_{\mathbf{l}}G_{\mathbf{k}'\mathbf{l}}(\omega)\\ &-(J/N)\sum_{\mathbf{l}}\langle(\boldsymbol{\sigma}\cdot\mathbf{S})_{\alpha\beta}C_{\mathbf{l}\alpha}{}^{\dagger}C_{\mathbf{l}\beta}\rangle\cdot\sum_{\mathbf{l}}G_{\mathbf{k}'\mathbf{l}}(\omega)-2(J/N)S(S+1)\sum_{\mathbf{l}}G_{\mathbf{k}'\mathbf{l}}(\omega)]-\Gamma_{\mathbf{k}\mathbf{k}'}(\omega),\end{aligned} \tag{2.12}$$

where

$$n_{\mathbf{k}} \equiv -\pi^{-1}\sum_{\mathbf{l}}\int_{-\infty}^{\infty}d\omega\, f(\omega)\ \mathrm{Im}G_{\mathbf{k}\mathbf{l}}(\omega), \tag{2.13}$$

$$m_{\mathbf{k}} \equiv -(2/\pi)\sum_{\mathbf{l}}\int_{-\infty}^{\infty}d\omega\, f(\omega)\ \mathrm{Im}\Gamma_{\mathbf{k}\mathbf{l}}(\omega). \tag{2.14}$$

Making use of the Eqs. (2.6), (2.7) and (2.12) in Eq. (2.4), we obtain

$$\begin{aligned}p(r) = -(g\mu_B h/3T)(4\pi\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\Big\{&6f_{\mathbf{k}}(\delta_{\mathbf{k}\mathbf{k}'}-\tfrac{1}{2}\langle C_{\mathbf{k}'\alpha}{}^{\dagger}C_{\mathbf{k}\alpha}\rangle)\\ &-2(J/N)[S(S+1)+\sum_{\mathbf{l}}\langle\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{S})_{\alpha\beta}C_{\mathbf{l}\alpha}{}^{\dagger}C_{\mathbf{l}\beta}\rangle]\left[\sum_{\mathbf{l}'}\int d\omega\, f(\omega)\ \mathrm{Im}\frac{G_{\mathbf{k}'\mathbf{l}'}(\omega)}{\omega-\epsilon_{\mathbf{k}}}\right]\\ &-\frac{J}{N}\sum_{\mathbf{l}}\int d\omega\, f(\omega)\ \mathrm{Im}\frac{n_{\mathbf{l}}+1}{\omega-\epsilon_{\mathbf{k}}}\Gamma_{\mathbf{k}'\mathbf{l}}(\omega)+\frac{3}{2}\frac{J}{N}\sum_{\mathbf{l}}\int d\omega\, f(\omega)\ \mathrm{Im}\frac{m_{\mathbf{l}}}{\omega-\epsilon_{\mathbf{k}}}G_{\mathbf{k}'\mathbf{l}}(\omega)\Big\}.\end{aligned} \tag{2.15}$$

We wish to emphasize that the last term of Eq. (2.12) and $\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$ have cancelled each other in Eq. (2.15). This clearly shows that the spin polarization of the conduction electrons is not given by

$$\bar{p}(r) = (-\mu_B g h/3T)\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle, \tag{2.16}$$

as implied in Refs. 2 and 5. The difference between $\bar{p}(r)$ as defined by (2.16) will become apparent when we discuss the connection between the spin polarization and the susceptibility in Sec. III.

The second term in (2.15) is very similar to the Ruderman-Kittel type of oscillatory polarization. Let us introduce the function $C'(T)$ by

$$\begin{aligned}C'(T) &\equiv \tfrac{1}{3}(g\mu_B)[S(S+1)+\sum_{\mathbf{l}}\langle\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{S}^{\mathrm{imp}})_{\alpha\beta}C_{\mathbf{l}\alpha}{}^{\dagger}C_{\mathbf{l}\beta}\rangle]\\ &= \tfrac{1}{3}(g\mu_B)[S(S+1)+\langle \mathbf{S}^{\mathrm{el}}\cdot\mathbf{S}^{\mathrm{imp}}\rangle].\end{aligned} \tag{2.17}$$

Furthermore, we express the Green's function $G_{\mathbf{k}'\mathbf{k}}(\omega)$ in terms of the non-spin-flip scattering amplitude $t(\omega)$:

$$G_{\mathbf{k}\mathbf{k}'}(\omega) = \delta_{\mathbf{k}\mathbf{k}'}/(\omega-\epsilon_{\mathbf{k}})+(\omega-\epsilon_{\mathbf{k}})^{-1}t(\omega)(\omega-\epsilon_{\mathbf{k}'})^{-1}. \tag{2.18}$$

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[7] See, for example, D. N. Zubarev, Usp. Fiz. Nauk **71**, 71 (1960) [English transl.: Soviet Phys.—Usp. **3**, 320 (1960)].

We note that the free-particle energies $\epsilon_{\mathbf{k}}$ are measured with respect to the Fermi energy.

Then, the second term in (2.15) takes the form

$$p_0(r) \equiv \frac{g\mu_B h}{T}(4\pi\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]C'(T)$$

$$\times\left[\sum_{1'}\int d\omega\, f(\omega)\ \mathrm{Im}\,\frac{G_{\mathbf{k}'\mathbf{1}'}(\omega)}{\omega-\epsilon_{\mathbf{k}}}\right]$$

$$=\frac{J}{2N}h\frac{C'(T)}{T}\Omega^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]$$

$$\times\int d\omega\, f(\omega)\ \mathrm{Im}\left[\frac{1-i\pi t(\omega)}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})}\right]. \quad (2.19)$$

If we neglect all scattering effects in the integral in (2.19) [i.e., $t(\omega)\equiv 0$; the justification for this is given in Appendix A], we obtain

$$p_0(r)\simeq 3\pi N_0\frac{J\rho}{2N}h\frac{C'(T)}{T}\frac{j_1(2k_Fr)}{(k_Fr)^2}. \quad (2.20)$$

Here $N_0=k_F^3/6\pi^2$ is the average density of electrons (per spin) and $j_1(z)$ denotes the spherical Bessel function of order 1. If we consider $h[C'(T)/T]$ to be the "effective" value of the z component of the impurity spin we may interpret the expression (2.20) as the Ruderman-Kittel polarization induced by an impurity spin, the magnitude of which is modified due to its interaction with the conduction electrons. The temperature dependence of $C'(T)$ is closely related to the temperature dependence of the susceptibility. The relation will be worked out in Sec. III.

The physical meaning of the remaining terms in (2.15) is not so obvious. In order to get some insight into their structure, we shall resort to perturbation theory. We obtain an expansion in powers of J simply by iterating Nagaoka's approximate equations of motion, which are

$$G_{\mathbf{k}\mathbf{k}'}(\omega)=(\omega-\epsilon_{\mathbf{k}})^{-1}[\delta_{\mathbf{k}\mathbf{k}'}-(J/2N)\sum_{\mathbf{l}}\Gamma_{\mathbf{l}\mathbf{k}'}(\omega)], \quad (2.21\text{a})$$

$$\Gamma_{\mathbf{k}\mathbf{k}'}(\omega)=(\omega-\epsilon_{\mathbf{k}})^{-1}\{(J/2N)[m_{\mathbf{k}}-S(S+1)]\sum_{\mathbf{l}}G_{\mathbf{l}\mathbf{k}'}(\omega)$$

$$-(J/2N)(2n_{\mathbf{k}}-1)\sum_{\mathbf{l}}\Gamma_{\mathbf{l}\mathbf{k}'}(\omega)\}. \quad (2.21\text{b})$$

