# Electric Field Dependence of Positronium Formation in Matter

WERNER BRANDT* AND HOWARD FEIBUS†‡
*Department of Physics, New York University, New York, New York*
(Received 1 May 1968)

Positronium (Ps) can be formed whenever positrons, on slowing down in a substance, pass through an Ore gap, i.e., a range of energies, just below the ionization threshold of the substance, of width comparable to the Ps binding energy. We have studied the time evolution of the energy distribution of the positrons below the ionization threshold in an external electrostatic field, and find: (1) The field induces diffusion out of the Ore gap, which decreases Ps formation. (2) The over-all heating effect of the field on the positrons increases Ps formation. Depending on the magnitude of the relevant cross sections, (1) can outweigh (2), at small fields, and cause a significant decrease in the Ps formation. At field strengths larger than a characteristic value, (2) becomes the dominant effect. In this way measurements of the field dependence of Ps formation can give access to the cross sections for Ps formation and for positron-energy loss and scattering in matter.

## 1. INTRODUCTION

WHEN Deutsch and Brown[1] studied the Zeeman effect and the hyperfine splitting of positronium (Ps) in gaseous argon, they noted an *increase* of the Ps yield in the presence of an electric field. They conjectured that thermalized positrons in a sufficiently strong field can gain enough energy during their lifetime that they pass the threshold for Ps formation in the gas, and thus increase the Ps yield. Detailed experiments by Marder *et al.*[2] on a number of gases and their analysis by Teutsch and Hughes[3] confirmed the effect and this interpretation.

In one heavy gas, sulfur hexafluoride ($SF_6$), the Ps yield showed a significant *decrease* with increasing field, which has remained unexplained. At higher fields, Obenshain and Page[4] found a minimum in the Ps yield followed by a rise to values higher than the field-free value. The field-induced decrease in $SF_6$ was an isolated anomaly until Bisi *et al.*[5] observed a large drop of the Ps yield in some molecular solids (polyethylene, Teflon) at fields up to 100 kV/cm.

It is the purpose of this paper to explain the initial decrease of the Ps yield in an electric field, and to investigate what new information can be extracted from the quantitative aspects of this phenomenon. We find that a decrease always occurs in principle, but it is large enough to be observed only in systems where scattering processes are not totally overshadowed by the Ps formation in the relevant range of positron energies. Conversely, measurements of the decrease and subsequent increase of the Ps yield in an electric field give access to the cross sections for Ps formation and for positron scatter and energy loss in the target substance.

The physical processes that cause the field dependence of the Ps yield are examined in Sec. 2. A comprehensive Ps yield function is derived from simple arguments which show why and when an initial decrease occurs. The formal solution as presented in the Appendix confirms and justifies the results of Sec. 2. The yield function and its application are discussed in Sec. 3.

## 2. Ps YIELD FUNCTION

Positrons can form Ps in a substance if their kinetic energy $E$ falls into an energy range (1), the so called Ore gap (cf. Fig. 1). It is bounded by an upper $E_u$ nearly equal to the internal ionization threshold of the substance, and a lower $E_l$ such that the gap $E_{\mathrm{Ore}}=E_u-E_l$ is equal to the Ps ionization energy in the substance. Positrons that fall below the Ore gap into the energy range (2) between zero and $E_l$ annihilate as free particles.[6]

After their last ionizing collision at energies above $E_u$, the positrons are injected with a probability

$$g(E_0)\,dE_0 \qquad (1)$$

somewhere into the energy range between zero and $E_u$ with an initial energy $E_0$, such that

$$\int_0^{E_u} g(E_0)\,dE_0=1.$$

Starting from $E_0$, the positrons lose energy at some mean rate

$$-dE(t)/dt=\gamma_r E(t)=\Lambda\gamma_{sc}E(t), \qquad (2)$$

where $\gamma_r$ is the rate of energy loss characteristic of the

* Supported in part by the U.S. Atomic Energy Commission and the Defense Atomic Support Agency.
† Permanent address: Bell Telephone Laboratories, Inc., Whippany, N.J.
‡ Portions of this article are based on a thesis presented to New York University by Howard Feibus in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
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energies of the Ore gap, $\Lambda$ is the mean fractional energy lost per scattering collision, and $\gamma_{sc}$ is the number of scattering collisions per unit time. The positrons with an initial energy $E_0 > E_l$ can form Ps only as long as $E(t, E_0) > E_l$. At later times they become part of the population in range (2). Therefore the actual Ps yield, $\varphi_\eta$, is less than the fraction

