Sum Rule for the Landau Fermi-Liquid Parameters in Metals

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(Received 4 June 1968)

We have derived an exact sum rule among the Legendre polynomial expansion coefficients of the Landau f function. The relation explicitly depends on both band-structure and phonon-anisotropy effects. It is based on the exclusion principle and correctly takes into account the effects of the long-range self-consistent field. We compare the predictions of the sum rule with the sum of the first few experimentally determined Landau parameters for Na. The results of the comparison are consistent with the existing data.

1. INTRODUCTION

IN the last three years various new spin-wave and plasma-wave resonances have been observed in the alkali metals.[1,2] Spin waves with nonzero wave vectors were first observed by Schultz and Dunifer[1] in Na and K, while high-frequency plasmalike waves were first seen by Walsh and Platzman[2] in these same metals. These resonances are of interest because they involve the interaction between particles in a nontrivial way and, at the same time, are at sufficiently low frequencies that they can be given a precise interpretation in terms of the Landau theory of a Fermi liquid.[3] In this theory, the interaction between two quasiparticles is characterized by an effective interaction function $f(\mathbf{k}\sigma, \mathbf{k}'\sigma')$. It has been shown that all long-wavelength low-energy excitations of a normal Fermi system can be completely described in terms of the Landau function and an effective mass for the quasiparticles.[3] The theory thus gives a phenomenologically self-consistent interpretation of all low-energy experiments on a Fermi liquid.

Silin[4] extended the Landau theory, originally formulated for neutral Fermi systems, to include the charged Fermi gas by properly introducing the self-consistent field into the transport equation. Subsequently, he considered the problem of resonances in an infinite metallic medium. The boundary conditions for finite bulk samples, the effect of collisions, and the influence of orbital motion were considered, for the conduction electron spin resonance problem, by Platzman and Wolff[5] for application to the alkali metals. Since the metals considered have spherical Fermi surfaces and since the samples examined were polycrystalline, the Landau function was assumed to depend only on the angle $\theta_{\mathbf{k},\mathbf{k}'}$ between the wave vectors of the two interacting quasiparticles. In this case, the equations for the resonant frequencies of the normal modes of the system may be written in terms of the coefficients of the expansion of the f function in Legendre polynomials of the angle $\theta_{\mathbf{k},\mathbf{k}'}$. Several of the lower-order coefficients of this expansion have been determined from the experimental data.

The objective in this paper is to find an exact relationship among the Legendre polynomial expansion coefficients of the Landau f function for a real metal. Such a relationship in the form of a sum rule is known for the uncharged Fermi liquid.[6] It depends on the fact that the forward scattering amplitude for two fermions of the same spin vanishes. We have considered this sum rule for the alkali metals, taking into account both the Coulomb and the electron-phonon interactions. The main difference between charged and uncharged systems arises from the long-range nature of the Coulomb force. In the charged case, the interaction between two quasiparticles is not entirely given by the Landau function; rather, the long-range part is described in terms of the macroscopic self-consistent field in the crystal. Only the remaining short-range interactions are described by the Landau function. Consequently, the part of the forward scattering amplitude described by the Landau parameters, which sums to zero for the uncharged system, now depends on the long-wavelength limit of the self-consistent field.

When anisotropic interactions are included one finds that the contributions to the forward scattering amplitude from the self-consistent field and from the short-range interactions depend on the state on the Fermi surface into which the two quasiparticles are scattered. They also depend on the direction that $\mathbf{k}$–$\mathbf{k}'$ makes with the crystal axes. One thus obtains a relation that holds at each point on the Fermi surface and for all possible directions in which one can bring the wave vectors of the two scattered particles together. By averaging this relation over these variables, one obtains a relation which, when applied to the alkali metals, is quite similar to the isotropic sum rule.

