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Transport Theory of a Partially Degenerate Plasma*

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The transport coefficients of a high temperature, weakly coupled plasma of nondegenerate ions and partially degenerate electrons (i.e., stellar matter) are calculated by means of a Chapman-Enskog solution of the quantum Lenard-Balescu kinetic equation. Both electron-electron and electron-ion collisions are included. The leading term of the transport coefficients, the Coulomb logarithm, is evaluated exactly and relatively simply for all degrees of electron degeneracy. Dynamic shielding, and other corrections to the term of order unity which follows the larger logarithm, have been neglected. The method is applicable to any partially degenerate system with weak, long-range particle interactions.

The transport coefficients of weakly coupled plasmas (i.e., average interaction energy per particle much less than average electron or ion kinetic energy) have been calculated in the following limits: (1) classical electrons,[1] i.e., $r_0 \gg \lambda_{th}$, where $r_0 = e^2kT$ is the classical distance of closest approach, $\lambda_{th} = \hbar/(2mkT)^{1/2}$ is the thermal DeBroglie wavelength, e is the electron charge, m is the electron mass, k is Boltzmann's constant, and T is the temperature; (2) nondegenerate quantum-mechanical electrons,[2] i.e., $\lambda_{th} \gg r_0$, $\alpha \equiv \mu/kT \ll 0$, where μ is the chemical potential; (3) highly degenerate electrons,[3] i.e., $\alpha \gg 1$. In stellar interiors, electrons are always weakly coupled and quantum mechanical because of the high temperature, but all degrees of electron degeneracy occur. Heavy ions are always nondegenerate, and will be assumed weakly coupled in this paper. We shall obtain expressions for the transport coefficients valid for any degree of electron degeneracy.

This problem has already been considered by Mestel and Lee.[4] However, their approach is drastically simplified by the neglect of electron-electron (ee) collisions, i.e., by the adoption of the Lorentz model. It is well known[1] that the neglect of ee collisions is entirely unjustified in the nondegenerate regime (unless ions are heavily charged), and results in the overestimation of thermal conductivity by a factor of 4.5 and of electrical conductivity by 1.7, in the case of singly charged ions. Recently, Lampe[3] has shown that ee collisions are surprisingly important in thermal conductivity even when electrons are highly degenerate (if the ions are not too highly charged). The inclusion of ee collisions greatly increases the difficulty of the calculation.

It should also be noted that at the time of Mestel's and Lee's work, the shielding of Coulomb collisions was not entirely understood, and it was customary to cut off the interaction at a distance $n^{-1/3}$, which is of the order of the mean interparticle separation, rather than at the Debye shielding length D. Use of the correct shielded Coulomb interaction, which automatically cuts off the interaction essentially at D, also somewhat reduces the conductivities.

The method used here is applicable to the transport theory of any system with weak, long-range particle interactions. The results, expressed in terms of standard Fermi integrals and some closely related quadratures, are analytic and relatively simple in form.

The correct kinetic equation for weakly coupled quantum-mechanical plasma with any degree of degeneracy is the quantum Lenard-Balescu equation:[5]

$$\begin{aligned}
&\partial f_e/\partial t + \vec{v}_1 \cdot \partial f_e/\partial \vec{x}_1 + \dot{\vec{v}}_1 \cdot \partial f_e/\partial \vec{v}_1 \\
&= 4m^{-2}\int d^3p_2 d^3P \sigma_{ee} \delta(E_1 + E_2 - E_1' - E_2') \\
&\times\{f_e(\vec{p}_1')f_e(\vec{p}_2')[1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_1)][1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_2)] \\
&- f_e(\vec{p}_1)f_e(\vec{p}_2)[1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_1')][1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_2')]\} \\
&+ m^{-2}\int d^3\vec{p}_2 d^3P \sigma_{ei} \delta(E_1 + E_2 - E_1' - E_2')\{f_e(\vec{p}_1')f_i(\vec{p}_2') \\
&\times[1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_1)] - f_e(\vec{p}_1)f_i(\vec{p}_2)[1 - \tfrac{1}{2}(2\pi\hbar)^3 f_e(\vec{p}_1')]\}, \qquad (1)
\end{aligned}$$