$$\varphi_1 = \int_{E_l}^{E_u} g(E_0)\, dE_0$$

deposited initially in the Ore gap. Clearly, the larger the ratio

$$\lambda_1 \equiv \gamma_f/\gamma_r = \sigma_f/\sigma_r \tag{3}$$

of Ps formation to positron energy loss, the closer $\varphi_\eta(\lambda_1)$ will be to $\varphi_1$. Here and in the following, we denote rates by $\gamma$ and cross sections by $\sigma$. They are interrelated as $\gamma = nv\sigma$, where $n$ is the density of scatterers and $v$ the positron velocity; in the present context it suffices to set $v \simeq v_l$, the positron velocity corresponding to $E_l$.

The positrons initially in range (2) reach thermal equilibrium at a mean temperature $T_+$ before annihilating. If $k_B T_+$ is sufficiently large compared to $E_l$, the tail of their distribution $\sim \exp(-E/k_B T_+)$ can reach into the Ore gap and contribute to the Ps yield with a probability $W(T_+)$. We determine the contributions of $\phi_\eta$ and $W$ to the total Ps yield as follows. Consider a positron with an initial wave vector $\mathbf{k}_0$. Under the influence of an electric field $\mathcal{E}$, the probability of finding the positron at $\mathbf{k}$ is given by the solution of the Boltzmann equation:

$$(\partial f/\partial t) + \hbar^{-1} e\mathcal{E}\nabla_k f + (df/dt)\,|_{sc} - (\gamma_a + \gamma_f) f = 0. \tag{4}$$

The third term accounts for the influence of scattering processes that occur at the rate $\gamma_{sc}$. The last term removes particles at rates $\gamma_a$ and $\gamma_f$ from the distribution through free-particle annihilation and Ps formation, respectively. The field makes the distribution anisotropic in $k$ space. We expand $f$ in Legendre polynomials in the usual way and, on retaining the first two terms, cull from Eq. (4) the equation

$$\frac{\partial f_0}{\partial t} = k^{-2}\frac{\partial}{\partial k}\left[\frac{k^2 e^2 \mathcal{E}^2}{3\hbar^2 \gamma_{sc}}\frac{\partial}{\partial k} + (\tfrac{1}{2}\gamma_r) k^3\right] f_0 - (\gamma_a + \gamma_f) f_0 \tag{5}$$

for the isotropic part $f_0$. Since $\gamma_f$ is large only in the Ore gap, two Sturm-Liouvile equations of this kind must be solved, one for each of the distinct energy domains (1) and (2), subject to the condition that the eigensolutions match at the boundary $E_l$. Insistence on the exact boundary conditions precludes solutions other than by computational means. Therefore, we in effect decouple the two equations by appropriate boundary conditions in each domain. The final result turns out to be insensitive to the detailed choice. It is made such as to render the problem soluble while approximating the physical situation as closely as possible. The functions $\varphi_\eta$ and $W$ then are independent of one another, and the total Ps yield becomes

$$\Phi = \varphi_\eta + (1 - \varphi_\eta) W. \tag{6}$$

Teutsch and Hughes[3] took $\varphi_\eta$ to be field-independent and derived the function $W(T_+)$ in various approximations. The effective positron temperature $T_+$ depends on the strength of the applied field $\mathcal{E}$. The temperature of the substance, $T$, and $E_l$ are normally such that $k_B T/E_l < 10^{-2}$. The contribution of $W(T_+)$ to $\Phi$, therefore, becomes significant only when $T_+(\mathcal{E}) \gg T$, and the dependence of $\Phi$ on $T$ can be neglected. In terms of the effective positron mass $m_+^*$, the positron temperature in a field is given by $T_+(\mathcal{E}) = 2e^2\mathcal{E}^2/3k_B m_+^* \gamma_r \gamma_{sc}$. Since we are interested in comparing $k_B T_+(\mathcal{E})$ with $E_l$, it is convenient to use the reduced field variable

$$\begin{aligned}\rho(\mathcal{E}) &= [k_B T_+(\mathcal{E})/E_l]^{1/2} \\ &= (2e^2/3m_+^* \gamma_r \gamma_{sc} E_l)^{1/2}\mathcal{E},\end{aligned} \tag{7}$$

which is linear in $\mathcal{E}$. We have rederived $W$ from Eq. (5) for a slightly different set of assumptions than those considered by Teutsch and Hughes, as shown in the Appendix. We find

$$W(\rho) = \Gamma(\rho)/[\Gamma(\rho) + 1], \tag{8a}$$

where

$$\Gamma(\rho) = \lambda_2[1 + (2/\pi^{1/2}\rho)\exp(-\rho^{-2}) - \operatorname{erf}(\rho^{-1})]. \tag{8b}$$