[1] S. Schultz and G. Dunifer, Phys. Rev. Letters **18**, 283 (1967); see also G. Dunifer, S. Schultz, and P. H. Schmidt, J. Appl. Phys. **39**, 397 (1968).
[2] W. M. Walsh, Jr., and P. M. Platzman, Phys. Rev. Letters **15**, 784 (1965); P. M. Platzman and W. M. Walsh, Jr., *ibid.* **19**, 514 (1967); **20**, 89(E) (1968).
[3] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956); **32**, 59 (1957) [English transls.: Soviet Phys.—JETP **3**, 920 (1956); **5**, 101 (1957)].
[4] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 1227 (1957); **35**, 1243 (1958) [English transls.: Soviet Phys.—JETP **6**, 945 (1958); **8**, 870 (1959)].
[5] P. M. Platzman and P. A. Wolff, Phys. Rev. Letters **18**, 280 (1967).
[6] J. C. Wheatley, in *Quantum Fluids*, edited by D. F. Brewer (John Wiley & Sons, Inc., New York, 1966), p. 183.

In Sec. 2 we derive the sum rule for the simple case of the electron gas. This result should be useful for checking the consistency of calculations in this theoretical system. It also enables us to define clearly the limits of the interaction function used to obtain the sum rule. In Sec. 3 we include the electron-phonon interactions and the effects due to anisotropy in the phonon spectrum. These are probably the most important anisotropic effects in the alkali metals. In the last section, we apply the sum rule to Na and give a few conclusions.

2. ELECTRON GAS

To derive the sum rule, it is necessary to make contact with the microscopic definition of the functions that appear in the Landau theory. We will follow the notation used by Noziéres[7] where possible. The sum rule is a direct result of the Pauli exclusion principle, which demands that the four-point vertex function be antisymmetric with respect to an interchange of the labels of the two outgoing fermion lines. As a consequence, the full vertex function shown in Fig. 1 satisfies the equation

$$^0\Gamma(p\uparrow, p'\uparrow; \tilde{\omega}) = -{}^0\Gamma((p+p'-\tilde{\omega})/2\uparrow, (p+p'+\tilde{\omega})/2\uparrow; p'-p), \tag{1}$$

where $p\equiv(\mathbf{k}, \omega)$ and $p'\equiv(\mathbf{k}', \omega')$. The quantity $\tilde{\omega}\equiv(\mathbf{q}, \epsilon)$ is the total momentum of the particle-hole pair. Setting $\tilde{\omega}=p'-p$ we obtain the result

$$^0\Gamma(p\uparrow, p'\uparrow; p'-p)=0. \tag{2}$$

If we take the limit of Eq. (2) as $|\mathbf{k}-\mathbf{k}'|$ goes to zero, after setting ω and ω' equal to zero, we find

$$\lim_{|\mathbf{k}-\mathbf{k}'|\to 0}\ \lim_{\omega,\omega'\to 0} {}^0\Gamma(p\uparrow, p'\uparrow; p'-p)={}^0\Gamma^{\infty}(\mathbf{k}\uparrow, \mathbf{k}\uparrow)=0. \tag{3}$$

This result is simply the statement that the forward scattering amplitude for two fermions is zero.

In the unchanged system, the Landau interaction function is related to the vertex function $\Gamma(p, p'; \tilde{\omega})$ with p and p' on the energy shell and on the Fermi surface,[7] i.e.,

$$f(\mathbf{k}\uparrow, \mathbf{k}'\uparrow)=2\pi i z_{\mathbf{k}} z_{\mathbf{k}'} \lim_{q/\epsilon\to 0}\ \lim_{\tilde{\omega}\to 0} {}^0\Gamma(p\uparrow, p'\uparrow; \tilde{\omega}) = 2\pi i z_{\mathbf{k}} z_{\mathbf{k}} {}^0\Gamma^0(\mathbf{k}\uparrow, \mathbf{k}'\uparrow). \tag{4}$$

Here $z_{\mathbf{k}}$ is the quasiparticle renormalization factor. Because of the nonanalyticity of Γ in the small $\mathbf{q},\epsilon$ limit, $f(\mathbf{k}\uparrow, \mathbf{k}'\uparrow)$ is not directly related to

$$^0\Gamma(p\uparrow, p'\uparrow; p'-p).$$

However, it is well known that the function[8]

$$g(\mathbf{k}\uparrow, \mathbf{k}'\uparrow)=2\pi i z_{\mathbf{k}} z_{\mathbf{k}'} {}^0\Gamma^{\infty}(\mathbf{k}\uparrow, \mathbf{k}'\uparrow) \tag{5}$$

is related to the Landau interaction function by the integral equation

$$g(\mathbf{k}\sigma, \mathbf{k}'\sigma')=f(\mathbf{k}\sigma, \mathbf{k}'\sigma') - \sum_{\mathbf{k}''\sigma''} f(\mathbf{k}\sigma, \mathbf{k}''\sigma'')\,\delta(\epsilon_{k''}-\mu)\,g(\mathbf{k}''\sigma'', \mathbf{k}'\sigma'). \tag{6}$$