where f_e and f_i are the electron and ion phase-space density distributions. In this paper we shall be interested only in the first term in the expansion of the transport coefficients[6] in λ_{th}/D – the so-called Coulomb logarithm, which is given exactly although we assume the interactions to be statically shielded.[7] (Corrections due to dynamic shielding contribute only to the smaller term, of order unity, that follows the logarithm.) The collision cross sections for ee and electron-ion (ei) collisions, which then depend only on the momentum transfer $q \equiv |\vec{p}_1' - \vec{p}_1|$, are, respectively,

$$\sigma_{ee}(q) = m^2e^4(q^2 + \hbar^2/D_e^2)^{-2}, \qquad (2a)$$

$$\sigma_{ei}(q) = 4m^2Z^2e^4(q^2 + \hbar^2/D^2)^{-2}, \qquad (2b)$$

where the shielding lengths[7] (taking into account electron degeneracy) are given by

$$D^{-2} = D_e^{-2} + 4\pi ne^2Z/kT, \qquad (3a)$$

$$D_e^2 = (kT/4\pi ne^2)[4\pi^2\hbar^3 n/(2mkT)^{3/2}F_{-1/2}], \qquad (3b)$$

where n is the electron number density, Z is the ionic charge, and F_j is the Fermi integral,

$$F_j(\alpha) = \int_0^\infty dx\, x^j (1 + e^{x-\alpha})^{-1}. \qquad (4)$$

The Lenard-Balescu equation has the form of a Boltzmann equation for electrons whose scattering is given by the Born approximation for the shielded interaction. We solve for the transport coefficients by the Chapman-Enskog method.[8]

We now quote from Ref. 3 the formal Chapman-Enskog results, using the first three polynomials,

$$P_0(x) = 1, \qquad (5a)$$

$$P_1(x) = x + h_1, \qquad (5b)$$

$$P_2(x) = x^2 + h_2 x + h_3, \qquad (5c)$$

where $h_1 = -\frac{5}{3} F_{3/2}/F_{1/2}$, (6a)

$$h_2 = (27F_{7/2}F_{1/2} - 35F_{5/2}F_{3/2})/(25F_{3/2}^2 - 21F_{5/2}F_{1/2}), \qquad (6b)$$

$$h_3 = (49F_{5/2}^2 - 45F_{7/2}F_{3/2})/(25F_{3/2}^2 - 21F_{5/2}F_{1/2}). \qquad (6c)$$

These polynomials, which reduce to the Sonine polynomials in the nondegenerate case, have been chosen to preserve the orthogonality relations of the Sonine polynomials:

$$\int_0^\infty d^3p\, p^2 f^-(p^2) f^+(p^2) P_i(p^2) P_j(p^2) = 0, \quad i \neq j,$$

where f^- and f^+ are the Fermi functions,

$$f^-(p^2) = (1 + e^{p^2/2mkT - \alpha})^{-1},$$

$$f^+(p^2) = 1 - f^-(p^2).$$

The transport coefficients may be defined by[9]

$$\vec{J} = eS_{11}'(e\vec{E} + \nabla P/n) + eS_{12}'\nabla T/T, \qquad (7a)$$

$$\vec{Q} = -S_{21}'(eE + \nabla P/n) - S_{22}'\nabla T/T - \tfrac{5}{3}(\vec{J}/e)\,\mathcal{E}, \qquad (7b)$$

where $\vec{J}$ is the electric current, $\vec{Q}$ is the heat flux, $\vec{E}$ is the electric field, $\mathcal{E}$ is the mean kinetic energy per electron, $P = \frac{2}{3}n\mathcal{E} + kTn/Z$ is the pressure. $S_{12}' = S_{21}'$ will be called the thermoelectric coefficient. The electrical conductivity is $\sigma = e^2 S_{11}'$, while the thermal conductivity (with the conventional constraint $\vec{J} = 0$) is