The appearance of the parameter

$$\lambda_2 \equiv \gamma_f/\gamma_a = \sigma_f/\sigma_a, \tag{9}$$

indicates that if Ps formation is much more likely than

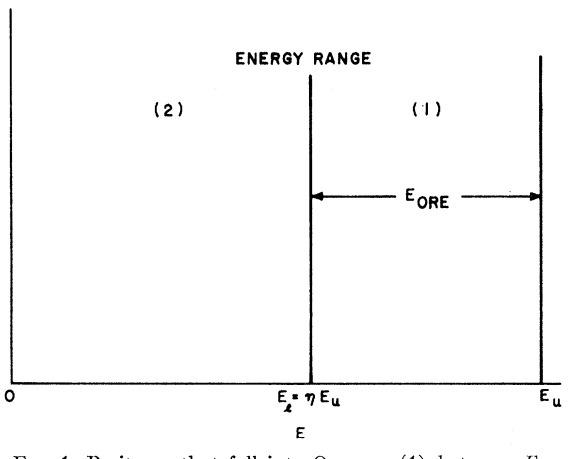


FIG. 1. Positrons that fall into Ore gap (1) between $E_u$ and $E_l$ can form positronium. Positrons in range (2) annihilate as free particles. In many Ps-forming substances $E_u \sim V_{ion}$, the internal ionization energy, and $E_l \sim V_{ion} - V_{Ps}$, where $V_{Ps} \sim \frac{1}{2}$ Ry is the ionization energy of Ps in the substance. Typically, the value of the parameter $\eta \equiv E_l/E_u$ ranges from 0.4 to 0.7 and, in the absence of an electric field, the Ps yield is $\sim 1 - \eta$.

free-positron annihilation, $W$ can approach unity at large fields and, by Eq. (6), so can the total Ps yield $\Phi$. The function $W(\rho)$ is shown in Fig. 2 for various values of $\lambda_2$. It is vanishingly small as long as $\rho<0.1$, begins to contribute to $\Phi$ when $\rho>0.2$, and rises rapidly at larger fields to its saturation value $\lambda_2/(\lambda_2+1)$. This is the normal field effect observed in monoatomic gases.

The decrease at small fields, then, must be found in a field dependence of $\varphi_\eta$. We solve Eq. (5) for the positrons initially in the Ore gap with the boundary condition that all positrons leaving the Ore gap become part of the particle population in range (2). This is a good assumption even for positrons that attain energies in the field in excess of $E_u$ since they lose most of their energy promptly in an ionizing collision. The calculation is given in the Appendix. Presently, we derive the field dependence of $\varphi_\eta$ from a simple physical argument.

Under the influence of an electric field, the positrons in the Ore gap execute a random walk in energy space about their mean trajectory $E(t, E_0)$, at the rate $\gamma_{sc}$ and to first order in the field in steps of size

$$\Delta E(\mathcal{E}) = \pm ve\mathcal{E}/\gamma_{sc}. \tag{10}$$

In other words, the field-induced motion is a diffusion in energy space away from $E(t, E_0)$, characterized by a diffusion constant

$$D(\mathcal{E}) = \tfrac{1}{2} \mid \Delta E(\mathcal{E}) \mid^2 \gamma_{sc}. \tag{11}$$

The probability per unit time, $q_{l,u}$, that a positron crosses the lower or the upper boundary of the Ore gap to join the population in (2) is then

$$q_{l,u} = \frac{\mid E(t, E_0) - E_{l,u} \mid}{\pi^{1/2} t[4D(\mathcal{E})t]^{1/2}} \times \exp\left[ - \frac{\mid E(t, E_0) - E_{l,u} \mid^2}{4D(\mathcal{E})t} - (\gamma_f + \gamma_a)t \right]. \tag{12}$$

The last exponential term accounts for the removal of positrons through Ps formation and direct annihilation, with a probability per unit time

$$q_{f,a} = \gamma_{f,a} \exp[-(\gamma_f + \gamma_a)t]. \tag{13}$$

The Ps yield from positrons initially in the Ore gap becomes

$$\varphi_\eta(\mathcal{E}) = \int_{E_l}^{E_u} dE_0 g(E_0) \times \int_0^{E(t,\, E_0)=E_l} dt[q_f(t) - q_l(t, E_0, \mathcal{E}) - q_u(t, E_0, \mathcal{E})]. \tag{14}$$