For isotropic systems with spin-independent forces this equation can be solved by expanding g and f in Legendre polynomials of $\cos[\theta_{\mathbf{k},\mathbf{k}'}]$, where $\theta_{\mathbf{k},\mathbf{k}'}$ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Breaking the f function into singlet f_s and triplet f_a parts and defining

$$A_l=\frac{m^*k_F}{2\pi^2}\int\frac{d\Omega}{2\pi} f_s(\cos\theta)P_l(\cos\theta)$$

and

$$B_l=\frac{m^*k_F}{2\pi^2}\int\frac{d\Omega}{2\pi} f_a(\cos\theta)P_l(\cos\theta), \tag{7}$$

we obtain

$$g_{sl}=A_l/(1+A_l)$$

and

$$g_{al}=B_l/(1+B_l). \tag{8}$$

Here g_{sl} and g_{al} are related to g in the same manner as the A_l and B_l are to f. Utilizing Eqs. (3)–(8) we find that

$$\sum_{l=0}^{\infty}(2l+1)\left[\frac{A_l}{1+A_l}+\frac{B_l}{1+B_l}\right]=0. \tag{9}$$

This is the exclusion principle sum rule for the uncharged system.[9] It has been applied by several authors[6,10] to liquid He^3.

In the electron gas Eq. (4) is not valid because it is necessary to separate out the long-range interactions that are described by the self-consistent field. The separation is shown diagrammatically in Fig. 1. For-

[7] P. Noziéres, *Theory of Interacting Fermi Systems* (W. A. Benjamin, Inc., New York, 1964).

[8] We denote the scattering amplitude by $g(\mathbf{k}\sigma, \mathbf{k}'\sigma')$ rather than by $A(\mathbf{k}\sigma, \mathbf{k}'\sigma')$ as Nozières (Ref. 7) has done, to avoid conflict with the notation A_n for the Legendre coefficients of the f function.

[9] We have tacitly assumed throughout this paper that the series expansion of the Landau function in Legendre polynomials is convergent. It has been shown [C. Herring, in *Exchange Interactions Among Itinerant Electrons*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1966)] that in second-order perturbation theory the f function has a logarithmic infinity in the back scattering directions. The summation of the particle-particle ladder graphs removes this infinity [A. A. Adsikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963)] so that we feel that as long as the system is normal and away from all instabilities the series will at least be asymptotic.

[10] D. J. Amit, J. W. Kane, and H. Wagner, Phys. Rev. Letters 19, 425 (1967).

mally

$$^0\Gamma(p\uparrow, p'\uparrow, \tilde{\omega}) = {}^0\tilde{\Gamma}(p\uparrow, p'\uparrow, \tilde{\omega}) - \frac{iV_q}{2\pi}\left(\frac{\tilde{\Lambda}_4(p;\tilde{\omega})\,\tilde{\Lambda}_4(p';\tilde{\omega})}{1+(iV_q/2\pi)\tilde{S}_{44}(\tilde{\omega})}\right). \quad (10)$$

The functions on the right-hand side of this equation are defined in Ref. 7. Each function is a sum of an infinite set of irreducible diagrams. The diagrams are irreducible with respect to the breaking of a single Coulomb line V_q. The quantity $\tilde{\Lambda}_4$ is the proper three-point vertex function while $\tilde{S}_{44}$ is the proper polarization propagator. Taking the limit as defined in Eq. (3), using the Ward identities,[7]

$$\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to\infty}\tilde{\Lambda}_4(p,\tilde{\omega}) = \frac{|\mathbf{v}_F|}{z_k}\frac{d|\mathbf{k}_F|}{d\mu} \quad (11)$$

and

$$\lim_{\tilde{\omega}\to 0}\lim_{q/\epsilon\to\infty}\tilde{S}_{44}(\tilde{\omega}) = -2\pi i(dn/d\mu), \quad (12)$$

we find

$$2\pi i z_k^2\,{}^0\tilde{\Gamma}^\infty(\mathbf{k}\uparrow, \mathbf{k}\uparrow) = -(\pi^2/m^*)(dk_F/d\mu). \quad (13)$$

