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Spin Waves in He^3 in the Paramagnon Model*

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We have estimated the effect on the long-wavelength spin-wave spectrum of the large spin fluctuations in a nearly ferromagnetic Fermi liquid, in order to investigate the possibility of observing spin waves in liquid He^3. The study is based on a general formula, which we derive from the spin conservation law, for the long-wavelength spin-wave dispersion curve and the paramagnon model. With the parameters of the model fixed by the He^3 susceptibility and the spin-diffusion coefficient data, the width of the spin-wave line and its shift from the Larmor frequency are estimated as functions of the temperature, the pressure, and the wavelength of the spin wave. The spin diffusion coefficient is obtained from the large damping limit of the dispersion formula.

I. INTRODUCTION

The rapid advances of experimental low-temperature physics leads us to expect that the spin wave, as well as many other phenomena, in liquid He^3 will be observable in the not too distant future. Some theoretical estimate on the spin-wave spectrum of liquid He^3 would therefore be of interest, and it is the purpose of this paper to make such an estimate.

Theoretical investigations of the spin waves in liquid He^3 and in some paramagnetic metals have been largely based on the kinetic equations in the Landau theory.[1] Recently it has been shown that, for He^3, some of the features of which the Landau theory does not easily give an adequate description can be qualitatively understood, in terms of the paramagnon model,[2] as the consequences of the large spin fluctuations implied by the fact that the He^3 is nearly ferromagnetic. One thus expects that the large spin fluctuations would have important effects on the spin-wave spectrum, and the paramagnon model should provide a reasonable qualitative description. We are aware of the fact that, unlike the Landau theory, which is a self-consistent phenomenological theory within its domain of application, the paramagnon model is very crude and almost certainly not rigorously self-consistent in its present form.[3] However, because of its simplicity and qualitative success so far, we shall base our investigation on the paramagnon model in spite of its crudeness and ambiguities, which will be discussed in some detail.

Much of our discussion will center around the width of the spin-wave line, to which the previous investigations[1] paid little attention. The physical picture is clarified and the mathematical complication reduced considerably by exploiting the fact that the interaction in the model conserves the total spin and is of very short range. A qualitative discussion of the physical processes involved will precede the analysis of diagrams leading to our results.

An important feature of the spin-wave spectrum is that it must be an infinitely sharp line at the Larmor frequency, i.e., the precessing frequency of a free spin, for $k \to 0$, where k is the wave number of the spin wave, if the interaction conserves the spin. This is so because the total spin of the system ($k=0$ implies integrating over the volume of the whole system) must not depend on an interaction which does not change the total spin. This point is discussed in Sec. II. There the spin conservation law together with the assumption that the interaction has a very short range enables one to derive a very useful formula for the long-wavelength spin-wave spectrum.

Section III includes a discussion of the basic features, the limitation, and the ambiguities of the paramagnon model. The spin-wave spectrum in the random phase approximation (RPA) is examined. To describe the decay of the spin wave, one must go beyond the RPA. A qualitative analysis is given of the decay processes involving one or many paramagnons. The order of magnitude of the spin-wave linewidth is estimated.

Section IV is devoted to an analysis of the simplest diagrams describing the effect of the paramagnons on the spin-wave spectrum. With the help of an extrapolation procedure, we derive the formula (5.1) for the spin-wave dispersion curve. The spin diffusion coefficient is found from a limiting case of the dispersion formula. No kinetic theory is used.

The result of Sec. IV is applied to He^3 in Sec. V. The parameters are determined by using the He^3 data given by Wheatley[4] on the static susceptibility and the spin diffusion coefficient. Numerical estimates show that, while it seems difficult to observe the spin wave in He^3 at present, it will not be in the near future when a temperature of lower than 1 mdeg and a periodic driving field of a wavelength shorter than ~ 0.3 cm can be easily achieved.

II. FORMULATION

The Hamiltonian of a Fermi system in a static dc magnetic field is

$$H=H_0+H_{\text{int}}, \qquad (2.1)$$

where we have separated the noninteracting (H_0) and the interacting (H_{int}) parts of H. The Hamiltonian H_{int} will not be made explicit for the moment.

$$H_0=\sum_{\vec{p}}[a_{\uparrow\vec{p}}{}^\dagger a_{\uparrow\vec{p}}(\epsilon_{\vec{p}}+\tfrac{1}{2}\omega_L)+a_{\downarrow\vec{p}}{}^\dagger a_{\downarrow\vec{p}}\times(\epsilon_{\vec{p}}-\tfrac{1}{2}\omega_L)],$$

$$\epsilon_{\vec{p}}=p^2/2m-\mu. \qquad (2.2)$$

μ is the chemical potential and ω_L is the Larmor frequency ($\hbar=1$). Let the spin density and the spin current operators be defined as

$$S_{\vec{k}}{}^- \equiv \sum_{\vec{p}} a_{\downarrow\vec{p}}{}^\dagger a_{\uparrow\vec{p}+\vec{k}},$$

$$S_{-\vec{k}}{}^+=(S_{\vec{k}}{}^-)^\dagger,$$

$$S_{\vec{k}}{}^z=\tfrac{1}{2}\sum_{\vec{p}}(a_{\uparrow\vec{p}}{}^\dagger a_{\uparrow\vec{p}+\vec{k}}-a_{\downarrow\vec{p}}{}^\dagger a_{\downarrow\vec{p}+\vec{k}}),$$

$$J_{\mu\vec{k}}{}^-=(1/m)\sum_{\vec{p}}(p_\mu+k_\mu/2)a_{\downarrow\vec{p}}{}^\dagger a_{\uparrow\vec{p}+\vec{k}},$$

$$J_{\mu-\vec{k}}{}^+=(J_{\mu\vec{k}}{}^-)^\dagger. \qquad (2.3)$$

Greek subscripts denote the components of 3-vectors. We shall restrict ourselves to the case where the interaction conserves the total spin and is velocity-independent, so that the spin density operators commute with H_{int}. We then have

$$[S_{\vec{k}}{}^-,H]=[S_{\vec{k}}{}^-,H_0]=kJ_{3,\vec{k}}{}^-+\omega_L S_{\vec{k}}{}^-. \qquad (2.4)$$

Here $J_{3\vec{k}}{}^\pm$ denotes the component of $\vec{J}_{\vec{k}}{}^\pm$ along $\vec{k}$. Then an rf field perpendicular to the dc field is applied to the system. The response will be described by the spin-density response function χ defined by

$$\chi(\vec{k},\omega)=i\int dt\, e^{i\omega t}\langle[S_{\vec{k}}{}^-(t),S_{-\vec{k}}{}^+]\rangle\,\theta(t). \qquad (2.5)$$

As we shall see, there is a sharp singularity, interpreted as the spin wave, of χ near $\omega=\omega_L$ for small k. We also define χ_3 and χ_{33} as

$$\chi_3(\vec{k},\omega)=i\int dt\, e^{i\omega t}\langle[S_{\vec{k}}{}^-(t),(J_{3,\vec{k}}{}^-)^\dagger]\rangle\theta(t),$$

$$=i\int dt\, e^{i\omega t}\langle[J_{3,\vec{k}}{}^-(t),S_{-\vec{k}}{}^+]\rangle\,\theta(t),$$

$$\chi_{33}(\vec{k},\omega)=i\int dt\, e^{i\omega t}\times\langle[J_{3,\vec{k}}{}^-(t),(J_{3,\vec{k}}{}^-)^\dagger]\rangle\theta(t). \qquad (2.6)$$

Equations (2.5) and (2.6) are defined for $\text{Im}\omega>0$ only. For $\text{Im}\omega<0$, we simply replace the retarded commutators by the corresponding advanced ones.