We insert Eqs. (12) and (13) and for $g(E_0) = E_u^{-1}$ obtain, to terms linear in $\mathcal{E}$,

$$\varphi_\eta(\mathcal{E}) = \varphi_\eta(0) - \frac{\lambda_2}{\lambda_2+1}\left[\frac{3N}{2(\lambda_1-1)}\right]^{1/2} \eta\rho(\mathcal{E}). \tag{15}$$

We have abbreviated $E_l/E_u \equiv \eta$, and denoted the yield at zero field by

$$\varphi_\eta(0) = \Phi(0) = \lambda_2/(\lambda_2+1)\{1 - \eta[N\lambda_1/(\lambda_1-1)]\}. \tag{16}$$

$\rho(\mathcal{E})$ is given by Eq. (7), $\lambda_1$ by Eq. (3), and $\lambda_2$ by Eq. (9). In most Ps forming substances $\lambda_1$ and $\lambda_2$ are so large that the factor $\lambda_2/(\lambda_2+1)$ and the factor

$$N = 1 - \lambda_1^{-1}\eta^{\lambda_1 - 1} \tag{17}$$

can be set equal to unity. Equation (6) with Eqs. (8a) and (15) gives the total dependence of the Ps yield on the applied field.

Our result accounts for the initial decrease. The function $\varphi_\eta$, and thus $\Phi$, diminishes linearly with increasing field in the range of small $\rho$ where $W(\rho)$ is vanishingly small (cf. Fig. 2). As the field increases further, $W$ begins to rise sharply, and $\Phi$ goes through a minimum. This is shown in Fig. 3 for the reduced yield function $\Phi(\rho)/\Phi(0)$. At large fields, $\Phi$ becomes independent of $\varphi_\eta$ and reaches its limiting value concurrently with $W$. The minimum disappears and we retrieve the result of Teutsch and Hughes if $\lambda_1 = \sigma_f/\sigma_r > 10^3$, i.e., under conditions where practically none of the positrons initially in the Ore gap can escape positronium formation.

## 3. DISCUSSION

In summary, the field dependence of the Ps yield is determined by two processes. One is the increased rate of escape of positrons from the Ore gap under the influence of a field; this *decreases* the yield at low fields. The other is the heating by the field of the thermalized

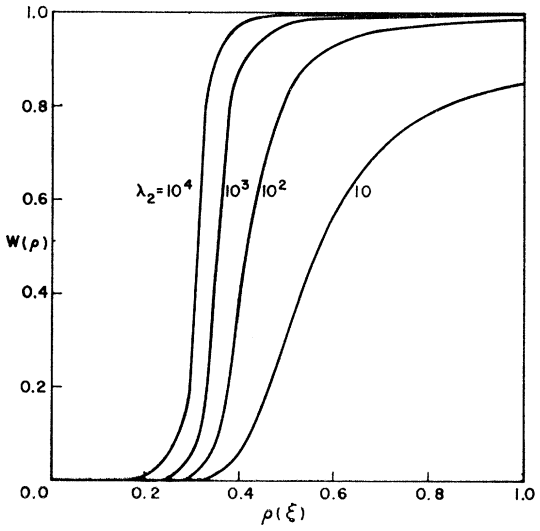


FIG. 2. Probability $W(\rho)$ that positrons initially below Ore gap can form Ps, Eq. (8). The variable $\rho(\mathcal{E})$ is proportional to the electric field, Eq. (7). The parameter $\lambda_2 = \sigma_f/\sigma_a$, Eq. (9), determines sharpness of the rise in $W$.

positrons such that the tail of their distribution reaches into the Ore gap; this *increases* the yield at high fields. Given $\sigma_f$ and $\sigma_a$, our analysis points to the cross sections for energy loss, $\sigma_r$, and for scattering, $\sigma_{sc}$, as the important parameters that determine the rate and the magnitude of the decrease of the Ps yield in an electric field. The initial slope $s$ of the yield function is given by

$$s=(d\Phi/d\mathcal{E})_{\mathcal{E}=0}$$
$$=-(e^2\eta/m_+{}^*\gamma_f\gamma_{sc}E_u)^{1/2}. \quad (18)$$

Since $E_u$ can be equated to ionization potential of the substance, and $\Phi(0)$ determines $\eta\simeq 1-\Phi(0)$, the initial slope is a direct measure of the product $(\gamma_f\gamma_{sc})^{1/2}$.