Here $\mathbf{v}_F = \mathbf{k}_F/m^*$ is the Fermi velocity, μ is the chemical potential, and n is the electronic density. The Landau f function is related to ${}^0\tilde{\Gamma}^0(\mathbf{k}\sigma, \mathbf{k}'\sigma')$ by a relation identical with Eq. (3). Since

$$dk_F/d\mu = m^*/k_F(1+A_0), \quad (14)$$

we find

$$\sum_{l=0}^{\infty}(2l+1)\left[\frac{A_l}{1+A_l}+\frac{B_l}{1+B_l}\right] = -(1+A_0)^{-1}. \quad (15)$$

This sum rule is a useful check on calculations of the f function for the electron gas.[11] Note that Eq. (15) does not depend on A_0.

3. ELECTRON-PHONON SYSTEM

The formulation of the Landau theory in the presence of electron-phonon interactions alone was first carried out by Prange and Kadanoff.[12] Recently, Prange and Sachs[13] have extended this formulation to include both Coulomb and phonon interactions between electrons.[14] In this section we will show that, in order to apply Eq. (3) to a "real" metal it is essential to explicitly include electron-phonon effects, allowing for the anisotropy of the phonon spectrum.

Again, we consider the full four-point vertex function in the neighborhood of the Fermi surface. In the alkali metals, this allows us to restrict our attention to a single band, the conduction band. All quantities will refer to this band and we will not introduce band indices. The momentum variables will be restricted to the first zone. As in the previous section, we must separate out the part of the four-point vertex function that is included in the macroscopic self-consistent field. When phonons are present there is a contribution to the macroscopic field from the variations in ion density. The total vertex function may be written as

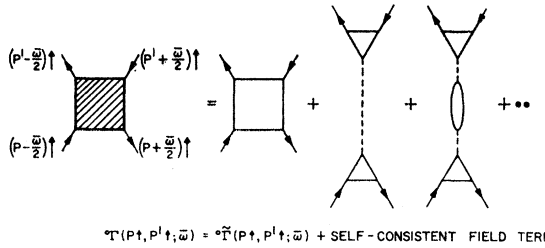

FIG. 1. The separation of the four-point vertex function into the short-range part $\tilde{\Gamma}$ and the macroscopic self-consistent field part.

$$^0\Gamma(p\uparrow, p'\uparrow; \tilde{\omega}) = {}^0\tilde{\Gamma}(p\uparrow, p'\uparrow; \tilde{\omega}) - \frac{iV_q}{2\pi}\frac{\tilde{\Lambda}_4(p;\tilde{\omega})\,\tilde{\Lambda}_4(p';\tilde{\omega})}{1+(iV_q/2\pi)\tilde{S}_{44}(\tilde{\omega})} - (i/2\pi)g_\lambda(p,\tilde{\omega})g_\lambda(p',\tilde{\omega})\mathfrak{D}_\lambda(\tilde{\omega}), \quad (16)$$

where $\tilde{\Gamma}$, $\tilde{\Lambda}_4$, and $\tilde{S}_{44}$ are defined as before but are irreducible with respect to breaking either a single Coulomb line or a phonon line with momentum $\tilde{\omega}$, where $\mathbf{q}$ is restricted to the first zone. Note that diagrams that are reducible with respect to a Coulomb line with momentum $\mathbf{q}+\mathbf{K}$ are included in $\tilde{\Gamma}$, $\tilde{\Lambda}_4$, and $\tilde{S}_{44}$. ($\mathbf{K}$ is any reciprocal-lattice vector.) This means that the macroscopic field describes only that part of the induced field that has the same wavelength as the applied field.