The results of this iteration may be expressed as follows:

$$G_{\mathbf{k}\mathbf{k}'}(\omega)=G_{\mathbf{k}\mathbf{k}'}^{(0)}(\omega)+G_{\mathbf{k}\mathbf{k}'}^{(1)}(\omega)+G_{\mathbf{k}\mathbf{k}'}^{(2)}(\omega)+\cdots, \quad (2.22)$$

$$\Gamma_{\mathbf{k}\mathbf{k}'}(\omega)=\Gamma_{\mathbf{k}\mathbf{k}'}^{(0)}(\omega)+\Gamma_{\mathbf{k}\mathbf{k}'}^{(1)}(\omega)+\Gamma_{\mathbf{k}\mathbf{k}'}^{(2)}(\omega)+\cdots, \quad (2.23)$$

where

$$G_{\mathbf{k}\mathbf{k}'}^{(0)}(\omega)=\delta_{\mathbf{k}\mathbf{k}'}(\omega-\epsilon_{\mathbf{k}})^{-1},$$

$$\Gamma_{\mathbf{k}\mathbf{k}'}^{(0)}\equiv 0,\qquad G_{\mathbf{k}\mathbf{k}'}^{(1)}(\omega)\equiv 0, \quad (2.24)$$

$$\Gamma_{\mathbf{k}\mathbf{k}'}^{(1)}(\omega)=-(J/2N)S(S+1)[(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})]^{-1}, \quad (2.25)$$

$$G_{\mathbf{k}\mathbf{k}'}^{(2)}(\omega)=-i\pi(J\rho/2N)^2S(S+1)[(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})]^{-1}. \quad (2.26)$$

In view of (2.15), it is clear that to find $p(r)-p_0(r)$ to order J^2, we do not need $\Gamma_{\mathbf{k}\mathbf{k}'}^{(2)}(\omega)$. Furthermore, we need only calculate $n_{\mathbf{k}}$ and $m_{\mathbf{k}}$ to first order in J. It is a trivial matter to verify that

$$n_{\mathbf{k}}^{(0)}=f_{\mathbf{k}},\qquad n_{\mathbf{k}}^{(1)}\equiv 0.$$

On the other hand, using Eqs. (2.24) and (2.25), we find

$$m_{\mathbf{k}}^{(0)}=0,\qquad m_{\mathbf{k}}^{(1)}=(J\rho/N)S(S+1)\ \mathrm{Re}\,g(\epsilon_{\mathbf{k}}), \quad (2.27)$$

where

$$g(\epsilon_{\mathbf{k}})\equiv\rho^{-1}\sum_{\mathbf{k}'}[f_{\mathbf{k}'}/(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+i\delta)]$$

$$=\int_{-D}^{D}\frac{d\epsilon_{\mathbf{k}'}\, f_{\mathbf{k}'}}{\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+i\delta}.$$

The latter equality holds for a constant density of states in a conduction band of width $2D$. For values of $\epsilon_{\mathbf{k}}$ such that $|\epsilon_{\mathbf{k}}|\ll D$, the function $g(\epsilon_{\mathbf{k}})$ has the same analytic structure as the digamma function (see, for example, Ref. 8). For $|\epsilon_{\mathbf{k}}|\gg D$, however, it behaves like $1/\epsilon_{\mathbf{k}}$. Frequently we shall make use of an approximation for $g(\epsilon_{\mathbf{k}})$ due to Hamann,[9] namely,

$$g(\epsilon_{\mathbf{k}})\simeq-\ln[(\epsilon_{\mathbf{k}}+iT)/iD]. \quad (2.28)$$

This expression exhibits the correct behavior in the limit $T\rightarrow 0$ as well as in the limit $\epsilon_{\mathbf{k}}\rightarrow 0$.

Subtracting $p_0(r)$ from Eq. (2.15) making use of Eqs. (2.24)–(2.28), we obtain to second order in J

$$p(r)-p_0(r)=-(g\mu_Bh/3T)(4\Omega)^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]$$

$$\times\Bigg\{6\delta_{\mathbf{k}\mathbf{k}'}\cdot f_{\mathbf{k}}(1-f_{\mathbf{k}})+2(J^2/N^2)\rho S(S+1)[\epsilon_{\mathbf{k}'}-\epsilon_{\mathbf{k}}]^{-1}$$

$$\times\left[f_{\mathbf{k}'}\ln\left(\frac{\epsilon_{\mathbf{k}}'^2+T^2}{D^2}\right)^{1/2}-f_{\mathbf{k}}\ln\left(\frac{\epsilon_{\mathbf{k}}^2+T^2}{D^2}\right)^{1/2}\right]$$

$$+\frac{1}{2}\frac{J^2}{N^2}\rho S(S+1)\int d\omega\, f(\omega)\ \mathrm{Im}\,\frac{i[f(\omega)+1]}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})}\Bigg\}$$

$$\equiv p_{\mathrm{Pauli}}+p_1(r)+p_2(r). \quad (2.29)$$

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[8] P. J. Davis, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, New York, 1965), p. 258.

[9] D. R. Hamann, Phys. Rev. **158**, 570 (1967).

In deriving (2.29) we have carried out the ω integration in the first three terms utilizing the approximation (2.28). In the last term it is advisable to do the **k**, **k′** summations *first*, carrying out the ω integration afterwards. The first term denoted by p_{Pauli} is the *uniform* Pauli polarization

$$p_{\text{Pauli}}=(g\mu_B h/2T)\rho,$$

which is of no interest in the present context. To proceed with the evaluation of the other terms in Eq. (2.29) we replace the Fermi functions by their zero-temperature limits. This is possible since it turns out that no divergence occurs in the low-temperature limit of (2.29). The neglected temperature dependence involves terms of order T/ϵ_F and can therefore be disregarded completely in the temperature region we are interested in. Straightforward integration then leads to the following results:

$$\begin{aligned}p_1(r)=&(-g\mu_B h/T)\tfrac{1}{4}\pi N_0(J\rho/N)^2S(S+1)\\&\times\{2[j_1(2k_Fr)/(k_Fr)^2]\ln(T/D)+2[j_0(2k_Fr)/(k_Fr)^3]\\&+[(k_Fr)^2]^{-1}\sum_{\nu=0,1}\chi_\nu{}^2[\mathrm{Si}[2r(k_F+\chi_\nu)]\\&+\mathrm{Si}[2r(k_F-\chi_\nu)]]n_1(2r\chi_\nu)+[(k_Fr)^2]^{-1}\sum_{\nu=0,1}\chi_\nu{}^2\\&\times[\mathrm{Ci}[2r(k_F+\chi_\nu)]-\mathrm{Ci}[2r(k_F-\chi_\nu)]]j_1(2r\chi_\nu)\},\quad(2.30)\end{aligned}$$

where

$$\chi_\nu\equiv[k_F{}^2+(-1)^\nu 2imT]^{1/2},$$

and

$$\begin{aligned}p_2(r)=&(g\mu_B h/T)\pi^2N_0\tfrac{1}{4}(J\rho/N)^2\\&\times S(S+1)[(k_Fr)^2]^{-1}n_1(2rk_F).\quad(2.31)\end{aligned}$$

Here Si and Ci are the sine and cosine integral functions (for their definitions see Ref. 10), while j_m and n_m are the spherical Bessel and Neumann functions of of order m.