$$\kappa = (S_{11}'S_{22}' - S_{12}'^2)/TS_{11}'$$

The first two Chapman-Enskog polynomial approximations to κ, σ, and S_{12}' are, respectively,

$$\kappa^{(1)} = (128\pi^2/27)(2\pi\hbar)^{-6} mk(kT)^5 \times (21F_{5/2} - 25F_{3/2}^2/F_{1/2})^2/a_{11}, \qquad (8a)$$

$$\kappa^{(2)} = \kappa^{(1)}(1 - a_{21}^2/a_{11}a_{22})^{-1}, \qquad (8b)$$

$$\sigma^{(0)} = 384\pi^2(2\pi\hbar)^{-6} e^2 m(kT)^4 F_{1/2}^2/a_{00}, \qquad (9a)$$

$$\sigma^{(1)} = \sigma^{(0)}(1 - a_{10}^2/a_{00}a_{11})^{-1}, \qquad (9b)$$

$$S_{12}'^{(1)} = Ca_{10}/(a_{00}a_{11} - a_{10}^2), \qquad (10a)$$

$$S_{12}'^{(2)} = C(a_{10}a_{22} - a_{20}a_{21})/A^{(2)}, \qquad (10b)$$

$$C \equiv (128\pi^2/3)(2\pi\hbar)^{-6} m(kT)^5 \times (21F_{5/2}F_{1/2} - 25F_{3/2}^2), \qquad (10c)$$

$$A^{(2)} \equiv \begin{vmatrix} a_{00} & a_{10} & a_{20} \\ a_{10} & a_{11} & a_{21} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}, \qquad (10d)$$

where

$$a_{ij} = 2m^{-2}(2\pi\hbar)^{-3}\int d^3p\, \vec{p}P_i(p^2)\cdot[\vec{I}_{ee}(\vec{p}P_j) + \vec{I}_{ei}(\vec{p}P_j)],$$

and (11)

$$\vec{I}_{ee}[\vec{p}U(p_1^2)] = 8m^{-2}(2\pi\hbar)^{-3}\int d^3p_2\, d^3q\, \sigma_{ee}(q) \times \delta(E_1 + E_2 - E_1' - E_2') f_1^- f_2^- f_1'^+ f_2'^+ \times (\vec{p}_1U_1 + \vec{p}_2U_2 - \vec{p}_1'U_1' - \vec{p}_2'U_2'), \qquad (12)$$

$$\vec{I}_{ei}[\vec{p}U(p_1^2)] = (n/Z)m^{-2}(2\pi MkT)^{-\frac{3}{2}}\int d^3p_2 d^3q\, \sigma_{ei}(q) \times \delta(E_1 + E_2 - E_1' - E_2') f_1^- f_1'^+ f_2^0 (\vec{p}_1U_1 - \vec{p}_1'U_1') \qquad (13)$$

are collision integrals for ee and ei collisions, respectively. M is the ion mass, $f_2^0 \equiv \exp(-p_2^2/2mkT)$ is the Boltzmann function, $f_1^\pm$ is shorthand for $f^\pm(p_1^2)$, and $U_1 \equiv U(p_1^2)$ is any function of p_1^2.

We note (see Ref. 3) that a_{ij} is a bilinear and symmetric functional of the two functions P_i and P_j, and that I_{ee} ($\vec{p}$ const) = 0 because of momentum conservation. It is convenient to write the $a_{ij} = a_{ji}$ explicitly as a sum over the different terms in P_i and P_j, Eqs. (5), (6), and also to separate the contributions from ee and ei collisions:

$$a_{00} = b_{00\mathrm{ei}}, \qquad (14a)$$

$$a_{10} = b_{10\mathrm{ei}} + h_1 b_{00\mathrm{ei}}, \qquad (14b)$$

$$a_{20} = b_{20\mathrm{ei}} + h_2 b_{10\mathrm{ei}} + h_3 b_{00\mathrm{ei}}, \qquad (14c)$$