The last term represents the contribution to the self-consistent field due to the phonons. $\mathfrak{D}_\lambda$ is the exact propagator for phonons of polarization λ. The $g_\lambda(p,\tilde{\omega})$ is the renormalized electron-phonon interaction

$$g_\lambda(p,\tilde{\omega}) = \frac{g_\lambda{}^0(\mathbf{q})\,\tilde{\Lambda}_4(p,\tilde{\omega})}{1+iV_q/2\pi\tilde{S}_{44}(\tilde{\omega})} \quad (17)$$

with

$$g_\lambda{}^0(\mathbf{q}) = -(N_a/2\Omega_{q\lambda}M)^{1/2}V^0(\mathbf{q})\,\mathbf{q}\cdot\boldsymbol{\epsilon}_{q\lambda}. \quad (18)$$

In this equation V^0 is the unscreened ionic pseudopotential,[15] $\Omega_{q\lambda}$ and $\boldsymbol{\epsilon}_{q\lambda}$ are the exact frequency and polarization vectors of the λ phonon mode, and N_a is the number of atoms per unit volume.

The full four-point vertex function ${}^0\Gamma$ satisfies Eq. (3) as before. Setting $\omega=\omega'=0$ and denoting the direction of $\mathbf{k}-\mathbf{k}'$ by $\hat{q}$ we examine the limit as $|\mathbf{k}-\mathbf{k}'|\to 0$ of Eq. (16). The generalization for nonuniform systems of the Ward identities Eqs. (11) and (12) has been considered by Sham and Kohn.[16] The results are formally

[11] T. M. Rice (to be published).
[12] R. E. Prange and L. P. Kadanoff, Phys. Rev. **134**, A566 (1964).
[13] R. E. Prange and A. Sachs, Phys. Rev. **158**, 672 (1967).
[14] B. L. Jones and J. W. McClure, Phys. Rev. **143**, 133 (1966), have included band structure effects ignoring phonon interactions.
[15] V. Heine, P. Nozières, and J. W. Wilkins, Phil. Mag. **13**, 741 (1966).
[16] L. J. Sham and W. Kohn, Phys. Rev. **145**, 561 (1966).

the same as Eqs. (11) and (12) but now Eq. (11) varies with position on the Fermi surface. Using the result that the

$$\lim_{q\to 0} V_0(\mathbf{q}) \longrightarrow -4\pi e^2 Z/q^2$$

and defining

$$g(\mathbf{k}\uparrow,\mathbf{k}'\uparrow) = 2\pi i z_{\mathbf{k}} z_{\mathbf{k}'} \lim_{\bar{\omega}\to 0} \lim_{q/\epsilon\to 0} {}^0\Gamma(p\uparrow, p'\uparrow; \bar{\omega}), \quad (19)$$

we find

$$\lim_{|\mathbf{k}-\mathbf{k}'|\to 0} g(\mathbf{k}\uparrow,\mathbf{k}'\uparrow) = -\left(|\mathbf{v}_F| \frac{d|\mathbf{k}_F|}{d\mu}\right)^2 \left(\frac{dn}{d\mu}\right)^{-1}$$

$$\times\left[1-\frac{N_a Z^2}{M}\left(\frac{dn}{d\mu}\right)^{-1} \sum_\lambda \left(\frac{\hat{q}\cdot\boldsymbol{\epsilon}_\lambda(\hat{q})}{v_\lambda(\hat{q})}\right)^2\right] \equiv g(\mathbf{k},\hat{q}). \quad (20)$$

Here the quantities $v_\lambda(\hat{q})$ are the phonon velocities at small $|\mathbf{q}|$. This is the general form of the sum rule for any point on the Fermi surface. Equation (20) includes all the anisotropy effects that are present in metals. The first term in round brackets on the right-hand side of Eq. (20) contains all Fermi-surface anisotropy effects. The second term in the square brackets contains the effects of phonon anisotropy.

Before discussing in detail the anisotropy effects in Eq. (20), it is instructive to consider the sum rule for the jellium model. In this model the ions are replaced by a uniform background of positive charge whose mass density is Mn/Z. In this case it is straightforward to show that the right-hand side of Eq. (20) vanishes identically. The sum rule then reduces to the form for short-range interactions Eq. (9). This is not surprising since, in this model, in the long-wavelength static limit there is perfect screening of the long-range part of the interaction.[17]

Let us now examine the right-hand side of Eq. (20) in the alkali metals, as a function of $\mathbf{k}$ and $\hat{q}$. The dependence of the second term in brackets on $\hat{q}$ can be determined from the measured elastic constants. For Na and K this term varies by almost a factor of 2 according to the direction of $\hat{q}$ relative to the crystal axes. Thus $g(\mathbf{k},\hat{q})$ depends strongly on $\hat{q}$. The dependence on $\mathbf{k}$ is very much weaker. In Na and K the Fermi surface is spherical to better than 0.2%,[18] and no anisotropy has been observed in the effective mass to an accuracy of 2.0%.[19] Prange and Sachs[13] have shown that $d|\mathbf{k}_F|/d\mu$ is independent of electron-phonon effects so that we expect this quantity to be almost isotropic. Thus, to a good approximation, we can neglect the $\mathbf{k}$ dependence of $g(\mathbf{k},\hat{q})$ and assume that it only depends on $\hat{q}$.