The temperature dependence of $p_1(r)$ contained in the arguments of the Si and Ci functions is very weak. Provided that

$$\Delta_Tr\equiv k_Fr(T/\epsilon_F)\ll1,$$

which is a very weak condition since $T/\epsilon_F\sim10^{-4}-10^{-3}$, we may expand $p_1(r)$ in terms of Δ_Tr. Retaining only the zeroth-order term, we obtain

$$\begin{aligned}p_1(r)\simeq&(-g\mu_B h/T)\tfrac{1}{2}\pi N_0S(S+1)(J\rho/N)^2\\&\times\{[j_1(2k_Fr)/(k_Fr)^2][\mathrm{Ci}(4k_Fr)-\ln((\gamma D/2\epsilon_F)k_Fr)]\\&+n_1(2k_Fr)\ \mathrm{Si}(4k_Fr)+j_0(2k_Fr)/(k_Fr)^3\},\quad(2.32)\end{aligned}$$

where $\ln\gamma=0.577$ (Euler-Mascheroni constant). The $\ln(T/D)$ term appearing in (2.30) has cancelled with the leading term of the asymptotic expansion of the function $\mathrm{Ci}[2r(k_F-\chi_\nu)]$, which is given by

$$\begin{aligned}\mathrm{Ci}[2r(k_F-\chi_\nu)]=&\ln[(-1)^\nu i(\gamma T/2\epsilon_F)k_Fr]\\&+O[(k_F\cdot r)^2(T/\epsilon_F)^2],\end{aligned}$$

for $k_Fr\ll(\epsilon_F/T)$. Collecting our results, we have

$$\begin{aligned}p_1(r)+p_2(r)&=p(r)-p_0(r)-p_{\text{Pauli}}\\&=-(g\mu_B h/T)\tfrac{1}{2}\pi N_0S(S+1)(J\rho/N)^2\\&\times\left\{\frac{j_1(2k_Fr)}{(k_Fr)^2}\left[\mathrm{Ci}(4k_Fr)-\ln\left(\frac{\gamma D}{2\epsilon_F}k_Fr\right)\right]\right.\\&\left.+\frac{n_1(2k_Fr)}{(k_Fr)^2}[\mathrm{Si}(4k_Fr)-\tfrac{1}{2}\pi]+\frac{j_0(2k_Fr)}{(k_Fr)^3}\right\}.\quad(2.33)\end{aligned}$$

With the further restriction $k_Fr\gg1$, this may be approximated by

$$\begin{aligned}p(r)-p_{\text{Pauli}}-p_0(r)\simeq&(g\mu_B h/T)S(S+1)\tfrac{1}{2}\pi N_0(J\rho/N)^2\\&\times[\cos2k_Fr/(k_Fr)^3]\ln[(\gamma D/2\epsilon_F)k_Fr].\quad(2.34)\end{aligned}$$

Although the right-hand side of Eq. (2.34) differs from the RKKY polarization through the presence of the logarithmic factor, it does not show a totally dissimilar behavior. In particular, we do not find any indication of a nonoscillatory contribution to the polarization as Nagaoka[2] did in the quantity that he identified with the electron spin polarization. Since the result in (2.34) was obtained by second-order perturbation theory, one might object that higher-order terms could still lead to a qualitatively different result. Although we have not been able to rule out this possibility entirely, we give an argument in Appendix B based on nonperturbative methods which makes it seem very improbable.

We should also mention that our result (2.34) agrees with that given in the paper by Falk and Fullenbaum[6] *if* we replace $p_0(r)$ by the usual first-order RKKY polarization. As we shall see, however, the temperature dependence entering $p_0(r)$ through $C'(T)$ is important in explaining the available experimental data.

Recently Heeger *et al.*[11] have calculated the polarization on the basis of the Applebaum-Kondo[12] approach. They also found the modified RKKY polarization $p_0(r)$. However, in addition to that, they also obtained a negative term which varies as $(\sin2k_Fr/r)^2$. At the present time, the existence of such a term has not been confirmed experimentally.

III. RELATION OF THE STATIC SUSCEPTIBILITY TO $p(r)$

Taking the second derivative of the free energy (Z is the partition function)

$$F=-(1/\beta Z)\ \mathrm{Tr}\exp[-\beta(H_{\text{kin}}+H_{sd}+H_h)]$$

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[10] W. Gautschi and W. F. Cahill, Ref. 8, pp. 227–252.

[11] A. J. Heeger, L. B. Welsh, M. A. Jensen, and G. Gladstone (to be published).

[12] J. Applebaum and J. Kondo, Phys. Rev. Letters **19**, 906 (1967).

with respect to the magnetic field h, we find the following expression for the static susceptibility of a system described by the Kondo Hamiltonian:

$$\chi=[(\mu_B g)^2/3T]\{\langle \mathbf{S}^{\rm el}\cdot \mathbf{S}^{\rm el}\rangle+2\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle+S(S+1)\}. \tag{3.1}$$

Introducing the polarization as defined in Sec. II, we may rewrite (3.1) as

$$\begin{aligned}\chi=\chi_{\rm Pauli}+\mu_B g\{&(\mu_B g/3T)\,S(S+1)\\&+(\mu_B g/3T)\,\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle+h^{-1}\int p_0(r)\,r^2dr\\&+h^{-1}\int[p(r)-p_0(r)-p_{\rm Pauli}]r^2dr\},\end{aligned} \tag{3.2}$$

where $\chi_{\rm Pauli}$ is the Pauli susceptibility of a free-electron gas. Going back to Eq. (2.15) one can see that

$$\int[p(r)-p_0(r)-p_{\rm Pauli}]r^2dr \tag{3.3}$$

vanishes. The arguments for this have been given in Ref. 2 so that we need not repeat them here.

From Eq. (2.20) we have

$$h^{-1}\int p_0(r)\,r^2dr=\frac{\mu_B g}{3T}\frac{J\rho}{N}[S(S+1)+\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle], \tag{3.4}$$

and hence[13]

$$\begin{aligned}\chi=\chi_{\rm Pauli}+[(\mu_B g)^2/3T][1+(J\rho/N)]&\\ \times[S(S+1)+\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle]&\\ \simeq\chi_{\rm Pauli}+\mu_B g[C'(T)/T].&\end{aligned} \tag{3.5}$$

In obtaining the last line of Eq. (3.5), we have neglected $J\rho/N$ in comparison with unity, and we have introduced $C'(T)$ as defined by Eq. (2.15). Evidently, $p_0(r)$ and χ have the same temperature dependence, the source of this being the term $\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle$.

From Eq. (2.6), one notes that

$$\begin{aligned}\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle&=\int d^3r\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}(r)\rangle\\&=-\pi^{-1}\sum_{\mathbf{k}}\int d\omega\, f(\omega)\ {\rm Im}\Gamma_{\mathbf{kk}}(\omega).\end{aligned} \tag{3.6}$$

Inserting the first- and second-order expressions for $\Gamma_{kk'}(\omega)$ [as given in Eqs. (2.25) and (D3)] in Eq. (3.6) we find to second order

$$\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle=S(S+1)[(J\rho/N)+(J\rho/N)^2\ln(T/D)], \tag{3.7}$$

and hence, to the same order,

$$\begin{aligned}\chi=\chi_{\rm Pauli}+[(\mu_B g)^2/3T]S(S+1)&\\ \times[1+(J\rho/N)+(J\rho/N)^2\ln(T/D)].&\end{aligned} \tag{3.8}$$

This result has been found earlier by several authors.[2,6,9] The logarithmic term in Eq. (3.8) shows that for $T<T_k$ $[T_k=D\exp(-|N/J\rho|)]$ the perturbation series no longer converges.