$$a_{11} = b_{11\mathrm{ei}} + 2h_1 b_{10\mathrm{ei}} + h_1^2 b_{00\mathrm{ei}} + b_{11\mathrm{ee}}, \qquad (14d)$$

$$a_{21} = b_{21\mathrm{ei}} + h_1 b_{20\mathrm{ei}} + h_2 b_{11\mathrm{ei}} + (h_1h_2 + h_3) b_{10\mathrm{ei}} + h_1h_3 b_{00\mathrm{ei}} + b_{21\mathrm{ee}} + h_2 b_{11\mathrm{ee}}, \qquad (14e)$$

$$a_{22} = b_{22\mathrm{ei}} + 2h_2 b_{21\mathrm{ei}} + h_2^2 b_{11\mathrm{ei}} + 2h_3 b_{20\mathrm{ei}} + 2h_2h_3 b_{10\mathrm{ei}} + h_3^2 b_{00\mathrm{ei}} + b_{22\mathrm{ee}} + 2h_2 b_{21\mathrm{ee}} + h_2^2 b_{11\mathrm{ee}}, \qquad (14f)$$

where

$$b_{ij\,\mathrm{ei(ee)}} = 2m^{-2}(2\pi\hbar)^{-3}\int d^3p\,\vec{p}W^{2i}\cdot\vec{I}_{\mathrm{ei(ee)}}(\vec{p}W^{2j}), \quad (15)$$

and $W^2 \equiv p_1{}^2/2mkT$. We must now calculate the coefficients $b_{ij\mathrm{ee}}$ and $b_{ij\mathrm{ei}}$.

By far the more difficult part of the calculation is the evaluation of the ee collision integrals $b_{ij\mathrm{ee}}$. We note as limiting cases the known results in the nondegenerate and highly degenerate limits. In the nondegenerate regime, the one-polynomial thermal conductivity neglecting ei collisions [i.e., Eq. (8a) with $a_{11} = b_{11\mathrm{ee}}$] is [2]

$$\kappa_{\mathrm{ee}}{}^{(1)} = \tfrac{75}{32}\pi^{-1/2}k(kT)^{5/2}/m^{1/2}e^4\ln(D_e/\lambda_{\mathrm{th}}\sqrt{2}). \quad (16)$$

Equation (16) is also correct in the classical limit, except that $\lambda_{\mathrm{th}}\sqrt{2}$ is replaced by r_0 in the argument of the logarithm. The transition between the classical and nondegenerate quantum regimes has been studied by Williams and DeWitt.[2]

In the highly degenerate limit,[10] $\kappa_{\mathrm{ee}}{}^{(1)}$ depends sensitively on the parameter D_e/λ, where

$$\hbar/\lambda \equiv 2\pi^{-1/2}(2mkT)^{1/2}F_{1/2}/F_1$$

is the momentum such that collisional momentum transfers $q > \hbar/\lambda$ are essentially prevented by the conservation laws and the Pauli exclusion principle. In the nondegenerate regime, $\lambda = \lambda_{\mathrm{th}}$; in the highly degenerate limit, $\hbar/\lambda = 16\pi^{-1/2}mkT/3p_F$ is essentially the thermal width of the Fermi surface, where p_F is the Fermi momentum. The two limits $D_e/\lambda \gg 1$ and $D_e/\lambda \ll 1$ (in both cases requiring that $\alpha \gg 1$ and the electrons be weakly coupled) are entirely distinct.[10] We consider here only the limit $D_e/\lambda \gg 1$, $\alpha \gg 1$, in which case

$$\kappa_{\mathrm{ee}}{}^{(1)} = (5\pi^3/72)[k^3T^2p_F/me^4\ln(D_e/\lambda)]. \quad (17)$$

The problem to be discussed is the transition between the limiting forms (16) for $\alpha \ll -1$ and (17) for $\alpha \gg 1$, where $D_e/\lambda \gg 1$ throughout. The argument of the logarithm depends on the degree of degeneracy in a well-defined way through D_e and λ. The two limits (16) and (17) differ otherwise only in that a factor p_F in the latter is replaced by $(135/4\pi^{7/2})(mkT)^{1/2} = 0.63(mkT)^{1/2}$ in the former. Dimensionally, this is easy to understand, but the numerical coefficient 0.63 is smaller by a factor of 3 or 4 than one might expect on the basis of the simplest heuristic arguments (e.g., $\frac{3}{5}p_F{}^2 \to$ mean squared electron momentum $\langle p^2\rangle_{\mathrm{av}}$ for intermediate degeneracy $\to 3mkT$ for nondegenerate electrons). Thus it is not obvious how to interpolate between the two limits.