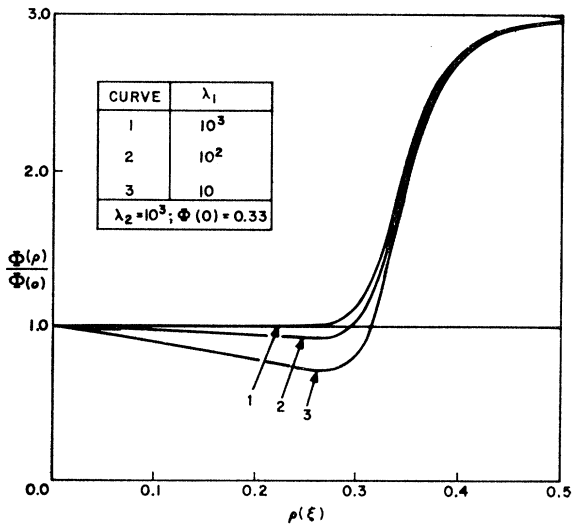


FIG. 3. Reduced Ps yield function $\Phi(\rho)/\Phi(0)$ shows the decline of $\varphi_\eta$, Eq. (15), until $\rho(\mathcal{E})$ is so large that the rise in $W(\rho)$ becomes dominant, Eq. (8) and Fig. 2. The variable $\rho(\mathcal{E})$ is proportional to $\mathcal{E}$, Eq. (7). The parameter $\lambda_1=\sigma_f/\sigma_r$, Eq. (3), determines the magnitude of the field-induced minimum. The parameter $\lambda_2=\sigma_f/\sigma_a$, Eq. (9), determines the sharpness of the rise at high fields.

The magnitude of $\Phi_{min}$ and any of the values of $\mathcal{E}$ at which $\Phi$ reaches $\Phi_{min}$ or attains some other specified characteristics on the rising part of the curve, determine $\lambda_1$ and $\lambda_2$ separately. One need only use the fact that $\rho$ is related to the applied $\mathcal{E}$ through the measured slope $s$ as $\rho=(\sqrt{\frac{3}{2}})\,|\,s\,|\,\lambda_1{}^{1/2}\mathcal{E}$. Thus, from an experimental Ps yield function, one can extract information about the cross sections for Ps formation $\sigma_f$, for energy loss $\sigma_r$, and for scattering $\sigma_{sc}$ of positrons with energies comparable to $E_1$.

We have analyzed the available measurements on $SF_6$ in this way. Figure 4 shows the fit we obtain with the accurate data by Marder *et al.*[2] on the initial decline at 2 atm. The experimental method used by Obenshain and Page[4] gave less accurate information on the change of the Ps yield at higher fields. Their data are shown in

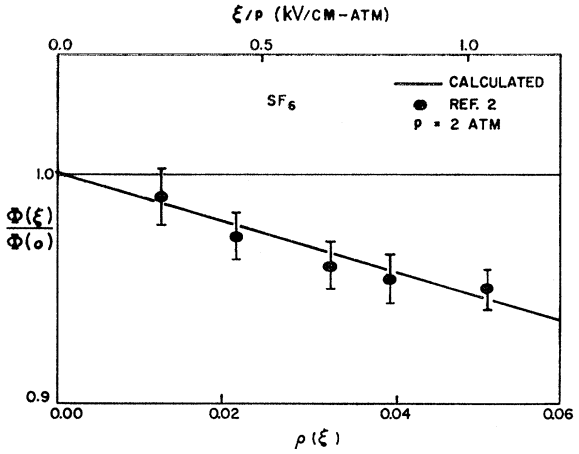


FIG. 4. Field dependence of the Ps yield in $SF_6$ at 2 atm as observed by Marder *et al.* (Ref. 2). The solid line shows the decrease of the Ps yield with a slope given by Eq. (18) as determined by the constants listed in Table I.