The fact that the interaction term H_{int} commutes with $S_{\vec{k}}$ enables us to write the continuity equation or the spin conservation law. By (2.4),

$$i(\partial/\partial t)S_{\vec{k}}{}^-=kJ_{3,\vec{k}}{}^-+\omega_L S_{\vec{k}}{}^-. \qquad (2.7)$$

By Eqs. (2.5), (2.6), and (2.7), one easily verifies that

$$(\omega-\omega_L)\chi=k\chi_3-\Delta n, \qquad (2.8)$$

$$(\omega-\omega_L)\chi_3=k(\chi_{33}-n/2m), \qquad (2.9)$$

where n is the total density, and

$$\Delta n\equiv\langle[S_{\vec{k}}{}^-,S_{-\vec{k}}{}^+]\rangle=n_\downarrow-n_\uparrow. \qquad (2.10)$$

Combining (2.8) and (2.9), we have

$$\chi=[k^2/(\omega-\omega_L)^2](\chi_{33}-n/2m)-[\Delta n/(\omega-\omega_L)]. \qquad (2.11)$$

Thus, in the limit $k\to0$, the spin-wave spectrum is an infinitely sharp line at $\omega=\omega_L$. For small k, we expect a small correction to this spectrum of $O(k^2)$.

More useful identities can be derived if the interaction is of so short range that it can be approximated by a point interaction of the type

$$H_{\text{int}} = I \sum_{\vec{p},\vec{p}',\vec{q}} a_{\uparrow\vec{p}+\vec{q}}^{\dagger} a_{\downarrow\vec{p}'-\vec{q}}^{\dagger} a_{\downarrow\vec{p}'} a_{\uparrow\vec{p}} . \quad (2.12)$$

In this approximation, it is useful to introduce the "dielectric" function $\epsilon(k,\omega)$ as

$$\epsilon \equiv 1 - I\chi' , \quad (2.13)$$

so that $\chi = \chi'/\epsilon$, (2.14)

where χ' is the "irreducible" spin response function which includes the diagrams that cannot be separated into two pieces, each containing one external vertex, by cutting the two fermion lines at the ends of one interaction line. Figures 1(a) and (b) show the simplest irreducible diagrams. We emphasize that the above construction for ϵ and χ' makes sense only because an interaction line can be regarded as a point in space and in time. Corresponding to χ_3 and χ_{33}, we define the irreducible χ_3' and χ_{33}' in the same fashion. We then have

$$\chi_3 = \chi_3'/\epsilon ,$$

$$\chi_{33} = \chi_{33}' - I\chi_3'^2/\epsilon \quad (2.15)$$

Using (2.13), (2.14), and (2.15), one derives[5] from (2.8) and (2.9)

$$(\omega - \omega_L - I\Delta n)\chi' = k\chi_3' - \Delta n ,$$

$$(\omega - \omega_L - I\Delta n)\chi_3' = k(\chi_{33}' - n/2m) , \quad (2.16)$$

and $$\chi' = k^2(\chi_{33}' - n/2m)/(\omega - \omega_L - I\Delta n)^2 - \Delta n/(\omega - \omega_L - I\Delta n) . \quad (2.17)$$

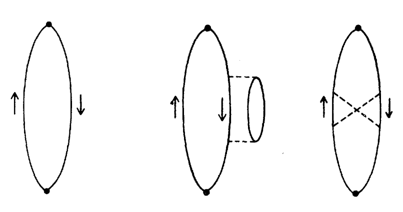
Thus, for small k, the singularities of χ', and hence those of ϵ, are near $\omega = \omega_L + I\Delta n$ instead of near ω_L. The spin-wave spectrum can now be deduced from the zero of ϵ near $\omega = \omega_L$ where χ' is expected to be well behaved. Substituting (2.17) in (2.13), one finds, setting $\epsilon = 0$,

$$\omega - \omega_L + [Ik^2/(\omega - \omega_L - I\Delta n)] \times [-\chi_{33}'(k,\omega) + n/2m] = 0 . \quad (2.18)$$

In the limit of small k, the zero of ϵ near ω_L is given by

$$\omega = \omega_L + (k^2/\Delta n)[(n/2m) - \chi_{33}'(0,\omega_L)] . \quad (2.19)$$

The ω_L in $\chi_{33}'(0,\omega_L)$ is understood to have an infinitesimal positive imaginary part. We have thus replaced the calculation of $\chi(k,\omega)$ at small k near $\omega = \omega_L$ by the easier calculation of a single number $\chi_{33}'(0,\omega_L)$. Equation (2.19) is applicable whenever the interaction conserves the total spin and has a very short range. Equation (2.19) is applied to the paramagnon model in the following sections.



(a) (b)

FIG. 1. (a) The free electron-hole pair bubble diagram. (b) The second-order correction to (a). The self-energy correction is counted in the string sum and the ladder sum of Fig. 4(a). Similarly the "exchange" diagram on the right is counted twice in summing Fig. 4(c). See Appendix.

III. QUALITATIVE FEATURES

1. Basic Features of the Paramagnon Model

The paramagnon model[2] is a rather crude model which allows one to estimate the various implications of the observed very large enhancement of the static susceptibility. A point interaction of the form (2.12) is assumed

$$H_{\text{int}} = I \sum_{\vec{p},\vec{p}',\vec{q}} a_{\uparrow\vec{p}+\vec{q}}^{\dagger} a_{\downarrow\vec{p}'-\vec{q}}^{\dagger} a_{\downarrow\vec{p}'} a_{\uparrow\vec{p}} , \quad (3.1)$$

where I is a phenomenological parameter. The static susceptibility is proportional to the static uniform limit of the spin response function χ. [See (2.5), ignoring the dc field in this case.] In the so-called RPA, χ' is [see (2.13) and (2.14)] approximated by χ_0, given by Fig. 1(a),

$$\chi_0(\vec{q},\nu) = -\sum_{\vec{p}} (f_{\vec{p}} - f_{\vec{p}+\vec{q}})/(\nu - \epsilon_{\vec{p}+\vec{q}} + \epsilon_{\vec{p}}) ,$$

$$\approx N(0)(1 - s^2 + \tfrac{1}{2}\pi i s + q^2/12), \quad (3.2)$$

for small q and small $s \equiv \nu/qv_F$. The quantity $N(0) \equiv mk_F/2\pi^2$ is the density of one-particle states at the Fermi surface. The quantities m, k_F are respectively the mass of a fermion and the Fermi momentum $v_F \equiv k_F/m$. By (2.13) and (2.14),

$$\chi_{\text{RPA}}(\vec{q},\nu) = \chi_0(\vec{q},\nu)/[1 - I\chi_0(\vec{q},\nu)] . \quad (3.3)$$

In the limit $s \to 0$, $q \to 0$, we have the static susceptibility in RPA:

$$\chi_{\text{RPA}} \to N(0)/K_0^2 ,$$

$$K_0^2 \equiv 1 - \bar{I} \equiv 1 - IN(0) . \quad (3.4)$$

If $\bar{I}$ is very close to 1, the susceptibility becomes very large. The large enhancement of the susceptibility is thus described by the large factor K_0^{-2}, which is identified with the observed enhancement factor; and the contact between the model and experimental data is thus established. For small q and small s, Eqs. (3.2) and (3.3) show that χ_{RPA} is large, and thus implies that there are large spin fluctuations of long wavelength, low energy, and small speed of propagation (i.e., small s), which are referred to as the "paramagnons" in the literature. The density of pair states in RPA is proportional to $\mathrm{Im}\chi_{\mathrm{RPA}}(\vec{q}, \nu)$, which has a peak near $\nu = qv_F K_0^2$, as can be seen easily from (3.2) and (3.3).