The two-polynomial corrections to all the transport coefficients vanish in the highly degenerate limit[3] as α^{-2}, but they are important in nondegenerate plasmas[2] for any value of Z (e.g., $\kappa^{(2)}$ is larger than $\kappa^{(1)}$ by a factor of 2.4 for singly charged ions). Thus we must evaluate all of the b_{ij}, $i,j \le 2$, to correctly interpolate in the partially degenerate regime.

We now proceed with the calculation of $\vec{I}_{\mathrm{ee}}(\vec{p}W^2)$ and $\vec{I}_{\mathrm{ee}}(\vec{p}W^4)$. Only one approximation is necessary: The coefficient of the leading term in the b_{ij}, i.e., the coefficient of $\ln(D_e/\lambda)$, is determined exactly by those collisions with small momentum transfer, $q \ll (3mkT)^{1/2}$. Thus we expand the integrand of Eq. (12) in $q(3mkT)^{-1/2}$, keeping orders through q^3, which gives the logarithmic term. This procedure gives the coefficient exactly, but involves the following slight complication. The integrand of Eq. (12) dies off exponentially for $q \gtrsim \hbar/\lambda$. This "automatic upper cutoff" at $q \approx \hbar/\lambda$, in an integral which is essentially of the form $\int dq\, q^3\sigma(q)$, gives D_e/λ as the argument of the logarithm. In making the small q approximation, this upper cutoff is lost. Therefore we must restrict the integration over q to $q \le \hbar/\lambda$ in order to prevent a divergence and get the correct argument D_e/λ for the logarithm.[11]

If we neglect higher orders in q, and make use of energy and momentum conservation during collision, Eq. (12) for $\vec{I}_{\mathrm{ee}}(\vec{p}W^2)$ and $\vec{I}_{\mathrm{ee}}(\vec{p}W^4)$ can be put (after some algebra) in the forms

$$\begin{aligned}\vec{I}_{\mathrm{ee}}(\vec{p}W^j) &= 16(2\pi\hbar)^{-3}m^{-1}(2mkT)^2(\vec{p}_1/p_1{}^3)f_1{}^-\\ &\times\int dq\,q^2\sigma(q)\int_{-W}^{W}dV(e^{W^2+2VQ-\alpha}+1)^{-1}\\ &\times\int_{V^2+2VQ}^{\infty}dX(e^{X-\alpha}+1)^{-1}e^{X-2VQ-\alpha}\\ &\times(e^{X-2VQ-\alpha}+1)^{-1}K_j. \end{aligned}\quad (18a)$$

where
$$K_2 = 2V^3 - 3VW^2 - Q(V^2+W^2) + XV, \quad (18b)$$

$$\begin{aligned}K_4 &= 5VW^4 - Q(4V^4 + 8V^2W^2 + 2W^4)\\ &\quad + X(4V^3 + 2QV^2) + X^2V, \end{aligned}\quad (18c)$$

$$Q^2 \equiv q^2/2mkT,$$

$$V \equiv W(\vec{p}_1\cdot\vec{P})/p_1P,$$

$$X \equiv p_2{}^2 - (\vec{p}_2\cdot\vec{P}/P)^2 + (\vec{p}_1\cdot\vec{P} + P^2)^2/P^2.$$