Recently Zittartz[14] has pointed out that it is possible to evaluate the right-hand side of Eq. (3.6) *exactly* in the limit $T\rightarrow 0$ if one uses the Fowler-Hamann-Bloomfield solution (see Refs. 15, 16) for the forward scattering matrix of the Kondo problem. The result is

$$\lim_{T\rightarrow 0}\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle=-(S+\tfrac{1}{2})+O(J\rho/N), \tag{3.9}$$

where S is the magnitude of the impurity spin. For $S=\frac{1}{2}$, it follows that

$$C'(T)=-\tfrac{1}{4}.$$

This would lead to a negative value of the susceptibility of zero temperature and must therefore be wrong. Since it has been proved by Mattis[17] that for antiferromagnetic coupling the gound state of the Kondo Hamiltonian is a singlet state, one would expect

$$\langle \mathbf{S}^{\rm el}\cdot \mathbf{S}^{\rm imp}\rangle_{\rm ground\ state}=-\tfrac{3}{4}\qquad \text{for } S=\tfrac{1}{2}. \tag{3.10}$$

Furthermore, we do not see any reason why the equality

$$\lim_{T\rightarrow 0}\langle \mathbf{S}^{\rm el}\cdot \mathbf{S}^{\rm imp}\rangle=\langle \mathbf{S}^{\rm el}\cdot \mathbf{S}^{\rm imp}\rangle_{\rm ground\ state}$$

should not hold.

The calculations that lead to the result (3.9) do not contain approximations other than Nagaoka's decoupling procedure and the simplifications made in deriving the integral equation by Falk and Fowler.[18] Therefore the contradiction between the Eqs. (3.9) and (3.10) forces us to conclude that these approximations do not provide the correct ground state of the Kondo Hamiltonian. The question of what the low-temperature behavior of $\langle \mathbf{S}^{\rm imp}\cdot \mathbf{S}^{\rm el}\rangle$ and χ is must be left to the future.

The correlation function $\langle \mathbf{S}^{\rm el}(r)\cdot \mathbf{S}^{\rm imp}\rangle$ which occurs in Eq. (3.6) is of some intrinsic interest. It describes the static correlation between the impurity spin and the electron spin density at a distance r from the impurity. Nagaoka[2] calculated this function for $T=0$ using his self-consistent solution of the equations of motion. He found that for $r\ll v_F/\Delta$ [where $\Delta\equiv D\exp(-N/|J|\rho)$]

$$\langle \mathbf{S}^{\rm el}(r)\cdot \mathbf{S}^{\rm imp}\rangle=-a(\sin k_F r/r)^2,$$

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[13] The susceptibility formula given in the second paper of Ref. 2 is incorrect. There should be a factor 2 in front of the first term of the right-hand side of Eq. (B12).

[14] J. Zittartz (private communication).

[15] M. Fowler, Phys. Rev. **160**, 463 (1967).

[16] P. E. Bloomfield and D. R. Hamann, Phys. Rev. **164**, 856 (1967).

[17] D. C. Mattis, Phys. Rev. Letters **19**, 1474 (1967).

[18] D. S. Falk and M. Fowler, Phys. Rev. **158**, 567 (1967).

where a is a positive constant. This result shows that the conduction electrons are aligned antiparallel to the impurity spin over a distance of order $v_F/\Delta \simeq 10^{-4}$ cm. We have computed $\langle \mathbf{S}^{\text{el}}(r)\cdot\mathbf{S}^{\text{imp}}\rangle$ to second order in J, the details being given in Appendix D. Besides oscillatory terms, we also find a negative definite term. However, in contrast to Nagaoka's result, ours varies as $1/r^3$. As discussed in Appendix D it is this part which, when integrated over r, yields the logarithmic singularity in the second-order expression for the susceptibility.

IV. DISCUSSION

Several authors[11,19–23] have investigated the magnetic properties of dilute magnetic alloys experimentally. Apart from conventional measurements of the susceptibility, NMR and Mössbauer techniques were used to get information about the local magnetic fields in alloy systems. Since our theoretical calculations in Sec. II were based on the linear response approach, we cannot say anything about the possible nonlinear magnetic field dependence of the electron spin polarization (see Appendix D). The NMR experiments, however, yield information about the temperature dependence of $p(r)$ as well as the magnetic field dependence. According to Sugawara[19] the linewidth of the NMR signal should be proportional to $\langle S_z^{\text{imp}}\rangle$. If the polarization is given, for example, by the RKKY formula

$$p(r) = -3\pi N_0(J\rho/2N)\langle S_z^{\text{imp}}\rangle[j_1(2k_Fr)/(k_Fr)^2], \quad (4.1)$$

NMR enables us to effectively study some spatial average of $p(r)$. If we neglect the terms $p_1(r)$ and $p_2(r)$ for the moment (and remembering p_{Pauli} does not contribute to the linewidth since it is position-independent), we would expect from (2.20) that the measured temperature dependence would be given by

$$\langle S_z\rangle_{\text{eff}} = hC'(T)/T.$$

Comparing this with the susceptibility formula (3.5) we see that the linewidth and the susceptibility should exhibit the same temperature dependence. This is exactly what the experiments show.[11,23]

So far we have neglected the terms $p_1(r)$ and $p_2(r)$. For high temperatures where perturbation theory is applicable, this is certainly justified since $p_0(r)$ is of first order in the expansion parameter $(J\rho/N)$, whereas p_1 and p_2 are of second order. Reintroducing the bare impurity spin into (2.31) and (2.32) by

$$\langle S_z^{\text{imp}}\rangle_{\text{bare}} = -(\mu_B gh/3T)S(S+1),$$

we see that in general the ratio

$$[p_1(r)+p_2(r)]/p_0(r)$$

is of the order of

$$\frac{(J\rho/N)\langle S_z^{\text{imp}}\rangle_{\text{bare}}}{\langle S_z^{\text{imp}}\rangle_{\text{eff}}}. \quad (4.2)$$

As we have discussed in Sec. III the value of $\langle S_z^{\text{imp}}\rangle_{\text{eff}}$ cannot be predicted theoretically with the help of Nagaoka's theory and thus no prediction can be made as to the magnitude of the ratio given in (4.2). Recent experimental data[11,22] shows that for temperatures below 0.5°K, the electron spin polarization becomes temperature-independent. The experimental conditions at these temperatures were such that the Brillouin function representing $\langle S_z^{\text{imp}}\rangle_{\text{bare}}$ is temperature- and field-independent. Therefore the strong field dependence of the polarization which was observed in the same temperature range (see Refs. 11 and 22) must be taken as an indication that even at very low temperatures $p_0(r)$ is still predominant. It also shows that if at these low temperatures $\langle S_z^{\text{imp}}\rangle_{\text{eff}}$ can be described by the Brillouin function, then the effective magnetic moment of the spin must be very small.

ACKNOWLEDGMENTS

We would like to express our sincere gratitude to Dr. A. Griffin for suggesting this problem, for discussions, and for helpful comments on the manuscript. A report from Dr. A. J. Heeger and his co-workers sent to us prior to publication is gratefully acknowledged.

APPENDIX A

We want to show that part of the integral in (2.19),

$$\phi(r) = \frac{1}{\Omega^2}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \times \int d\omega\, f(\omega)\,\text{Im}\,\frac{-i\pi t(\omega)}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})}, \quad (\text{A1})$$

is of second order in J so that it may always be neglected compared to terms of order zero in J. The argument we give here has been developed by Suhl in a similar context. Suhl shows—for the details we refer to Refs. 4 and 5—that under the condition

$$k_Fr \gg 1,$$

$\phi(r)$ can be expressed as follows:

$$\begin{aligned}\phi(r) &= \int_0^\infty e^{2iKr}\frac{\partial}{\partial K}[Kf(K)(-i\pi\rho t(K))+\text{c.c.}]dK \\ &= -(1/r^3)\int_0^\infty dK\{\sin 2Kr(\partial/\partial K) \\ &\quad\times[Kf(K)\pi\rho\,\text{Re}\,t(K)] \\ &\quad+\cos 2Kr(\partial/\partial K)[Kf(K)\pi\rho\,\text{Im}\,t(K)]\}. \quad (\text{A2})\end{aligned}$$

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[19] T. Sugawara, J. Phys. Soc. (Japan) 14, 643 (1959).