The interaction is not weak in this model. Whether it is consistent to identify the RPA result K_0^{-2} with the observed enhancement factor, in calculating various quantities by including a small subset of diagrams, is still a question. The self-consistency observed in the Landau theory is clearly too much to be expected from this model where there is only one parameter $\bar{I}$. However, it seems that this model is not grossly inconsistent, since the correction to the zero temperature susceptibility[6] from the simplest paramagnon effect turns out not to diverge as $K_0^2 \to 0$.[7] There are infinite numbers of diagrams which would give $O(1)$ corrections to $\bar{I}$, and, almost certainly, to any $O(1)$ quantity one calculates. Thus, when one goes beyond RPA in calculating a certain quantity, it becomes unclear how one could absorb the infinite subsets of diagrams in various parameters in the most nearly consistent way. Here we shall not consider this difficult problem. Instead, we shall observe the following rules:

(A) One must discard all the terms which are of the same order of magnitude as that given by RPA, and keep only those which are infinitely larger as $K_0^2 \to 0$.

(B) One must make sure that these retained terms would not lead to any gross inconsistency such as generating more divergent terms. For example, a divergent correction to $\bar{I}$ would.

These seem to be the simplest rules which a sensible calculation in this crude model should satisfy, although they are not claimed to be consistent or complete. It is also clear that m, the mass which appears in the RPA formulas, may not be interpreted as the bare mass or the effective mass. It must be absorbed into other quantities when comparison with experiment is made. This point will become clearer later. The calculation of the spin-wave spectrum will be carried out with the limitation of the model and the above rules in mind.

2. Range of the Parameters and Units

Besides K_0^2, the other parameters are the temperature T and the Larmor frequency ω_L in the dc field. Because the large spin fluctuation is mainly of low energy, the model must be restricted to T which is small compared to the paramagnon characteristic temperature $\sim K_0^2 T_F$ (T_F = Fermi temperature). We are mainly interested in applying the results to He^3, where ω_L/K_0^2 is much smaller than the Fermi energy. In the following, for simplicity, we shall always measure momenta in units of k_F, energies and temperatures in units of k_F^2/m. In this system of units, k_B = Boltzmann's constant = 1,

$$k_F = 1, \quad v_F \equiv k_F/m = 1, \quad m = 1,$$

$$N(0) = 1/2\pi^2, \quad n = 1/3\pi^2, \quad I = \bar{I}/N(0) = 2\pi^2\bar{I}. \quad (3.5)$$

For the clarity of discussion, we shall occasionally write out these parameters explicitly. The restrictions we impose on the parameters are then

$$K_0^2 \ll 1, \quad T/K_0^2 \ll 1, \quad \omega_L/K_0^2 \ll 1. \quad (3.6)$$

The function $\chi_0(q, \nu)$ given by (3.2) appears often in the intermediate stages of calculations. q and ν usually turn out to be the momentum and the energy of a paramagnon. Since the paramagnons are mainly of long wavelength, one may use the limiting form of $\chi_0(q, \nu)$ for very small q. Furthermore, since the paramagnon spectrum peaks near $\nu/q \sim K_0^2$, one may use the form of $\chi_0(q, \nu)$ for small ν/q. The algebra is then simplified considerably. An upper cut off $\bar{p}_1$ for the momentum of the paramagnons served as an adjustable parameter in some of the early works,[2] we shall also adopt it here for the same purpose.

3. Spin-Wave Spectrum in the RPA

In RPA, one ignores the interaction in calculating χ_{33}', i.e., only the intermediate states of one electron-hole pair is included. The corresponding diagram is given in Fig. 1(a). Substituting this approximation χ_{33}' in (2.19), one obtains the spin-wave dispersion curve in RPA. Explicitly, we have

$$\chi_{33}'(k, \omega) = -T\sum_{\vec{p}, \epsilon} (p_3 + \tfrac{1}{2}k)^2 G_\downarrow(\vec{p}, \epsilon)\, G(\vec{p}+\vec{k}, \epsilon+\omega)$$

$$= -\sum_{\vec{p}} (p_3 + \tfrac{1}{2}k)^2 (f_{\downarrow\vec{p}} - f_{\uparrow\vec{p}+\vec{k}}) \times 1/(\omega - \epsilon_{\uparrow\vec{p}+\vec{k}} + \epsilon_{\downarrow\vec{p}}), \quad (3.7)$$

where $G_{\downarrow,\uparrow}(\vec{p}, \epsilon) \equiv (\epsilon - \epsilon_{\downarrow,\uparrow\vec{p}})^{-1}$, (3.8)

with $\epsilon_{\downarrow,\uparrow\vec{p}} = \epsilon_{\vec{p}} \mp \tfrac{1}{2}(\omega_L + I\Delta n)$. (3.9)

The G's are the unperturbed Green's function in the enhanced dc field. The functions $f_{\uparrow,\downarrow\vec{p}}$ are given by

$$f_{\uparrow,\downarrow\vec{p}} = (\exp\epsilon_{\uparrow,\downarrow\vec{p}}/T + 1)^{-1}. \quad (3.10)$$

Δn is directly proportional to the static susceptibility:

$$\Delta n = (\omega_L/K_0^2)N(0)[1 + 0(T^2/K_0^4)]. \quad (3.11)$$

The temperature-dependent term[8] in (3.11) can be ignored here, since we shall find a much stronger T-dependent term in $\chi_{33}'(k, \omega)$. ϵ and ω are respectively half-odd-integral and integral multiples of $2\pi i/T$. Equation (3.7) is then continued analytically to real ω (in the upper half plane) and ω is set to equal to ω_L. We find, since $\omega_L \ll 1$ [see (3.6)],

$$\chi_{33}'(0, \omega_L) = \tfrac{1}{3}N(0)(\omega_L + I\Delta n)/I\Delta n. \quad (3.12)$$

Substituting (3.11) and (3.12) in (2.19), one finds

$$\omega = \omega_L - \tfrac{1}{3}k^2 K_0{}^4 \omega_L{}^{-1} \bar{I}^{-1}. \qquad (3.13)$$

To have some idea about the degree of ambiguity of the paramagnon model, let us compare (3.13) with the corresponding results derived from the Landau theory,[1] which gives at $T=0$ (ignoring z_1),

$$\omega = \omega_L + \tfrac{1}{3}k^2(1+z_0)^2 k_F{}^2/\omega_L z_0 m^{*2}, \qquad (3.14)$$

where m^* is the effective mass of the quasiparticles and z_0 is defined according to the convention of the last of Ref 1. With m and k_F restored, Eq. (3.13) reads,

$$\omega = \omega_L + \tfrac{1}{3}k^2 K_0{}^4 k_F{}^2/\omega_L(-\bar{I})m^2. \qquad (3.15)$$

The zero-temperature susceptibility measurement determines $m/K_0{}^2$ in the paramagnon model and $m^*/(1+z_0)$ in the Landau theory. Thus we see that (3.15) would agree with (3.14) only if $z_0 = -\bar{I}$. Since

$$-z_0 = \bar{I} - K_0{}^2[(m^*/m)-1] = 1 - K_0{}^2 m^*/m \qquad (3.16)$$

and $m^*/m - 1 \sim \ln K_0{}^2$ (see Ref. 2), we see that (3.14) and (3.15) would agree in the limit of $K_0{}^2 \to 0$, i.e., when $-z_0 \approx \bar{I} \approx 1$.