Fig. 5, together with the last point of Fig. 4 for reference. The constants chosen to fit the calculated curves to these data are listed in Table I. The difference in accuracy between the two sets of experimental data leaves the absolute magnitude of the individual cross sections uncertain by a factor of 2, or so. However, the trends in their relative magnitudes are clear. Starting with the last column, the fractional energy loss per scattering collision, $\Lambda$, is just equal to $2m_+/M(SF_6)=6\times10^{-6}$, as it should be for classical momentum transfer

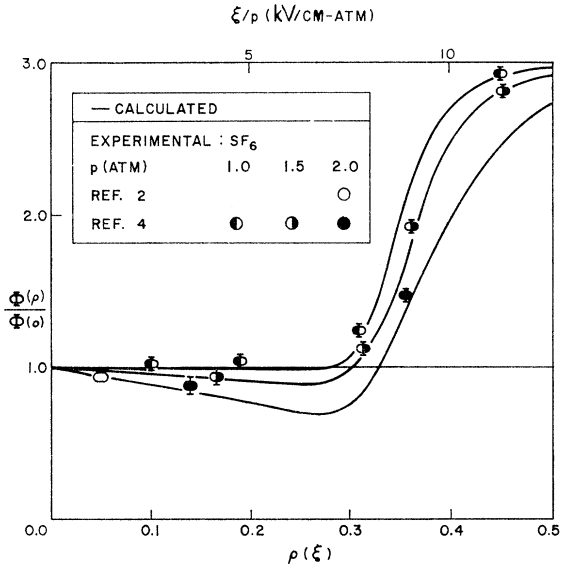


FIG. 5. Electric field dependence of Ps yield in $SF_6$. The experimental points are those of Obenshain and Page (Ref. 4) and one point from the series of measurements by Marder *et al.* (Ref. 2) shown in Fig. 4. The curves have been calculated with the constants listed in Table I.

TABLE I. Ps yield curves shown in Figs. 4 and 5 for $SF_6$ are calculated with the parameters listed here. The conversion factor from $\sigma$ to $\gamma/p$ is $\sigma$ (cm$^2$) $=1.8\times10^{-28}\gamma/p$ (sec$^{-1}$ atm$^{-1}$). $E_u=19.3$ eV; $\Phi(0)\simeq1-\eta=0.33$; $\gamma_a/p=4.1\times10^7$ sec$^{-1}$ atm$^{-1}$.

| $SF_6$ $p$(atm) | Cross sections ($10^{-16}$ cm$^2$) $\sigma_f$ | $\sigma_r$ | $\sigma_{sc}$ | $\Lambda$ |
|---|---|---|---|---|
| 1.0 | $3.8\times10^{-1}$ | $6.5\times10^{-4}$ | $1.1\times10^{2}$ | $6.0\times10^{-6}$ |
| 1.5 | $9.0\times10^{-2}$ | $6.5\times10^{-4}$ | $1.1\times10^{2}$ | $6.0\times10^{-6}$ |
| 2.0 | $1.1\times10^{-2}$ | $6.5\times10^{-4}$ | $1.1\times10^{2}$ | $6.0\times10^{-6}$ |

in elastic positron-molecule collisions. In the investigated pressure range, $\sigma_r$ and $\sigma_{sc}$ are insensitive to changes of the gas density. In contrast to the situation in light monoatomic gases, the difference between positron and electron scattering cross sections of large molecules at energies $\sim E_l$ are minor, and both should show the same trends with the molecular structure of the scatterer. Indeed, the value of $\sigma_{sc}$ listed in Table I is similar to the electron value.[7] The pressure dependence of $\Phi(\mathcal{E})$ stems from the change of $\sigma_f$; the Ps formation cross section decreases with decreasing free volume in the gas.

A similarly comprehensive analysis of the decline of the Ps yield in semicrystalline organic solids reported by Bisi *et al.*[5] is not possible because the field dependence was not pursued far enough to show the minimum our theory predicts. Moreover, the inhomogeneities in these materials may cause special effects that have not been discussed here. In solids the interpretation of the relation between $\sigma_r$ and $\sigma_{sc}$ as given by Eq. (2) differs from that in gases. For example in nonpolar solids where the positron-phonon interaction is the dominant loss mechanism, we have $\sigma_r=\Lambda_{ph}\sigma_{ph}=\Lambda\sigma_{sc}$, where $\sigma_{sc}=\sigma_{ph}+\sigma_d$, the subscripts referring to positron-phonon and positron-defect scattering, respectively. Knowing $\sigma_r$ and $\sigma_{sc}$, we can separate $\sigma_{ph}$ and $\sigma_d$ if we introduce an effective phonon mass $M_{ph}^*$ by the relation

$$M_{ph}^*=(k_BT/c^2)\,y\coth y, \tag{19}$$

where $y=(m_+^*c^2\eta E_u)^{1/2}/k_BT$, $c$ being the sound velocity. When $y\ll1$, Eq. (19) becomes equal to the Davydov value $M_{ph}^*=kT/c^2$.[8] In the limit $y\gg1$ of interest here,

$$M_{ph}^*=m_+^*(\eta E_u/m_+^*c^2)^{1/2}. \tag{20}$$

Then $\Lambda_{ph}=2m_+^*/M_{ph}^*$ in formal analogy to the result in gases.