The integrations over X can be performed as a series expansion in Q. If we retain order q in the integrand over V, note that terms odd in V integrate to zero, and perform some partial integrations, Eqs. (18) can be put into the forms

$$\begin{aligned}\vec{I}_{\mathrm{ee}}(p\vec{W}^2) &= (16\pi^2/3)(2\pi\hbar)^{-3}m^{-1}(2mkT)^{3/2}\langle\sigma\rangle\\ &\times(\vec{p}_1/p_1{}^3)f_W{}^-f_W{}^+\{2W^3(f_W{}^+ - f_W{}^-)\ln f_W{}^+\\ &+\int_0^W dV[(3W^2 - 27V^2) + (f_W{}^+ - f_W{}^-)\\ &\times(18V^2W^2 - 22V^4)]f_V{}^-\}, \end{aligned}\quad (19a)$$

$$\begin{aligned}\vec{I}_{\mathrm{ee}}(p\vec{W}^4) &= (16\pi^2/15)(2\pi\hbar)^{-3}m^{-1}(2mkT)^{3/2}\langle\sigma\rangle\\ &\times(\vec{p}_1/p_1{}^3)f_W{}^-f_W{}^+\{-40W^3\ln f_W{}^+\\ &+24W^5(f_W{}^+ - f_W{}^-)\ln f_W{}^+\\ &-40W^3\int_W^\infty dV\,V^3f_V{}^-(f_W{}^+ - f_W{}^-)\end{aligned}$$

$$+\int_0^W dV f_V^- [(-175V^4 - 240V^2W^2 + 15W^4)$$

$$+ (f_W^+ - f_W^-)(-238V^6 + 150V^2W^4)]\} , \qquad (19b)$$

where $f_W^- \equiv [1+\exp(W^2-\alpha)]^{-1}$ and $f_W^+ \equiv 1 - f_W^-$. The $f_V^{\pm}$ are defined similarly, and

$$\langle\sigma\rangle \equiv \int_0^{\hbar/\lambda} dq\, q^3\sigma(q) = m^2e^4 \ln(D_e/\lambda) .$$

Equations (19) can now be used in (15) to calculate the $b_{ij\text{ee}}$. The condition $b_{12\text{ee}} = b_{21\text{ee}}$ provides a useful check on the algebra. After a considerable amount of integration, partial integration, and other manipulations, but with no approximations, we find

$$b_{11\text{ee}} = B(\tfrac{1}{2}G_2 + H_{03}), \qquad (20a)$$

$$b_{21\text{ee}} = B(79G_4 + 108H_{23} + 18H_{05})/12 , \qquad (20b)$$

$$b_{22\text{ee}} = B(238G_6 - 350H_{43} + 540H_{25} + 30H_{07})/15, \qquad (20c)$$

where

$$B \equiv 256\pi^3(2\pi\hbar)^{-6}m^{-1}e^4(2mkT)^{5/2}\ln(D_e/\lambda), \qquad (20d)$$

and the G_i and H_{ij} are integrals similar to the usual Fermi integrals F_i:

$$G_i \equiv -\int_0^\infty dV\, V^i (e^{V^2-\alpha}+1)^{-1}\ln(1+e^{\alpha - V^2}), \qquad (21a)$$

$$H_{ij} \equiv \int_0^\infty dV\, V^i (e^{V^2-\alpha}+1)^{-1} \times \int_V^\infty dW\, W^j (e^{W^2-\alpha}+1)^{-1}. \qquad (21b)$$

It remains to calculate the electron-ion collision integrals $b_{ij\text{ei}}$. Because of the large ion-to-electron mass ratio, the energy transfer in an ei collision is negligible compared with kT. Thus we may immensely simplify Eq. (13) by replacing p_1' by p_1 (the accuracy of this elastic scattering approximation is discussed in Sec. 4 of Ref. 3). However, as in the derivation of Eq. (18a), this approximation destroys the "automatic upper cutoff" in Eq. (13) at $q \sim \hbar/\lambda'$, where

$$\hbar/\lambda' \equiv (\pi mkT)^{1/2}F_1/F_{1/2}$$

is of the order of the average electron momentum. Thus to obtain the correct argument of the logarithm,[11] we must restrict the integration over q to $q < h/\lambda'$. Equation (13) then reduces to