For He^3 at 0.28 atm, $-z_0 \sim 0.6$, which is far away from 1. Since only the leading term in $K_0{}^2$ makes sense in the paramagnon theory, the above instance shows the ambiguity one should expect in fitting experimental data. Since ω is purely real in (3.13), the spin wave, which is a collective excitation, has an infinite lifetime in RPA. There is also a continuum of excited states given by the singularities of (3.7). Each of these states is a single pair. The excited levels in RPA are shown in Fig. 2.

4. Corrections to the RPA Spin-Wave Spectrum

It is clear from (2.13) and Fig. 2 that the spin wave described in the RPA cannot decay. This is strictly a feature of the RPA, which only counts for single-pair states. The states of two or more pairs fill the whole (ω, k) plane of Fig. 2, and the spin wave can decay into these continuum states. Among the pairs, one or more paramagnons may form, and as a result the density of low-lying states and the population in these states at finite

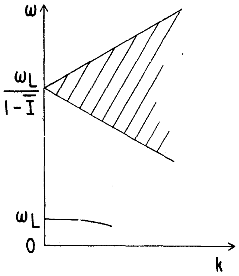


FIG. 2. The excited-state energy levels in RPA.

temperatures are increased. We thus expect the spin-wave line given by (3.13) to acquire a shift and a width which are enhanced by the formation of the paramagnons and are strongly temperature-dependent. Before proceeding to detail calculations, let us try to give a crude estimate of the decay rate of the spin wave using Fermi's "golden rule."

Consider first the final state of one paramagnon and one free pair [See Fig. 3(a)]. Here we regard a pair or a paramagnon as an individual particle. The density of states of a free pair with the momentum q' at the energy ν' is given by Im $\chi_0(q', \nu') \sim \nu'/q'$. Similarly, the density of states, for one paramagnon, is obtained from Im$\chi_{RPA}(q, \nu) \sim \nu/qK_0{}^4$ [See (3.2) and (3.3)], for small q and ν. Since the momentum of the initial state is very small, $\vec{q} \approx -\vec{q}'$, the density of final states at the initial energy ω_L is then, roughly,

$$\int_0^{\bar{p}_1} q^2 dq \int d\nu d\nu' \, \delta(\omega_L - \nu - \nu')\nu\nu' K_0{}^{-4}/q^2 \sim \omega_L{}^3 K_0{}^{-4}\bar{p}_1. \qquad (3.17)$$

We have ignored the complications in the density of states due to the presence of the dc field. Since q and ν are integrated over, these complications will contribute terms of higher order in ω_L.

Since the intermediate state is a free pair which has an energy $\omega_L/K_0{}^2$, we have a matrix element

$$\sim(\omega_L - \omega_L/K_0{}^2)^{-1} \sim K_0{}^2/\omega_L, \qquad (3.18)$$

ignoring all the constants of $O(1)$ such as I.

Thus, the decay rate is, by Fermi's "golden rule,"

$$(K_0^2/\omega_L)^2 \bar{p}_1 \omega_L{}^3 K_0{}^{-4} \sim \omega_L \bar{p}_1. \qquad (3.19)$$

At finite temperatures, one has to count for stimulated emissions and absorptions. Equation (3.17) must be supplemented by terms like

$$\bar{p}_1 \int d\nu d\nu' \delta(\omega_L - \nu - \nu')\nu\nu' K_0{}^{-4}(e^{\nu/T}-1)^{-1} \times [(e^{\nu'/T}-1)^{-1}+1], \sim \omega_L \bar{p}_1 T^2/K_0{}^4 \qquad (3.20)$$

The T^2 dependence is valid for $T \gg \omega_L$ or $T \ll \omega_L$. The width of the spin wave, according to the above estimate, would be

$$\bar{p}_1[O(\omega_L) + O(T^2/\omega_L)]. \qquad (3.21)$$

The process shown in Fig. 3(b) has a more complicated matrix element. It can be shown that it will also lead to a decay rate like (3.21). The result (3.21) is incorrect, however, since we know that, according to (2.19), the width must be proportional to the small quantity $k^2/\Delta n$. This is because we have overlooked the fact that Figs. 3(a) and (b) cannot be considered separately since a paramagnon, which is an interacting pair, and a free pair cannot be considered as different particles. There is a complete cancellation of the amplitudes as $k \to 0$ when the structure of the paramagnons is properly taken into account, as will be demonstrated later.

It turns out, as one might expect, that (3.21) is a correct estimate for the contribution of Figs.

3(a) and (b) to $\mathrm{Im}\chi_{33}'(0,\omega_L)$, i.e., to the decay rate if the spin wave were coupled to the electron hole pair via a longitudinal spin-current coupling (instead of a spin density coupling), where no strong cancellation between diagrams is expected. Since (3.21) is not small compared to ω_L, we see that the sharp spin-wave spectrum given by the RPA would disappear into the continuum were it not for the factor $k^2/\Delta n$, forced on by the spin conservation law.

Next, we consider processes involving many paramagnons. A sensible estimate of $\mathrm{Im}\chi_{33}'(0,\omega_L)$ due to multiparamagnon processes would be too involved to be included here. We shall only estimate the ω_L and T dependence of the upper limit of such a decay rate in the limit of small ω_L. Consider the process shown in Fig. 3(c), which describes the decay of the spin wave into many paramagnons. The self energy and vertex correction to these processes can be shown to have no effect on the order of magnitude of the amplitude. Suppose there are $n > 1$ paramagnons in the final state. The energy denominator of the intermediate states will depend on the momenta of the paramagnons and will have the form

$$[\omega_L - \omega_L/K_0^2 + O(p,q) + O(q^2)]^{-1} \equiv 1/d. \qquad (3.22)$$

Since the momenta q will be integrated over, the contribution of (3.22) should be much less than it would if the q-dependent terms are dropped,

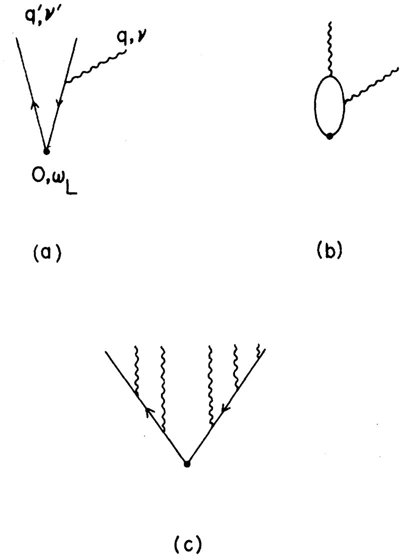


FIG. 3. The wavy lines represent paramagnons, i.e., either a sum of ladder diagrams or a sum of string diagrams. The heavy dot in each diagram represents the point where the spin wave decays. (a) The electron-hole pair in the final state has the total energy, momentum (ν', q'). The "square" of this diagram obtained by joining this diagram to the reversal of itself gives Fig. 4(a) and (b). (b) The free pair in (a) is joined to form another paramagnon. The "square" of this diagram gives Fig. 4(c).