The function $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ is quite anisotropic. This anisotropy is due primarily to the anisotropy of the phonon spectrum in the alkali metals as discussed above. It is not *a priori* obvious that this anisotropy will not make itself felt in single-crystal experiments on the alkali metals. We have estimated the effect of anisotropy on such experiments by assuming that all anisotropy effects in $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ come from the one-phonon exchange diagram.[13,20] We find that anisotropy effects are very small. The reason for the smallness of these effects seems to be that the sound velocity "only" varies by 40% and that the experiments measure an average of $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$ over the Fermi surface. The one-phonon-exchange part of $g(\mathbf{k}\sigma,\mathbf{k}'\sigma')$, for example, may be expanded as a series of cubic harmonics, i.e.,

$$g^{\mathrm{ep}}(\mathbf{k}\uparrow,\mathbf{k}'\uparrow) = g_0(|\mathbf{k}-\mathbf{k}'|)$$

$$+g_4(|\mathbf{k}-\mathbf{k}'|)K_4^\alpha[(\mathbf{k}-\mathbf{k}')/|\mathbf{k}-\mathbf{k}'|]+\cdots, \quad (21)$$

where K_4^α is the fourth-order cubic harmonic. The first two terms in this series are quite a good approximate representation of the function g^{ep}. Because of the size of the anisotropy in the sound velocity it can be shown, for example, that $g_4(0)/g_0(0)\cong 0.1$. This kind of estimate, along with a calculation that shows that the anisotropic terms g_4 enter the dispersion relation of the waves[1,2] in second order, leads us to believe that anisotropy effects are typically of the order of 1% corrections on the isotropic results.[21]

TABLE I. Experimental values of the Landau parameters in Na and K. m^*/m is taken from the cyclotron resonance data of Grimes and Kip (Ref. 19). The B_n have been determined by Schultz and co-workers (see Refs. 1 and 22) while the A_n for $n\geq 2$ have been measured by Walsh and co-workers (see Refs. 2 and 22). A_1 is determined from calculations of the phonon part and (m^*/m) as discussed in the text. The phonon average on the last line was determined from the elastic constants.

	Na	K
(m^*/m)	1.24±0.02	1.21±0.01
B_0	−0.21±0.05	−0.2±0.1
B_1	+0.01±0.03	+0.1±0.2
B_2	0.0±0.05	···
A_1	0.10	0.15
A_2	−0.05±0.01	−0.030±0.005
A_3	0.0±0.005	···
left-hand side of (23)	0.78±0.3	···
$\frac{m^* v_F^2}{3M}\sum_\lambda \left\langle\left(\frac{\epsilon_\lambda\cdot\hat{q}}{v_\lambda(\hat{q})}\right)^2\right\rangle_{\mathrm{av}}$	0.65	0.47

[17] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), p. 153.

[18] For Na, see, M. J. G. Lee, Proc. Roy. Soc. (London) **A295**, 440 (1966); and for K, see, M. J. G. Lee and L. M. Falicov, Proc. Roy. Soc. (London) **A304**, 319 (1968).

[19] C. C. Grimes and A. F. Kip, Phys. Rev. **132**, 1991 (1963).

[20] A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 1438 (1958) [English transl.: Soviet Phys.—JETP **7**, 996 (1958)].

[21] Experimentally, the magic angle at which the coefficient of the q^2 part of the spin spectrum is zero is determined to an accuracy ~1%. It can be shown that this angle does not depend on crystal orientation in a cubic crystal.