$$I_{\text{ei}}(pW^j) = 4\pi m e^4 Z \ln(D/\lambda')(\vec{p}_1/p_1^3) f_W^- f_W^+ W^j, \qquad (22)$$

so that Eq. (15) becomes

$$b_{ij\text{ei}} = 32\pi^2(2\pi\hbar)^{-3}nkTe^4Z\ln(D/\lambda') \times \int_0^\infty dx\, x^{i+j}e^{x-\alpha}(e^{x-\alpha}+1)^{-2}$$

$$= 32\pi^2(2\pi\hbar)^{-3}nkTe^4Z\ln(D/\lambda') \times \begin{cases}(1+e^{-\alpha})^{-1}; & i+j=0,\\ (i+j)F_{i+j-1}, & i+j \geq 1.\end{cases}$$

Equations (8), (9), (10), (14), (20), (21), (4), and (23) constitute the solution for thermal conductivity, electrical conductivity, and thermoelectric coefficient, valid for any degree of degeneracy (but, in the highly degenerate regime, only if $D_e/\lambda \gg 1$, and the ions are weakly coupled). The results contain many terms, as does any Chapman-Enskog calculation using several polynomials, but they are rather simple in form; in particular, the transition from nondegenerate to highly degenerate behavior is specified by Fermi integrals which do not depend on the details of the interaction. When the results of this paper are used in conjunction with Ref. 3, where the transport coefficients are evaluated when electrons are degenerate and D_e/λ is arbitrary, and with the work of Hubbard[12, 13] and the author on transport when the ions form a strongly coupled liquid or solid, transport coefficients of stellar matter are now known at all temperatures and densities (with only the requirement, always satisfied for stellar matter, that electrons be weakly coupled). Complete numerical tables of conductive opacities, as well as an extensive graphical presentation of the results of this paper and of Refs. 2, 3, and 12, and a new theory of the ion solid phase, are now being prepared.[13]

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[1]L. Spitzer, Physics of Fully Ionized Gases (Interscience Publishers, Inc., New York, 1962).

[2]R. H. Williams and H. E. DeWitt, University of California Lawrence Radiation Laboratory Report No. UCRL-71148, 1968 (unpublished).

[3]M. Lampe, Phys. Rev. 170, 306 (1968).

[4]L. Mestel, Proc. Cambridge Phil. Soc. 46, 331 (1950); T. D. Lee, Astrophys. J. 111, 625 (1950).

[5]R. Balescu, Statistical Mechanics of Charged Particles (Interscience Publishers, Inc., New York, 1963); R. Guernsey, Phys. Rev. 127, 1446 (1962).

[6]To be precise, λ_{th}/D is the expansion parameter in the nondegenerate regime, for ei collisions. The expansion parameters in general for ei and ee collisions are λ'/D and λ/D_e, respectively; λ', λ, and D_e are defined later.

[7]The "Debye shielding length" D_e of the partially degenerate electrons is obtained from $1 + (qD_e/\hbar)^2 = \epsilon(0,0)$, where $\epsilon(0,0)$, is the $\omega = 0$ (static) and $q = 0$ (small momentum transfer) limit of the electron dielectric function. D is the shielding length due to both the electrons and the nondegenerate ions. Electron-ion

collisions are shielded by both electrons and ions. However, in electron-electron collisions, $(E_1'-E_1)/q$ is of the order of the typical electron velocity; since this is much greater than the mean ion velocity, ions are completely ineffective in shielding ee collisions.

[8]S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge University Press, Cambridge, England, 1961).

[9]These definitions follow those of Chapman-Cowling, Chap. 8. The subsequent calculation is also closely analogous to that of Chapman-Cowling in the nondegenerate case. Transport coefficients are often defined in the following alternative way:

$$J = eS_{11}[e\vec{E} + T\nabla(\mu/T)] + eS_{12}\nabla T/T,$$

$$Q = -S_{21}[e\vec{E} + T\nabla(\mu/T)] - S_{22}\nabla T/T.$$

The S_{ij} are related to the S_{ij}' by $S_{11} = S_{11}'$; $S_{12} = S_{12}' + \frac{5}{3}\mathcal{E}S_{11}'$; $S_{22} = S_{22}' + (10/3)\mathcal{E}S_{12}' + (5\mathcal{E}/3)^2 S_{11}$.