i.e., we may write

$$1/d \ll K_0^2/\omega_L. \qquad (3.23)$$

Similar to (3.17), the density of final states is of the order[9]

$$\int_0^{\bar p_1} q^2 dq q_1^2 dq \cdots q_n^2 dq_n d\nu d\nu_1 \cdots d\nu_n \times \delta(\omega_L - \nu - \nu_1 \cdots \nu_n)(\nu/q)[\nu_1/(q_1 K_0^4)] \cdots \times [\nu_n/(q_n K_0^4)]\, \delta(\vec q + \vec q_1 \cdots + \vec q_n) \sim \omega_L(\omega_L^2/K_0^4)^n \bar p_1^{2n-1}. \qquad (3.24)$$

Again, using Fermi's "golden rule," we find the upper limit for $\mathrm{Im}\chi_{33}'(0,\omega_L)$ due to this process

$$|(K_0^2/\omega_L)^n|^2 \omega_L (\omega_L^2/K_0^4)^n \bar p_1^{2n-1} \sim \omega_L \bar p_1^{2n-1}. \qquad (3.25)$$

The extension to finite T may be estimated as before. We find

$$\sim \bar p_1^{2n-1}[\omega_L + O(T^2/\omega_L)], \quad \text{for } T \ll \omega_L,$$
$$\sim \bar p_1^{2n-1} T(T/\omega_L)^n, \quad \text{for } T \gg \omega_L, \qquad (3.26)$$

as the upper limit for $\mathrm{Im}\chi_{33}'(0,\omega_L)$. Comparing (3.26) to (3.24), we conclude that, for $T \ll \omega_L$, it is sufficient to consider the simplest processes, since $\bar p_1^2$ is supposed to be small and since $\bar p_1$ is already an adjustable parameter, but, for $T \gg \omega_L$, the multiparamagnon processes must be taken into account.

Besides the width, one also expects the spin-wave frequency to be shifted from that given by the RPA formula (3.13). This is obtained through the real part of the correction to $\chi_{33}'(0,\omega_L)$. One finds, after the more detailed study in the next section, that the correction to $\mathrm{Re}\chi_{33}'(0,\omega_L)$ turns out to be $O(1)$ for small ω_L. In view of the rules discussed previously, all such $O(1)$ terms must be absorbed into the RPA value of χ_{33}', which is of $O(1)$ [see (3.12)]. One might argue that the correction to $\mathrm{Im}\chi_{33}'(0,\omega_L)$ estimated above should also be discarded since it does not diverge as $K_0^2 \to 0$. In spite of this fact, however, the correction is nevertheless infinitely larger than the RPA value, which is identically zero and cannot be phenomenologically correct. Furthermore, in the above rough estimate, there seems to be no ambiguity about the leading powers of K_0^2. Except the uncertainty absorbed in $\bar p_1$, the qualitative conclusions about the imaginary part of $\chi_{33}'(0,\omega_L)$ seem to be definite, and must not be discarded.

In the next section, we carry out a more detailed study of the simplest processes in order to verify the conclusions of this section and, by an extrapolation procedure, to obtain a formula for the width valid also for $\omega_L \ll T$.

IV. CORRECTION TO THE RPA SPECTRUM

Led by the qualitative discussion in the previous section, we examine the diagrams given in Fig. 4. The strings of bubbles, as well as the ladders, may be regarded as paramagnons. They describe the spin fluctuations parallel to the dc field, whereas the ladders describe those perpendicular to the dc field.

Let the subscripts a, b, and c denote, respectively, the contributions of the diagrams shown in Fig. 4(a), (b), and (c).

It is verified explicitly in the Appendix that for $k=0$, the correction to χ' calculated from Fig. 4(a), (b), and (c) vanishes, i.e.,

$$\delta\chi' \equiv \chi_a' + \chi_b' + \chi_c' = 0, \tag{4.1}$$

as required by the identity (2.17), since the lowest approximation given by Fig. 1(a) already exhausts the right-hand side of (2.17). We emphasize that, in order to be consistent with the spin conservation law, all the diagrams in Fig. 4 must be kept. Since the momenta carried by the paramagnons are small, the momenta of the fermion lines on either side of the paramagnon line of Fig. 4(b) are about the same. Since all the fermions are close to the Fermi surface, their momenta all have a magnitude ~ 1. [Remember our unit system, see (3.5).] Thus, we have

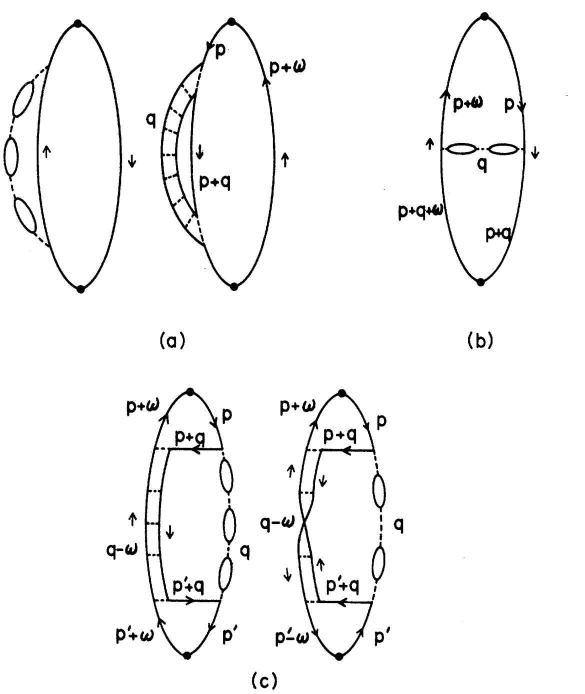


FIG. 4. The corrections to 1(a). (a) Self-energy corrections. There are two other such diagrams with the self-energy parts on the right side. (b) Vertex corrections. There must be an even number of bubbles on the string. (c) Vertex corrections. There are two other such diagrams with the position of the ladder and that of the string interchanged.

$$\chi_{33a}' + \chi_{33b}' \approx \tfrac{1}{3}(\chi_a' + \chi_b'), \tag{4.2}$$

since $p_3 p_3$ can be replaced by its average $\frac{1}{3}$. Combining (4.1) and (4.2), we have

$$\delta\chi_{33}' \equiv \chi_{33a}' + \chi_{33b}' + \chi_{33c}'$$
$$\approx \chi_{33c}' - \tfrac{1}{3}\chi_c'. \tag{4.3}$$

For the clarity and the continuity of discussion, we leave the detailed calculation of $\delta\chi_{33}'$ in the Appendix, part 2. Here we shall only outline the steps where approximations are made. The expression for $\delta\chi_{33}'(0,\omega)$ is

$$\delta\chi_{33}'(0,\omega) = -\tfrac{2}{3}(\omega - \omega_L - I\Delta n)^{-2}\Lambda(\omega), \tag{4.4}$$

where $\Lambda(\omega)$ is given by Eq. (A.29). $\Lambda(\omega)$ depends on ω_L explicitly and also through the dc field dependence of χ_0^{-+} [see (A.2)]. Under the condition $\omega_L/K_0^2 \ll 1$ [see (3.6)], we may expand $\Lambda(\omega)$ in powers of ω_L

$$\Lambda(\omega) = \Lambda_0(\omega) + \omega_L\Lambda_1(\omega) + \omega_L^2\Lambda_2(\omega) + \cdots . \tag{4.5}$$