A more useful relation than Eq. (20) can be obtained by averaging this equation over the allowed directions of $\hat{q}$ in the plane tangent to the Fermi surface and then over the Fermi surface. In the alkali metals this is equivalent, by our previous arguments, to averaging over all directions of $\hat{q}$. If we assume that this averaged value of $g(\mathbf{k}, \hat{q})$ can be obtained from a spherically symmetric f function we find using Eq. (8) that

$$\sum_{l=0}^{\infty}(2l+1)\left[\frac{A_l}{1+A_l}+\frac{B_l}{1+B_l}\right]=\frac{-1}{1+A_0}+\frac{Zm^*}{3M}v_F^2\sum_{\lambda}\left\langle\left(\frac{\epsilon_\lambda(\hat{q})\cdot\hat{q}}{v_\lambda(\hat{q})}\right)^2\right\rangle_{\mathrm{av}}, \quad (22)$$

where the average is over all directions of $\hat{q}$. It is important to note that the parameter A_0 does not enter Eq. (22) explicitly. We may for convenience rewrite Eq. (22) in the form

$$1+\frac{B_0}{1+B_0}+\sum_{l=1}^{\infty}(2l+1)\left[\frac{A_l}{1+A_l}+\frac{B_l}{1+B_l}\right]=\frac{Zm^*}{3M}v_F^2\sum_{\lambda}\left\langle\left(\frac{\epsilon_\lambda(\hat{q})\cdot\hat{q}}{v_\lambda(\hat{q})}\right)^2\right\rangle_{\mathrm{av}}. \quad (23)$$

We now apply Eq. (23) to the alkali metals.

4. NUMERICAL ESTIMATES AND CONCLUSIONS

A number of the Landau parameters have been determined experimentally.[1,2,22] These are listed in Table I. If we make the assumption that the series in Eq. (23) is well represented by the terms with $l<3$ we can compare the two sides of this equation. If Eq. (23) is badly violated, we are forced to consider two alternatives. Either the higher parameters are important, or the interpretation of the data in terms of the Landau theory is inconsistent. At present the accuracy on the B_n's for K is not good enough to allow us to make a meaningful comparison with the theory. Although A_1 cannot be determined directly from experiment, an estimate of its size can be obtained as follows. According to Migdal's result,[20] the phonon contribution to $g(\mathbf{k}\sigma, \mathbf{k}'\sigma')$ is accurately given by the one-phonon-exchange diagram. Utilizing this fact one of the authors[11] has calculated the phonon contribution to $g(\mathbf{k}\sigma, \mathbf{k}'\sigma')$. The calculation involves the use of the measured phonon spectrum and an estimate, based on a pseudopotential argument, of the electron-phonon coupling. The Coulomb mass enhancement is obtained by combining the observed effective mass from Azbel-Kaner resonance with the calculated phonon enhancement. The Coulomb A_1 is then determined by the Landau equation for the effective mass.[3] These two contributions are combined according to the prescription of Prange and Sachs[13] to give the numbers in Table I.

The phonon average entering the right-hand side was determined from the experimental elastic constants given by Trivisonno and co-workers.[23] For Na, we have extrapolated the reported values to 4.2°K. Comparing the last two lines of Table I for Na we find reasonable agreement between the two sides of Eq. (23). It is clear that the experimental error, especially for B_2 in Na, rules out an accurate test of the sum rule. However, there is no question that the use of higher frequency and better samples will, in the near future, lead to significant improvements in the experimentally determined Landau parameters. It is also clear that numbers for the first few A_n and B_n will become available for the other two alkalis, Rb and Cs, as well.

The agreement between theory and experiment is very encouraging. Experimentally, the moments seem to be falling off very rapidly. The fact that the first few moments of the Landau function, as determined from the experiments, mimic the exact sum rule result adds an element of consistency to this type of description of a normal metal. The interpretation of the data in terms of the Landau Fermi-liquid theory is at least consistent with the exclusion principle.

ACKNOWLEDGMENTS

The authors would like to acknowledge several useful conversations with J. M. Luttinger and C. Herring. They would also like to thank W. M. Walsh, Jr., and S. Schultz for supplying us with unpublished data and for several informative discussions.

[22] The parameters listed are the most accurate experimental numbers presently available and were communicated to the authors by S. Schultz and W. M. Walsh, Jr.

[23] For Na, see, M. E. Diederich and J. Trivisonno, J. Phys. Chem. Solids **27**, 637 (1966); For K, see, N. R. Marquardt and J. Trivisonno, J. Phys. Chem. Solids **26**, 273 (1964).