[10]In Ref. 3, the degenerate plasma is studied in detail for all values of D_e/λ. It is shown that, in a deep sense, the real transition from nondegenerate to highly degenerate behavior occurs not in the region $\alpha \sim 1$, but rather for $\alpha \gg 1$, $D_e/\lambda \sim 1$. This is true because, in our weakly coupled system, the ratio of momentum transfer in a typical collision to thermal width of the Fermi surface is λ/D_e, rather than α as in system with strong interactions. The transition studied in the present paper, for $-1 \lesssim \alpha \lesssim 1$, is relatively superficial (which is why the mathematics can be done so simply). A plasma obeys the Fermi liquid theory of Abrikosov and Khalatnikov, Rept. Progr. Phys. 22, 329 (1959), only if $D_e/\lambda \ll 1$.

[11]This simple procedure gives the coefficient of the logarithm exactly, but gives the argument of the logarithm correct to within a factor of order unity. Thus the ee collision contribution to the transport coefficients is accurate to within order $1/\ln(D_e/\lambda)$, and the ei contribution (discussed later) to within order $1/\ln(D/\lambda')$. There is little point in calculating the argument of the logarithm more accurately, since the effects of dynamic shielding, which have been neglected, also are of this order. In Refs. 2 and 3, dynamic shielding is included for the cases of nondegenerate and highly degenerate electrons, respectively, and the corrections of order $1/\ln(D_e/\lambda)$ and $1/\ln(D/\lambda')$ are found.

[12]W. B. Hubbard, Astrophys. J. 146, 856 (1966).

[13]W. B. Hubbard and M. Lampe, to be published.

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Asymptotic Behavior of the Pair Distribution Function of a Classical Electron Gas

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The asymptotic behavior of the pair distribution function of a classical electron gas is determined. The result is in disagreement with the form conjectured by Lie and Ichikawa.

1. INTRODUCTION

During recent years[1] the pair distribution function (PDF) of the classical electron gas has been extensively investigated by many authors using either the Mayer cluster-expansion method or the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equations. The PDF has been evaluated by diagram techniques by Bowers and Salpeter,[2] De Witt,[3] and others to order ϵ^2, where $\epsilon = (4\pi\beta^3\rho e^6)^{1/2}$ is the dimensionless plasma parameter, while Lie and Ichikawa[4] have reviewed the work of many authors who approached the problem via the kinetic equations. The result for the radial distribution function given by Bowers and Salpeter (BS) is

$$g^{BS}(r) = \exp(-\epsilon x^{-1}e^{-x}) + W_1(x), \tag{1.1}$$

where $$W_1(x) = -\tfrac{1}{8}\epsilon^2 x^{-1}[\tfrac{4}{3}(e^{-x} - e^{-2x}) + (3-x)[\ln 3 - E_1(x)]e^{-x} + (3+x)E_1(3x)e^{x}], \tag{1.2}$$

with $$x = r/\Lambda_D, \quad \Lambda_D = (4\pi\beta\rho e^2)^{-1/2}, \quad E_1(x) = \int_x^\infty (e^{-y}/y)dy. \tag{1.3}$$

In Eq. (1.1), $g(r)$ is defined by the relation

$$\rho^2 g(r) = n_2(\vec{r}_1, \vec{r}_2) = n_2(|\vec{r}_1 - \vec{r}_2|) = n_2(r), \tag{1.4}$$

where $n_2(r)$ is the PDF. At large distances, Eq. (1.1) has the asymptotic form

$$g(r) \sim 1 - \epsilon x^{-1}e^{-x} + \tfrac{1}{8}\epsilon^2 \ln 3\, e^{-x} + O(e^{-2x}). \tag{1.5}$$