So far, ω is a discrete imaginary energy variable. After performing the sum (A29) over ν, ω may be continued analytically to the real axis and set equal to ω_L above the real axis. Only then is one allowed to expand $\Lambda_0(\omega_L)$, $\Lambda_1(\omega_L)\ldots$ in powers of ω_L. Substituting the expanded $\Lambda_0(\omega_L)$, $\Lambda_1(\omega_L)$ ... in (4.5), one obtains the expansion in ω_L for $\Lambda(\omega_L)$. The expansion in T can be made and the leading K_0^2 dependence extracted. We find the dominant terms [see (A46)],

$$\mathrm{Im}\Lambda(\omega) = -\omega\pi\bar{p}_1 T^2/6K_0^4 + 9\omega\pi^2\zeta(3)T^3/2K_0^6$$
$$-\omega^3 b/4\pi^2 K_0^2,$$
$$b = \pi\bar{p}_1/6K_0^2 + O(T/K_0^4), \tag{4.6}$$

$$\mathrm{Re}\Lambda(\omega) = O(\omega^2/K_0^4). \tag{4.7}$$

The relevant algebra is included in the Appendix [see (A29)–(A46)]. By (4.4), we have

$$\mathrm{Im}\delta\chi_{33}'(0,\omega_L) = \pi\bar{p}_1 T^2/9\omega_L - \pi\zeta(3)T^3/2K_0^2\omega_L$$
$$+\omega_L b/6\pi^2 K_0^2,$$
$$\mathrm{Re}\delta\chi_{33}'(0,\omega_L) = O(1). \tag{4.8}$$

We have used (3.11) for Δn and kept only the leading T dependence in (4.8). In view of the rules of disposing $O(1)$ terms, the real part of $\delta\chi_{33}'(0,\omega_L)$ must be absorbed in the RPA value for $\mathrm{Re}\chi_{33}'(0,\omega_L)$. Since the RPA does not give a sensible prediction for $\mathrm{Im}\chi_{33}'(0,\omega_L)$, the imaginary part of $\delta\chi_{33}'(0,\omega_L)$ must be kept. These points have been discussed in the previous section. Combining (4.8) with the RPA expression for $\chi_{33}'(0, \omega_L)$, we find, using (2.19),

$$\omega = \omega_L - (k^2/3\omega_L)K_0^4(1 + iaT^2/\omega_L + ib\omega_L),$$

$$a \equiv 2\pi^3[\bar{p}_1 - 9\zeta(3)T/2K_0^2]/3K_0^2. \qquad (4.9)$$

Equation (4.9) will not be valid for $T \gtrsim \omega_L$ since, as was shown in the previous section, the processes involving a large number of paramagnons will dominate. Instead of attempting to sum the series of multiparamagnon diagrams, we shall simply extrapolate (4.9) by writing

$$\omega = \omega_L - (k^2/3\omega_L)K_0^4(1 - iaT^2/\omega_L - ib\omega_L)^{-1}, \qquad (4.10)$$

where a, and b are given in (4.9) and (4.6), respectively. This extrapolation is the simplest Padé approximation[10] for (4.9). The asymptotic condition of this approximation for large T^2/ω_L is provided by the observed fact that

$$\omega = \omega_L - iDk^2, \quad (DT^2)^{-1} \sim \text{const.}[1 + O(T)], \qquad (4.11)$$

where D is the spin diffusion coefficient. Since (4.10) reduces to (4.9) for $T \ll \omega_L$ and has the right T-dependence for $T^2 \gg \omega_L$, it should be a reasonably good approximation between these two limits. The diffusion coefficient implied by (4.10) is given by

$$D = \tfrac{1}{3}K_0^4/aT^2 \qquad (4.12)$$

or

$$(DT^2)^{-1} = 3a/K_0^4$$

$$= 2\pi^3\bar{p}_1/K_0^6 - 9\pi^3\zeta(3)T/K_0^8. \qquad (4.13)$$

In the next section, we shall fix the parameters K_0^2 and $\bar{p}_1$ from the observed susceptibility and spin diffusion coefficient. The formula (4.10) is then used to draw some qualitative conclusions about the long-wavelength spin wave in He^3.

V. APPLICATION TO He^3

In this section, using the experimental data obtained by Wheatley *et al.*,[4] for liquid He^3 at very low temperatures, the parameters in (4.10) will be fixed and the condition for observing the spin wave estimated.

We are only interested in the temperature range where the shift of the spin-wave frequency from ω_L is comparable to or greater than the width. This temperature range turns out to be around 1 mdeg or less. The Larmor frequency is in the megacycle range. Since the temperatures much less than 1 mdeg are not easily accessible at present, we have $k_B T \gg \hbar\omega_L$ (since $k_B/\hbar \sim 10^{12}\ \text{sec}^{-1}$). We therefore ignore the T-dependence of a and the $O(\omega_L)$ term in the denominator of (4.10). Restoring all parameters, (4.10) gives

$$\Delta\omega \equiv \omega - \omega_L = -k^2(2k_B T_F K_0^2/\hbar)^2/3k_F^2$$

$$\times[\omega_L - i\bar{p}_1\pi^3(k_B T)^2/(3k_B T_F K_0^2\hbar)]^{-1}. \qquad (5.1)$$

$k_F\,(=p_0/\hbar$ in Wheatley's notation) can be calculated from the particle density and $K_0^2 T_F$ is related to the temperature T^* defined by Wheatley from the susceptibility measurement by

$$K_0^2 T_F = 3T^*/2. \qquad (5.2)$$

Therefore, except for $\bar{p}_1$, all constants in (5.1) are known. To fix $\bar{p}_1$, we fit the spin diffusion coefficient data to (4.13), which, with all constants restored, reads

$$(DT^2)^{-1} = \pi^3\bar{p}_1\hbar k_F^2/4k_B(K_0^2T_F)^3$$

$$- 9\pi^3\zeta(3)\hbar k_F^2 T/16k_B(K_0^2T_F)^4. \qquad (5.3)$$

At temperatures low enough, the second term of (5.3) can be ignored. Notice that the parameter m has been absorbed in $K_0^2T_F = 3T^*/2$ as in the case of the static susceptibility.[8] Compared to T^{*3}, k_F^2 varies only slightly between the pressures 0.28 and 27 atm. (see Ref. 4, Table 6). If $\bar{p}_1$ is taken to be a constant, (5.3) gives a T^{*3} dependence for DT^2. Figure 5 shows the observed DT^2 as a function of T^* on a log-log scale and the T^{*3} dependence is evident.[11] (See Ref. 4, Tables 2 and 3.) Taking the intermediate value 8.4×10^7 cm^{-1} for k_F and $\bar{p}_1 = 0.3$, Eq. (5.3) gives the straight line in Fig. 5. This good fit of the dif-

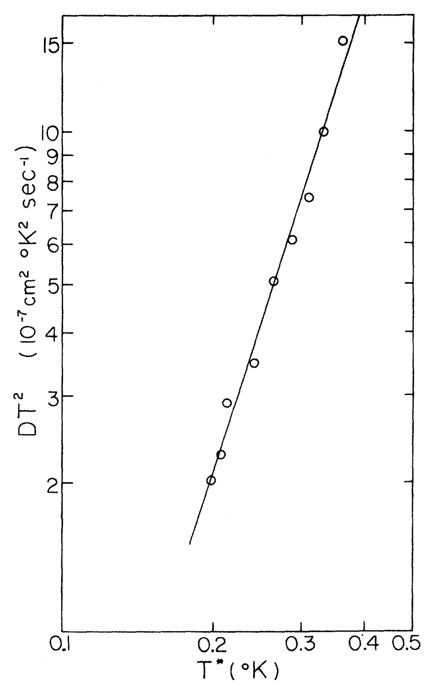


FIG. 5. The quantity DT^2 versus T^* on a log-log scale. D is the spin diffusion coefficient. The experimental points are taken from Ref. 4, Tables 2 and 3.

fusion coefficient data indicates that reliable estimates can be expected from (5.1).