[20] M. Daybell and W. Steyert, Phys. Rev. Letters 18, 398 (1967).

[21] C. M. Hurd, Phys. Rev. Letters 18, 1127 (1967).

[22] R. B. Frankel, N. A. Blum, B. B. Schwartz, and D. J. Kim, Phys. Rev. Letters 18, 1050 (1967).

[23] M. A. Jensen, A. J. Heeger, L. B. Welsh, and G. Gladstone, Phys. Rev. Letters 18, 997 (1967).

Now, according to Suhl's work[4,5] $\mathrm{Im}t(\omega)$ has a peak of width k_BT at a distance of the order of k_BT above the Fermi level. Since $\mathrm{Re}t(\omega)$ is connected to $\mathrm{Im}t(\omega)$ through the Kramers-Kronig relations, its main variation takes place in the same interval in which $\mathrm{Im}t(\omega)$ is peaked. As a consequence of this, we may restrict the integral in (A2) to the interval $[-\Delta, +\Delta]$, where $\Delta = ak_BT$ (a is a constant of the order of unity). Outside this interval the derivative in the integrand is essentially zero. For $k_Fr \ll \epsilon_F/k_BT$, the sine and cosine functions may then be taken outside the integral and we obtain

$$\phi(r) \simeq (1/r^3) k_F\pi\rho \times [\mathrm{Re}t(-\Delta)\sin 2k_Fr + \mathrm{Im}t(-\Delta)\cos 2k_Fr]. \quad \text{(A3)}$$

From the qualitative features of $t(\omega)$ as described in the Varenna lecture notes[4] we conclude that $t(-\Delta)$ can be replaced by its lowest-order perturbation theoretic value.

Thus,

$$\phi(r) \propto J^2 \qquad \text{for } 1 \ll k_Fr \ll \epsilon_F/k_BT.$$

This is a very weak condition and allows for all values of r that are of interest in this paper (apart from the susceptibility calculation, as discussed in Appendix D).

APPENDIX B: NONPERTURBATIONAL TREATMENT OF $p(r)-p_0(r)-p_{\mathrm{Pauli}}$

In Sec. II we evaluated $[p(r)-p_0(r)-p_{\mathrm{Pauli}}]$ in second-order perturbation theory. In order to get some insight into the behavior of this quantity for low temperatures where the perturbation expansion may be invalid, we rewrite (2.15) by introducing the t matrix, using

$$G_{\mathbf{kk}'}(\omega) = \delta_{\mathbf{kk}'}/(\omega-\epsilon_{\mathbf{k}}) + (\omega-\epsilon_{\mathbf{k}})^{-1} t(\omega)(\omega-\epsilon_{\mathbf{k}'})^{-1}, \quad \text{(B1)}$$

$$\Gamma_{\mathbf{kk}'}(\omega) = \left\{ \frac{n_{\mathbf{k}'}-\frac{1}{2}}{\omega-\epsilon_{\mathbf{k}'}} t(\omega) + \frac{J}{2N} \frac{m_{\mathbf{k}'}-S(S+1)}{\omega-\epsilon_{\mathbf{k}'}} [1-i\pi\rho t(\omega)] \right\} (\omega-\epsilon_{\mathbf{k}})^{-1}, \quad \text{(B2)}$$

the latter being completely equivalent to (2.21b).

This leads to

$$[p(r)-p_0(r)-p_{\mathrm{Pauli}}] = -\frac{g\mu_Bh}{T}(4\pi\Omega)^{-1} \sum_{\mathbf{k},\mathbf{k}'} \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \left\{ -\frac{6}{\pi} f_{\mathbf{k}} \int d\omega\, f(\omega)\, \mathrm{Im} \frac{t(\omega)}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})} \right.$$
$$- \frac{J}{N} \sum_{\mathrm{l}} \int d\omega\, \mathrm{Im} \frac{f(\omega)}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})} \left[\frac{(n_{\mathrm{l}}+1)(n_{\mathrm{l}}-\frac{1}{2})}{\omega-\epsilon_{\mathrm{l}}} t(\omega) + \frac{J}{2N} \frac{(n_{\mathrm{l}}+1)(m_{\mathrm{l}}-S(S+1))}{\omega-\epsilon_{\mathrm{l}}} (1-i\pi\rho t(\omega)) \right]$$
$$\left. + \frac{3}{2}\frac{J}{N} \int d\omega\, f(\omega)\, \mathrm{Im} \frac{m_{\mathbf{k}'}}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})} \right\}. \quad \text{(B3)}$$

We next note that

$$\sum_{\mathrm{l}} \Gamma_{\mathbf{k}\mathrm{l}}(\omega) = -(2N/J)[t(\omega)/(\omega-\epsilon_{\mathbf{k}})], \quad \text{(B4)}$$

as can be easily seen from the equations of motion (2.21a) and (2.21b). It follows that

$$m_{\mathbf{k}} = -\frac{2}{\pi} \int d\omega\, f(\omega)\, \mathrm{Im} \sum_{\mathrm{l}} \Gamma_{\mathbf{k}\mathrm{l}}(\omega)$$
$$= \frac{4N}{J\pi} \int d\omega\, f(\omega)\, \mathrm{Im} \frac{t(\omega)}{\omega-\epsilon_{\mathbf{k}}}. \quad \text{(B5)}$$

Using Eq. (B4) in the first term on the right-hand side of Eq. (B3) we obtain

$$p(r)-p_0(r)-p_{\mathrm{Pauli}} = -\frac{g\mu_Bh}{T}(4\pi\Omega)^{-1} \sum_{\mathbf{k},\mathbf{k}'} \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}] \left\{ -\frac{3}{2}\frac{J}{N} \frac{f_{\mathbf{k}}m_{\mathbf{k}}-f_{\mathbf{k}'}m_{\mathbf{k}'}}{\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}} - \frac{J}{N} \sum_{\mathrm{l}} \int d\omega\, \mathrm{Im} \frac{f(\omega)}{(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})} \right.$$
$$\left. \times \left[\frac{(n_{\mathrm{l}}+1)(n_{\mathrm{l}}-\frac{1}{2})}{\omega-\epsilon_{\mathrm{l}}} t(\omega) + \frac{J}{2N} \frac{(n_{\mathrm{l}}+1)(m_{\mathrm{l}}-S(S+1))}{\omega-\epsilon_{\mathrm{l}}} (1-i\pi\rho t(\omega)) \right] \right\}. \quad \text{(B6)}$$

We assume that the functions

$$\sum_{\mathrm{l}} \frac{(n_{\mathrm{l}}+1)(n_{\mathrm{l}}-\frac{1}{2})}{\omega-\epsilon_{\mathrm{l}}} \quad \text{and} \quad \sum_{\mathrm{l}} \frac{(n_{\mathrm{l}}+1)(m_{\mathrm{l}}-S(S+1))}{\omega-\epsilon_{\mathrm{l}}}$$

vary slowly with ω except in the interval

$$-\Delta \leq \omega \leq +\Delta,$$

where Δ is of order k_BT. That this is in fact so can be seen using (B5) and

$$n_{\mathbf{k}} = f_{\mathbf{k}} + \pi^{-1} \int d\omega\, f(\omega)\, \mathrm{Im} \frac{i\pi\rho t(\omega)}{\omega-\epsilon_{\mathbf{k}}}. \quad \text{(B7)}$$

Then we can apply Suhl's type of argument (see Appendix A) to the second term in Eq. (B6), which means that we may neglect all but the lowest-order perturbative term in (B6), we get the result given in Eq. (2.33). However, one is not so easily justified in treating the first term like this. From Eq. (B5), one can see that the lowest-order approximation to m_{k} [see Eq. (2.27)] is not very good. The result in Eq. (2.27) is obtained from Eq. (B5), if one replaces $t(\omega)$ by its lowest-order Born approximation:

$$t(\omega)=-4\pi\rho(J/2N)^2S(S+1).$$

In doing so one has completely neglected the resonance in $t(\omega)$, although the most important values of ω in the integrand in (B5) are those for which $-k_BT\leq\omega\leq+k_BT$. It is to be expected that a better approximation for m_{k} would change the result given in Eq. (2.33) quantitatively. However, the *oscillatory* character of $p(r)$ should survive, since this is ultimately related to the sharpness of the Fermi surface.