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[8] B. Davydov, Physik. Z. Sowjetunion **12**, 269 (1937).

## APPENDIX

In solving Eq. (4), it is convenient to define the function

$$H(k,k_0,\mathcal{E})=\int d\Omega_k\int dt\,\gamma(\mathbf{k})f(\mathbf{k},\mathbf{k}_0,\mathcal{E},t) \tag{A1}$$

in terms of the rate $\gamma=\gamma_a+\gamma_f+\gamma_{sc}$ at which the particle leaves the volume element $d^3k$ at $\mathbf{k}$. The integration over the solid angle $\Omega_k$ averages over all directions relative to $\mathcal{E}$ and retains only the isotropic part of $f$. With the abbreviation $p_j=\gamma_j/\gamma$, the Ps yield becomes

$$\Phi(\mathcal{E})=\int d^3k_0g(k_0)\int dk\,k^2p_f(k)H(k,k_0,\mathcal{E}). \tag{A2}$$

If $\mathcal{E}=0$, one can solve directly for $H$. On writing $\gamma_{sc}(\mathbf{k})=\int w(\mathbf{k},\mathbf{k}')\,d^3k'$, one obtains the integral equation

$$H(k,k_0)=H_s(k,k_0)+\int d^3k'\gamma^{-1}(k')w(k,k')H(k',k_0), \tag{A3}$$

where $H_s$ is the source term $(4\pi k^2)^{-1}\delta(k-k_0)$. We incorporate the fractional momentum loss per collision $(-\delta k/k)=\frac{1}{2}\Lambda$ by setting

$$w(k',k)=(4\pi k^2)^{-1}\alpha\gamma_{sc}(k')\delta(k'-\alpha k),$$

where $\alpha=1-(\frac{1}{2}\Lambda)$. The iteration solution of Eq. (A3) becomes

$$H(k,k_0)=\sum_n(4\pi k^2)^{-1}p_{sc}{}^n\delta(k-\alpha^nk_0) \tag{A4}$$

and therefore

$$\Phi(0)=p_f[\sum_n{}'\,p_{sc}{}^n-\eta\sum_n{}'\,(\alpha^2p_{sc})^n]. \tag{A5}$$

The primes mark the summation range bounded through the condition $\alpha^nk_u=k_l$. Summation yields Eq. (16).

If $\mathcal{E}\neq0$, we calculate $W(\mathcal{E})$ with the ansatz

$$f_0=C(k_0)h(k)\,\exp[(rk)^2-(\Gamma+\gamma_a)t], \tag{A6}$$

where $r^2=(\rho k_l)^{-2}=\frac{3}{4}(\Lambda\gamma_{sc}{}^2\hbar^2/e^2\mathcal{E}^2)$; it lends itself readily to satisfy the boundary conditions that the current in $k$ space be zero at $k=0$ and that $f_0$ be bounded everywhere. In terms of the separation constant $\Gamma$, Eq. (A6) converts Eq. (5) into the Sturm-Liouville equation

$$\frac{d}{dk}\left(k^2\frac{dh}{dk}\exp[-(rk)^2]\right)+4(r^2k^2/\gamma_r)[\Gamma-\gamma_f(k)]h(k)\,\exp[-(rk)^2]=0. \tag{A7}$$

As did Teutsch and Hughes in one of their approximations, Eq. (A7) can be solved simply if $\gamma_f$, while zero in range (2), is finite in range (1) but small enough for Ps formation to act merely as a perturbation on the thermalized positron distribution. The eigenvalues of the unperturbed equation are $\Gamma_n=n\gamma_r$ where $n=0$, 1,

2, ···. Since $\gamma_a \ll \gamma_f$, only the lowest perturbed eigenvalue $\Gamma$ contributes, for which we obtain directly Eq. (8b). The integration according to Eq. (A2) yields Eq. (8a).