The second term of (5.3) contains no adjustable parameter. It agrees with the result obtained by Pethick from the transport theory in the small K_0^2 limit.[12]

At 0.28 atm, (5.1) gives, with $\bar{p}_1 = 0.3$, $k_F = 8.4 \times 10^7\ \text{cm}^{-1}$, and $K_0^2 T_F = 0.54\,°\text{K}$,

$$\Delta\omega = \omega - \omega_L \simeq -10^6 k^2/(\omega_L - i10^{12}T^2)\ \text{sec}^{-1}, \quad (5.4)$$

with ω_L in sec^{-1}, T in °K and k in cm^{-1}. ω_L is related to the dc field H through

$$\omega_L = 2\pi \times 3.2 \times 10^6\ \text{sec}^{-1} \times H(\text{in kG}). \quad (5.5)$$

The large coefficient of the T^2 term must be compensated by a very low temperature in order to observe $\Delta\omega$. Under the condition that $T < 10^{-3}\,°\text{K}$, and $k > 20\ \text{cm}^{-1}$ and $\omega_L \sim 10^6\ \text{sec}^{-1}$ (which means $H \sim 50$ G) one could see a shift from ω_L of $\sim 200\ \text{sec}^{-1}$ with a frequency resolution better than 1 part in 10^4. Although this set of conditions is experimentally hard to obtain at present, it is not unfeasible.

In summary, we have estimated the spin-wave spectrum of He³ based on the spin density conservation law and the paramagnon model. The spin nonconserving forces, which would give a finite width for the spin wave at $k=0$ (i.e., the nuclear magnetic resonance line), are probably negligible since no such width has been observed in He.[3]

We have devoted much of the discussion to the qualitative features and the ambiguities of the paramagnon model. It seems that we have got around the important ambiguities and our conclusions seem to be reasonable.

ACKNOWLEDGMENT

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APPENDIX

1. Contribution of Fig. 4(a), (b), and (c) to χ'

According to (2.17), at $k=0$, one has exactly

$$\chi' = -\Delta n/(\omega - \omega_L - I\Delta n). \quad \text{(A1)}$$

The lowest-order approximation to χ' is given by Fig. 1(a), i.e.,

$$\chi_0^{-+}(k,\omega) = -T\sum_{\vec{p},\epsilon} G_\downarrow(p)G_\uparrow(p+k) = -\sum_{\vec{p}} \frac{f_{\downarrow\vec{p}} - f_{\uparrow\vec{p}+\vec{k}}}{\omega - \epsilon_{\uparrow\vec{p}+\vec{k}} + \epsilon_{\downarrow\vec{p}}}, \quad \text{(A2)}$$

where $\epsilon_{\uparrow,\downarrow\vec{p}}$, $f_{\uparrow,\downarrow\vec{p}}$ are given by (3.9) and (3.10). For $k=0$,

$$\chi_0^{-+}(0,\omega) = -\Delta n_0/(\omega - \omega_L - I\Delta n) = \chi'(0,\omega) + \delta\Delta n/(\omega - \omega_L - I\Delta n), \quad \text{(A3)}$$

where $\Delta n_0 \equiv \sum_{\vec{p}}(f_{\downarrow\vec{p}} - f_{\uparrow\vec{p}})$, $\delta\Delta n \equiv \Delta n - \Delta n_0$. (A4)

From Fig. 4(a), (b), and (c), one can write down the corrections to Fig. 1(a). Let χ_a', χ_b', and χ_c' be, respectively, the contribution to χ' from Fig. 4(a), (b), and (c). Consider Fig. 4(b) first, since it is the simplest.

$$\chi_b'(0,\omega) = T^2 \sum_{\vec{p},\epsilon,\vec{q},\nu} \eta(q)G_\downarrow(p)G_\uparrow(p+\omega)G_\downarrow(p+q)G_\uparrow(p+q+\omega), \quad \text{(A5)}$$

where q stands for $(\vec{q},\nu)$, p for $(\vec{p},\epsilon)$, and $\omega = (0,\omega)$.

$$\eta(q) = I\{[1 - I^2\chi_0^{+}(q)\chi_0^{-}(q)]^{-1} - 1\}, \quad \text{(A6)}$$

$$\chi_0^{\pm}(q) = -T\sum_{p,\epsilon} G_{\uparrow,\downarrow}(p)G_{\uparrow,\downarrow}(p+q) = -\sum_{\vec{p}}(f_{\uparrow,\downarrow\vec{p}} - f_{\uparrow,\downarrow\vec{p}+\vec{q}})/(\nu - \epsilon_{\vec{p}+\vec{q}} + \epsilon_{\vec{p}}) \approx \chi_0(q) \text{ for small } q, \omega_L. \quad \text{(A7)}$$

χ_0 is given by (A2) with zero magnetic field. Since, with the help of (3.9),

$$G_\downarrow(p)G_\uparrow(p+\omega)=[G_\downarrow(p)-G_\uparrow(p+\omega)]/(\omega-\omega_L-I\Delta n)\,, \tag{A8}$$

(A5) becomes

$$\chi_b{}'=-(\omega-\omega_L-I\Delta n)^{-2}T\sum_{\vec{q},\nu}\eta(q)[\chi_0{}^{+}(q)+\chi_0{}^{-}(q)-\chi_0{}^{-+}(q+\omega)-\chi_0{}^{+-}(q-\omega)]=-2(\omega-\omega_L-I\Delta n)^{-2}\times T\sum_{\vec{q},\nu}\eta(q)[\chi_0(q)-\chi_0{}^{-+}(q+\omega)]\,, \tag{A9}$$

where $\chi_0{}^{-+}(q)=\chi_0{}^{+-}(-q)$,

is given by (A2) with $(\vec{q},\nu)$ replacing $(\vec{k},\omega)$. For Fig. 4(a), we have

$$\chi_a{}'(0,\omega)=-T\sum_{\vec{p},\epsilon}[G_\downarrow{}^2(p)\Sigma_\downarrow(p)G_\uparrow(p+\omega)+G_\uparrow{}^2(p)\Sigma_\uparrow(p)G_\downarrow(p-\omega)] \tag{A10}$$

The second term comes from the self-energy correction of the other fermion line. The self-energies are given by

$$\Sigma_\downarrow(p)=-T\sum_{\vec{q},\nu}[\eta'(q)G_\downarrow(p+q)+\eta^{-+}(p)G_\uparrow(p+q)]\,, \tag{A11}$$

where $\eta'=-I^2\chi_0(1-I^2\chi_0{}^2)^{-1}$, $\eta^{-+}=-I^2\chi_0{}^{-+}(1-I\chi_0{}^{-+})^{-1}$. (A12), (A13)