APPENDIX C: RELATION TO WORK OF FALK AND FULLENBAUM AND OF SUHL

Our calculation of $p(r)$ was based on a linear response approach which provided us with the basic formula (2.4). An alternative definition of the conduction-electron spin polarization would be

$$p(r)=\tfrac{1}{2}[\langle n_\uparrow(r)\rangle-\langle n_\downarrow(r)\rangle]. \tag{C1}$$

Since

$$\langle n_\sigma(r)\rangle=-\pi^{-1}\sum_{\mathrm{k},\mathrm{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\times\int d\omega\, f(\omega)\,\mathrm{Im}G_{\mathrm{kk}'}{}^{(\sigma)}(\omega), \tag{C2}$$

all one needs is the one-electron Green's function $G_{\mathrm{kk}'}{}^{(\sigma)}(\omega)$ under the influence of an external magnetic field. Falk and Fullenbaum[5] computed this function using the equation-of-motion method up to second order in the exchange interaction J, with the result

$$G_{\mathrm{kk}'}{}^{(\sigma)}(\omega)=\frac{\delta_{\mathrm{kk}'}}{\omega-\epsilon_{\mathrm{k}\sigma}}-\frac{-J/2N}{(\omega-\epsilon_{\mathrm{k}\sigma})(\omega-\epsilon_{\mathrm{k}'\sigma})}\times\{\sigma\langle S_z{}^{\mathrm{imp}}\rangle+(i\pi J\rho/2N)[-\sigma\langle S_z{}^{\mathrm{imp}}\rangle+2\sigma\langle S_z{}^{\mathrm{imp}}\rangle\times f(\omega+\sigma\omega_0)+\sigma\langle S_z{}^{\mathrm{imp}}\rangle(J/N)g(\omega+\sigma\omega_0)]\}. \tag{C3}$$

Here

$$\epsilon_{\mathrm{k}\sigma}=(k^2/2m)-\mu_Bg\sigma h \tag{C4}$$

($\sigma=\pm$ and h is the external magnetic field), while $g(\omega)$ is defined by (2.28).

In the course of evaluating the integral in (C2), Falk and Fullenbaum drop the Zeeman term in the one-electron energy $\epsilon_{k\sigma}$. Examining the implications of this step, we find that in doing so one only neglects terms of relative order k_BT/ϵ_F. We recall that terms of this order have been neglected throughout this paper. After performing the required integrations in Eq. (C2), the result still depends on the magnetic field in a nonlinear way. For brevity, we write it in the form

$$p(r)=\langle S_z{}^{\mathrm{imp}}\rangle F_h(r), \tag{C5}$$

where $F_h(r)$ depends on the magnetic field as well as $\langle S_z\rangle$. If it is a good approximation to take

$$\langle S_z{}^{\mathrm{imp}}\rangle\propto\mu_Bgh/T,$$

the linear response approximation to (C1) is given by

$$p_{\mathrm{lin\ resp}}(r)=(\mu_Bgh/3T)S(S+1)F_{h=0}(r). \tag{C6}$$

This agrees precisely with our second-order results given in Sec. II if the explicit form of $F_{h=0}(r)$ is worked out.

Suhl's approach to the polarization[4,5] is also based on the formulas (C1) and (C2). Introducing the non-spin-flip and the spin-flip scattering amplitudes $t(\omega)$ and $\tau(\omega)$, the Green's function $G_{kk'}{}^{(\sigma)}$ can be written as

$$G_{\mathrm{kk}'}{}^{(\sigma)}(\omega)=G_{\mathrm{k}}{}^{0(\sigma)}(\omega)\delta_{\mathrm{kk}'}+G_{\mathrm{k}}{}^{0(\sigma)}(\omega)[t(\omega)+4\tau_T(\omega)\times(\mathbf{S}_T{}^{\mathrm{imp}}\cdot\mathbf{S}_T{}^{\mathrm{el}})+4\tau_z(\omega)(S_z{}^{\mathrm{imp}}\cdot S_z{}^{\mathrm{el}})]G_{\mathrm{k}'}{}^{0(\sigma)}(\omega), \tag{C7}$$

where the subscripts T and z indicate the transverse and z components of the spins and

$$G_{\mathrm{k}}{}^{0(\sigma)}(\omega)=(\omega-\epsilon_{\mathrm{k}\sigma})^{-1}. \tag{C8}$$

In the expression (C7) the average has only been taken over the orbital variables of the electrons, while the impurity spin still occurs as an operator. The first term on the right-hand side of Eq. (C7) yields the Pauli polarization. In the second term, the Zeeman energy may be neglected for the same reason as discussed in the preceding paragraph. Symmetry considerations show that $t(\omega)$ cannnot depend on the magnetic field linearly. Thus we get in linear response

$$p(r)=\tfrac{1}{2}[\langle n_\uparrow(r)-n_\downarrow(r)\rangle]=p_{\mathrm{Pauli}}-\pi^{-1}\sum_{\mathrm{k},\mathrm{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\int d\omega\, f(\omega)\times\mathrm{Im}[G_{\mathrm{k}}{}^{(0)}(\omega)\cdot4\tau_z(\omega)G_{\mathrm{k}'}{}^{(0)}(\omega)]\langle S_z{}^{\mathrm{imp}}\rangle. \tag{C9}$$

Suhl calculated $\langle S_z{}^{\mathrm{imp}}\rangle$ using the free-spin density matrix

$$\rho_{\mathrm{spin}}=\exp(-\beta\mu_BghS_z)/\mathrm{Tr}[\exp(-\beta\mu_BghS_z)]. \tag{C10}$$

This is the reason why his result for $p(r)$ is proportional to the bare impurity spin. Since in deriving (C9) nonlinear terms in the magnetic field [those contained in $t(\omega)$] have already been neglected, it is consistent to set

$$\langle S_z{}^{\mathrm{imp}}\rangle\simeq(g\mu_Bh/3T)S(S+1),$$

and to neglect the h dependence of $\tau_z(\omega)$. Without going into any further details, we note that Suhl's final result[4,5] is simply the RKKY polarization which is of first order in J. By iterating Suhl's integral equations for $t(\omega)$ and $\tau(\omega)$ one can easily obtain $\tau(\omega)$ to