We determine $\varphi_\eta$ when $\mathcal{E}\neq 0$ by making the transformation $\mathbf{k}\rightarrow\mathbf{k}+(e\mathcal{E}t/\hbar)$ in Eq. (A1) or, equivalently,

$$\delta(z'-z)\rightarrow(\hbar\gamma_{sc}/2e\mathcal{E})\left[E_1\left(\frac{|z'-z|}{e\mathcal{E}/\hbar\gamma_{sc}}\right)-E_1\left(\frac{z'+z}{e\mathcal{E}/\hbar\gamma_{sc}}\right)\right] \tag{A8}$$

in Eq. (A3), where

$$E_1(x)=\int_x^\infty y^{-1}\exp(-y)\,dy.$$

Following the work of Stewart and Gerjuoy[9] on loss-free media ($\Lambda=0$), one proves explicitly that Eq (A8) is the correct transformation in lossy media where $\alpha=1-(\frac{1}{2}\Lambda)\lesssim 1$. We solve Eq. (A3) with Eq. (A8) by iteration and retain only terms of order $\Lambda$. With $x\equiv(\hbar k\gamma_{sc}/e\mathcal{E})$, the result is

$$H(x,x_0)=\int d^3x_1\frac{\exp(-|\mathbf{x}-\mathbf{x}_1|)}{4\pi|\mathbf{x}-\mathbf{x}_1|^2}\left[\frac{\delta(x-x_0)}{4\pi x^2}+\sum_{n=1}^{\infty}p_{sc}{}^n\prod_{j=1}^{n}\int d^3x_{j+1}\frac{\exp(-|x_j-x_{j+1}|)}{4\pi|x_j-x_{j+1}|^2}\frac{\delta(x_{n+1}-\alpha^n x_0)}{4\pi x_{n+1}{}^2}\right]. \tag{A9}$$

One can simplify Eq. (A9) by considering its Fourier transform

$$H(q,x_0)=\sum_n p_{sc}{}^n\left(\frac{\tan^{-1}q}{q}\right)^{n+1}\frac{\sin\alpha^n qx_0}{\alpha^n qx_0}, \tag{A10}$$

because in the range of relevant values of $\alpha$ and $p_{sc}$ close to unity the inverse of Eq. (A10) can be written concisely in terms of the principal solution $q_0$ of the characteristic equation

$$1-p_{sc}q^{-1}\tan^{-1}q=0$$

in the form

$$H(x,x_0)=\sum_n H_n(x,x_0)$$

$$=\sum_n p_{sc}{}^n\frac{q_0}{8\pi\alpha^n xx_0}[\exp(-q_0|x-\alpha^n x_0|)-\exp(-q_0|x+\alpha^n x_0|)], \tag{A11}$$

where $q_0=[(1-p_{sc})]^{1/2}$.[10] This function $H(x,x_0)$ is normalized in the sense that

$$(1-p_{sc})\int d^3x\,H(x,x_0)=1, \tag{A12}$$

i.e., the two channels of disappearance account for exactly one particle.

We impose the conditions of absorbing boundaries by noting that each term $H_n$ in Eq. (A11) represents the outcome of a random walk in energy space of positrons surviving just $n$ steps before disappearing. The effect of energy losses on this particle is equivalent to translating its initial position from $x_0$ to $\alpha^n x_0$. A particle that crosses $x_l$ or $x_u$ reaches thermal equilibrium and disappears from the positron ensemble at the rate $\gamma_a$, and the approximation incorporated in Eq. (5) applies. In effect, then, the boundaries eliminate all sequences of steps that permit a particle to cross a boundary more than once. We expand

$$H_n(x,x_0;x_l,x_u)=H_n(x,x_0)-H_n(2x-x_l,x_0)-H_n(2x_u-x,x_0). \tag{A13}$$

On inserting into Eq. (A2) one obtains

$$\varphi_\eta=\frac{\lambda_2}{\lambda_2-1}\left[1-\eta\frac{N\lambda_1}{\lambda_1-1}-\eta\frac{\rho}{\lambda_1{}^{1/2}}\frac{N\lambda_1}{\lambda_1-1}-\tfrac{1}{2}\eta^{1/2}\frac{\rho}{\lambda_1{}^{1/2}}\frac{N\lambda_1}{\lambda_1-1+\lambda_1{}^{1/2}/\eta^{1/2}\rho}\left(1+\frac{1}{2}\frac{\eta^{1/2}\rho}{\lambda_1{}^{1/2}}\right)+O[\exp(-q_0x_l)]\right]. \tag{A14}$$

Eq. (A14) confirms the essential field dependence of $\varphi_\eta$ as given in Eq. (15).

[9] G. W. Stewart and E. Gerjuoy, Phys. Rev. **119**, 892 (1960).
[10] K. M. Case, F. de Hoffman, and G. Placzek, Los Alamos Scientific Laboratory Report, 1953 (unpublished).