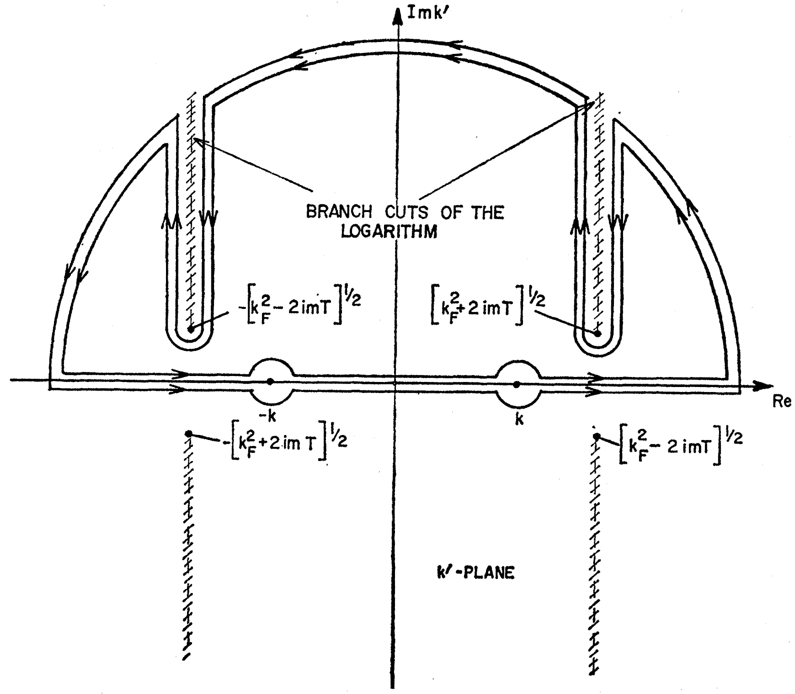

FIG. 1. Integration contour of the k' integration in Eq. (D8).

second order in J,

$$\tau(\omega)=\tfrac{1}{4}(J/2N)-i\pi\rho(J/2N)^2[f(\omega)-\tfrac{1}{2}]$$
$$-\tfrac{1}{4}(J/2N)^2\rho\ \ln[(\omega^2+T^2)/D^2]^{1/2}.$$

Inserting this expression into (C9), we find exactly the same result as Falk and Fullenbaum for $p(r)$.

Recently More and Suhl[24] have pointed out that (C7) does not use the most general form for the t matrix in the presence of the magnetic field. This is irrelevant in the present linear approximation, as we eventually only use the magnetic-field-independent t matrix.

APPENDIX D: CALCULATION OF THE IMPURITY-SPIN ELECTRON-SPIN CORRELATION FUNCTION TO SECOND ORDER IN J

It so happens that the correlation function $\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$ is directly connected to Nagaoka's $\Gamma_{kk'}(\omega)$ by the relation

$$\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$$
$$=-\pi^{-1}\Omega^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\int d\omega\, f(\omega)\ \mathrm{Im}\Gamma_{\mathbf{kk}'}(\omega) \tag{D1}$$
$$=-\pi^{-1}\Omega^{-1}\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\int d\omega\, f(\omega)$$
$$\times\mathrm{Im}[\Gamma_{\mathbf{kk}'}^{(1)}(\omega)+\Gamma_{\mathbf{kk}'}^{(2)}(\omega)+\cdots]. \tag{D2}$$

Here $\Gamma_{\mathbf{kk}'}^{(1)}(\omega)$ and $\Gamma_{\mathbf{kk}'}^{(2)}(\omega)$ are the first- and second-order iterative solutions to the Nagaoka equations. $\Gamma_{\mathbf{kk}'}^{(1)}(\omega)$ has already been given as (2.25), and $\Gamma_{\mathbf{kk}'}(\omega)$ is found to be

$$\Gamma_{\mathbf{kk}'}^{(2)}(\omega)=\tfrac{1}{2}(J/N)^2\rho S(S+1)(\omega-\epsilon_{\mathbf{k}})^{-1}$$
$$\times(\omega-\epsilon_{\mathbf{k}'})^{-1}[g(\epsilon_{\mathbf{k}})+\tfrac{1}{2}(i\pi)], \tag{D3}$$

where

$$g(\epsilon_{\mathbf{k}})=\int[f(\omega)/(\epsilon_{\mathbf{k}}-\omega+i\delta)]d\omega, \tag{D4}$$

as defined in Sec. II. The first-order term in (D2) yields the RKKY oscillations. The second-order term can be written as follows:

$$\langle \mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle^{(2)}$$
$$=(2\pi\Omega)^{-1}(J/N)^2\rho S(S+1)$$
$$\times\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}](\epsilon_{\mathbf{k}'}-\epsilon_{\mathbf{k}})^{-1}$$
$$\times\Big\{f_k\ln[(\epsilon_{k'}^2+T^2)/D^2]-f_{k'}\ln[(\epsilon_k^2+T^2)/D^2]^{1/2}$$
$$-\tfrac{1}{2}\pi\int d\omega\, f(\omega)\ \mathrm{Re}[(\omega-\epsilon_{\mathbf{k}})(\omega-\epsilon_{\mathbf{k}'})]^{-1}\Big\} \tag{D5}$$
$$=\tfrac{1}{2}(J/N)^2\rho S(S+1)[I_1(r)+I_2(r)]. \tag{D6}$$

Here we have made use of the approximation (2.28) for $g(\epsilon_k)$ wherever this does not cause divergences in the remaining sums over k and k'. The integral $I_2(r)$ can be evaluated in a straightforward manner. Neglect-

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[24] R. More and H. Suhl, Phys. Rev. Letters **20**, 500 (1968).

ing temperature-dependent corrections, which are of order k_BT/ϵ_F, we obtain

$$I_2(r)=+(3/8)\pi^2N_0\rho[n_1(2k_Fr)/(k_Fr)^2]. \tag{D7}$$

After integration over the angular variables $I_1(r)$ takes

$$I_1(r)=-\frac{\Omega}{r^3}m\int_{-\infty}^{\infty}\frac{k'dk'}{(2\pi)^2}e^{ikr}e^{ik'r}\ln\left(\frac{\epsilon_{k'}^2+T^2}{D^2}\right)^{1/2}$$
$$\times[P/(k'+k)+P/(k'-k)]. \tag{D8}$$

The integration over k' can be done by the contour integration method. The integration path is sketched in Fig. 1.

We obtain

$$I_1(r)=\frac{\Omega m}{r^2}i^{-1}\int_{-\infty}^{\infty}\frac{dk}{(2\pi)^3}ke^{ikr}\cos kr\,f_k\ln\left(\frac{\epsilon_k^2+T^2}{D^2}\right)^{1/2}$$
$$+(\Omega m/2r^2)i^{-1}\int_{-\infty}^{\infty}dk\,kf_k\sum_{\nu}\{\mathrm{Ei}[i(K_\nu+k)r]$$
$$+e^{2ikr}\,\mathrm{Ei}[i(K_\nu-kr)]\}, \tag{D9}$$

where

$$K_\nu=(-1)^\nu[k_F^2+(-1)^\nu 2imT]^{1/2}\qquad \nu=0,\,1.$$

In (D9) we replace f_{k} by its zero-temperature limit as we did before. After the remaining integration is performed, we have

$$I_1(r)=\tfrac{3}{2}\pi N_0\rho[(k_Fr)^2]^{-1}\{[\mathrm{Ci}(4k_Fr)+\mathrm{Ci}(2k_Fr)$$
$$-\ln((\gamma D/\epsilon_F)k_Fr)]j_1(2k_Fr)+[j_0(2k_Fr)/k_Fr]$$
$$+[\mathrm{Si}(4k_Fr)+\mathrm{Si}(2k_Fr)-\tfrac{1}{2}\pi]n_1(2k_Fr)$$
$$-\tfrac{1}{2}\pi n_1(2k_Fr)-(2k_Fr)^{-1}\}. \tag{D10}$$

Here we have neglected terms of order $(T\epsilon_F)k_Fr$, which means that the expression is correct only if $k_Fr\ll\epsilon_F/T$. Collecting our results we find

$$\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle=\tfrac{3}{4}\pi N_0(J\rho/N)S(S+1)[j_1(2k_Fr)/(k_Fr)^2]$$
$$+\tfrac{3}{4}\pi N_0(J\rho/N)^2[(k_Fr)^2]^{-1}\{[\mathrm{Ci}(4k_Fr)+\mathrm{Ci}(2k_Fr)$$
$$-\ln((\gamma D/\epsilon_F)k_Fr)]j_1(2k_Fr)+[j_0(2k_Fr)/k_Fr]$$
$$+[\mathrm{Si}(4k_Fr)+\mathrm{Si}(2k_Fr)-\tfrac{1}{2}\pi]n_1(2k_Fr)-(2k_Fr)^{-1}\}. \tag{D11}$$

The most remarkable term in the expression (D11) is the last one. It shows that there is a *nonoscillatory* contribution to the electron-spin impurity-spin correlation function. It is this term which is responsible for the "Kondo" $\log(\epsilon_F/T)$ term in the susceptibility. To see this more explicitly, we consider

$$\langle\mathbf{S}^{\mathrm{el}}\cdot\mathbf{S}^{\mathrm{imp}}\rangle=\int_0^\infty r^2dr\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle. \tag{D12}$$

The integrand has a nonintegrable singularity at $r=0$, which arises from the singular nature of the contact interaction in r space. At the upper integration limit only the last term of the expression (D11) leads to difficulties. We must, however, keep in mind that this expression is only valid if $k_Fr\ll\epsilon_F/T$. For $k_Fr>\epsilon_F/T$, it turns out that last term in (D11) has to be multiplied by $\exp[-(T/\epsilon_F)k_Fr]$. Thus we may write

$$\langle\mathbf{S}^{\mathrm{el}}\cdot\mathbf{S}^{\mathrm{imp}}\rangle$$
$$=\left(\int_0^{\epsilon_F/Tk_F}+\int_{\epsilon_F/Tk_F}^{\infty}\right)r^2dr\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$$
$$\simeq AJ+BJ^2-\tfrac{3}{4}\pi N_0(J\rho/N)S(S+1)$$
$$\times\left\{\int_0^{\epsilon_F/Tk_F}r^2dr\frac{1}{2(k_Fr)^3}\right.$$
$$\left.\times\int_{\epsilon_F/Tk_F}^{\infty}r^2dr\frac{\exp[-(T/\epsilon_F)k_Fr]}{2(k_Fr)^3}\right\}, \tag{D13}$$

where A and B are temperature-independent constants. As mentioned above, the (unphysical) singularity of the first integral at the lower integration limit need not bother us. However, from the upper integration limit, we get a $\log(\epsilon_F/T)$-type divergence. This is the singularity we encounter in the second-order expression for the susceptibility. The last integral in Eq. (D13) is independent of the temperature.

The results which we have presented in this Appendix were obtained by iterating Nagaoka's equations of motion. This is the most convenient way of computing the correlation function $\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$. However, we would like to remark that we get precisely the same results by means of a straightforward (but lengthy) second-order perturbation calculation of this correlation function.

APPENDIX E: CHARGE OSCILLATIONS AROUND A MAGNETIC IMPURITY

It is well known that a spin-dependent impurity potential in an electron gas leads to an oscillatory charge distribution around the impurity (Friedel oscillations), which already appears in first-order perturbation theory with respect to the impurity potential $V(r)$. As we shall demonstrate below, the spin-dependent exchange potential

$$H_{\mathrm{sd}}=(J/2N)\sum_{\mathrm{k,k'}}\mathbf{S}\cdot\boldsymbol{\sigma}_{\alpha\beta}C_{\mathrm{k}\alpha}{}^{\dagger}C_{\mathrm{k'}\beta} \tag{E1}$$

leads to charge oscillations only in second-order perturbation theory. However, one should not forget that the exchange potential H_{sd} is due to the presence of a paramagnetic ion in the electron gas and is therefore generally accompanied by a spin-independent potential. We shall therefore include this normal potential in our subsequent calculation of the charge oscillations $\rho(r)$.

To compute $\rho(r)$ we make use of the relation

$$\rho(r)=2e\sum_{\mathbf{k},\mathbf{k}'}\rho_{\mathbf{k}\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]$$
$$=-(2/\pi)\sum_{\mathbf{k},\mathbf{k}'}\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]$$
$$\times\int d\omega\, f(\omega)\,\mathrm{Im}G_{\mathbf{k}\mathbf{k}'}(\omega). \quad \text{(E2)}$$

$G_{\mathbf{k}\mathbf{k}'}(\omega)$ may be obtained from the equations of motion (2.21a) and (2.21b) generalized to take into account the spin-independent potential $V(r)$.[25] We also assume that $V(r)$ is a contact potential

$$V(r)=V\delta(r).$$

Then the term

$$(V/N)\sum_{\mathbf{l}}G_{\mathbf{l}\mathbf{k}'}(\omega)$$

has to be added to the right-hand side of Eq. (2.21a) and the term

$$(V/N)\sum_{\mathbf{l}}\Gamma_{\mathbf{l}\mathbf{k}'}(\omega)$$

has to be added to the right-hand side of Eq. (2.21b). A straightforward iteration yields to second order in J and V

$$G_{\mathbf{k}\mathbf{k}'}{}^{(2)}(\omega)=\frac{\delta_{\mathbf{k}\mathbf{k}'}}{\omega-\epsilon_{\mathbf{k}}}+(\omega-\epsilon_{\mathbf{k}})^{-1}$$
$$\times\left[\frac{V}{N}-i\pi\rho\frac{V^2}{N^2}-i\pi\rho\frac{J^2S(S+1)}{(2N)^2}\right](\omega-\epsilon_{\mathbf{k}'})^{-1}. \quad \text{(E3)}$$

Inserting this into (E2) we obtain

$$\rho(r)=\rho_0+6eN_0\pi$$
$$\times\left\{\frac{V\rho}{N}\frac{j_1(2k_Fr)}{k_Fr}-\tfrac{1}{2}\pi\left[\frac{V^2\rho^2}{N^2}+\frac{J^2S(S+1)}{4N^2}\right]\frac{n_1(2k_Fr)}{(k_Fr)^2}\right\}. \quad \text{(E4)}$$

Evidently there is no anomaly in $\rho(r)$ whatsoever in this order.

The potential $V(r)$ would of course also affect the results which have been presented in the main part of this paper. However, since in all the magnetic quantities only products of the type $J^m\cdot V^n$ ($m=1, 2\cdots$; $n=0, 1\cdots$) can occur, no drastic change in the nature of the results is expected. As an example, we quote the additional contribution to the correlation function $\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle$, which is found if we include the spin-independent potential $V(r)$ to lowest order. The result is

$$\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle-\langle\mathbf{S}^{\mathrm{el}}(r)\cdot\mathbf{S}^{\mathrm{imp}}\rangle_{v=0}$$
$$=\tfrac{3}{2}\pi^2N_0(JV/N)\rho^2S(S+1)[n_1(2rk_F)/(k_Fr)^2]. \quad \text{(E5)}$$

What we have said so far about the effect of a spin-independent potential on the magnetic properties is only true insofar as the potential of the single paramagnetic ion itself is concerned. If besides this paramagnetic ion other randomly distributed (and possibly magnetic) impurities are present in sufficient concentration that their effect on the conduction electrons can no longer be neglected, the situation may change considerably. As one example, the finite mean free path of the electrons is expected to somehow limit the range of the spin and charge density oscillations (see, for example, Ref. 26). In addition, the conduction-electron bulk density of states may be modified as a result of the Kondo effect, as it must in view of the anomalies in the electronic specific heat.

As the concentration of paramagnetic impurities is raised there must also be a certain critical concentration beyond which the interaction between the impurities can no longer be neglected. It is hard to estimate what this critical concentration might be since there is no characteristic relaxation time involved in our expressions for the spin-density oscillations. However, these expressions certainly cease to be valid beyond this critical concentration. Presumably the theory has to be modified so as to include an internal magnetic field which is produced by the impurity spins.

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[25] K. Fischer, Phys. Rev. **158**, 613 (1967).

[26] P. G. de Gennes, J. Phys. Radium **23**, 630 (1962).