$\Sigma_\uparrow(p)$ is obtained by interchanging + and −, ↑ and ↓ in (A11). Substituting (A11) in (A10) and going through the same algebra as before, we have

$$\chi_a{}'(0,\omega)=2(\omega-\omega_L-I\Delta n)^{-2}\,T\sum_{\vec{q},\nu}[\chi_0{}^{-+}(q)-\chi_0(q-\omega)][\eta^{-+}(q)-\eta'(q-\omega)]-\delta\Delta n/(\omega-\omega_L-I\Delta n)\,, \tag{A14}$$

where $\delta\Delta n\equiv T\sum_{\vec{p},\epsilon}[G_\downarrow{}^2(p)\Sigma_\downarrow(p)-G_\uparrow{}^2(p)\Sigma_\uparrow(p)]$ (A15)

is a contribution to $n_\downarrow-n_\uparrow$ due to the paramagnons. This point needs more clarification. The down-spin density $n_\downarrow$ may be expressed through the full Green's function, which is a geometric series in the self-energy; as

$$n_\downarrow=T\sum_{\vec{p},\epsilon}[G_\downarrow(p)+G_\downarrow{}^2(p)\Sigma_\downarrow(p)+G_\downarrow{}^3(p)\Sigma_\downarrow(p)^2+\cdots] \tag{A16}$$

$$=\sum_{\vec{p}}f_{\downarrow\vec{p}}+T\sum_{\vec{p},\epsilon}G_\downarrow{}^2(p)\Sigma_\downarrow(p)+\cdots \tag{A17}$$

and a similar expression for $n_\uparrow$. It is then clear that the Δn_0 in (A4) and (A15) are the first two terms of Δn. In calculating χ', we can simply include all the terms in (A16) by changing the Δn_0 in (A3) to Δn, i.e., dropping $\delta\Delta n$. We shall thus drop the last term of (A14). The modified (A3) then satisfies (A1). Combining (A14) and (A9), we find

$$\chi_a{}'+\chi_b{}'=2(\omega-\omega_L-I\Delta n)^{-2}T\sum_{\vec{q},\nu}[\chi_0{}^{-+}(q)-\chi_0(q-\omega)][\eta(q-\omega)-\eta'(q-\omega)+\eta^{-+}(q)]\,. \tag{A18}$$

By (A6), (A12), and (A13), one verifies that

$$\eta(q-\omega)-\eta'(q-\omega)+\eta^{-+}(q)=-I^2[\chi_0{}^{-+}(q)-\chi_0(q-\omega)]/(1-I\chi_0)(1-I\chi_0{}^{-+})\,. \tag{A19}$$

$\chi_c{}'(0,\omega)$ is slightly more complicated;

$$\chi_c{}'(0,\omega)=T^3\sum_{\vec{p},\epsilon,\vec{p}',\epsilon',\vec{q},\nu}[G_\downarrow(p)G_\uparrow(p+\omega)G_\downarrow(p+q)\eta'^{+-}(q-\omega)\eta'(q)G_\downarrow(p')G_\uparrow(p'+\omega)G_\downarrow(p'+q)$$
$$+G_\downarrow(p)G_\uparrow(p+\omega)G_\uparrow(p+q)\eta'^{-+}(q)\eta'(q-\omega)G_\downarrow(p')G_\uparrow(p'+\omega)G_\uparrow(p'+q)+G_\downarrow(p)G_\uparrow(p+\omega)G_\downarrow(p+q)$$
$$\times\eta'^{+-}(q-\omega)\eta''(q)G_\downarrow(p'-\omega)G_\uparrow(p')G_\uparrow(p'-q)+G_\downarrow(p)G_\uparrow(p+\omega)G_\uparrow(p+q)\eta'^{-+}(q)\eta''(q-\omega)G_\downarrow(p'-\omega)$$
$$\times G_\uparrow(p')G_\downarrow(p'-q)]\,, \tag{A20}$$

where $\eta'^{-+}(q)=-I/[1-I\chi_0{}^{-+}(q)]=\eta'^{+-}(-q)$, $\eta''(q)=I/[1-I^2\chi_0{}^2(q)]$. (A21)

Again, using (A8), we get after actually very little algebra,

$$\chi_c'(0,\omega)=2(\omega-\omega_L-I\Delta n)^{-2}T\sum_{\vec{q},\nu}[\chi_0(q-\omega)-\chi_0^{-+}(q)]^2[\eta'(q-\omega)-\eta''(q-\omega)]\eta'^{-+}(q). \tag{A22}$$

By (A12) and (A21), we have

$$[\eta'(q-\omega)-\eta''(q-\omega)]\eta'^{-+}(q)=I^2/[1-I\chi_0(q-\omega)][1-I\chi_0^{-+}(q)]. \tag{A23}$$

Comparing (A18) and (A22), and taking (A19) and (A23) into account, we see that

$$\chi_a'+\chi_b'+\chi_c'=0. \tag{A24}$$

One minor point: The second-order diagrams which are shown in Fig. 1(b) have been counted twice in (A18) and (A23). This is because they occur in both the ladder sum and the sum over the string of bubbles. Equation (A24) is not affected since it holds for every order of I. Our later results are not affected since we are only interested in the leading divergent term for $K_0^2\to 0$.

2. Calculation of $\delta\chi_{33}$

The expression for χ_{33c}' is obtained by multiplying (A20) by $\frac{1}{3}\vec{p}\cdot\vec{p}'$. Since $\hat{q}$ is the only preferred direction when one sums over $\vec{p}$ and $\vec{p}'$, $\vec{p}\cdot\vec{p}'$ may be replaced by $\vec{p}\cdot\hat{q}\,\vec{p}'\cdot\hat{q}$. One ends up with an expression like (A22), with $[\chi_0(q-\omega)-\chi_0^{-+}(q)]^2$ replaced by $[\chi_{30}(q-\omega)-\chi_{30}^{-+}(q)]^2$ and the whole expression divided by 3, where

$$\chi_{30}(q)=-\sum_{\vec{p}}\hat{q}\cdot\vec{p}\,(f_{\vec{p}}-f_{\vec{p}+\vec{q}})/(\nu-\epsilon_{\vec{p}+\vec{q}}+\epsilon_{\vec{p}}), \tag{A25}$$

$$\chi_{30}^{-+}(q)=-\sum_{\vec{p}}\hat{q}\cdot\vec{p}\,(f_{\vec{p}\downarrow}-f_{\uparrow\vec{p}+\vec{q}})/(\nu-\epsilon_{\uparrow\vec{p}+\vec{q}}+\epsilon_{\downarrow\vec{p}}). \tag{A26}$$

Thus we have

$$\chi_{33c}'-\tfrac{1}{3}\chi_c'=\tfrac{2}{3}(\omega-\omega_L-I\Delta n)^{-2}T\sum_{\vec{q},\nu}\{[\chi_{30}(q-\omega)-\chi_{30}^{-+}(q)]^2-[\chi_0(q-\omega)-\chi_0^{-+}(q)]^2\}$$
$$\times I^2/[1-I\chi_0(q-\omega)][1-I\chi_0^{-+}(q)]. \tag{A27}$$

χ_{30}, χ_{30}^{-+} may be expressed in terms of χ_0, χ_0^{-+}, with the aid of (2.16) [with zero dc field, since the field plays no role in (A25)],

$$\chi_{30}(q)=\chi_0(q)\nu/q,$$

and from (2.16) with the field, we have

$$\chi_{30}^{-+}(q)=\chi_0^{-+}(q)(\nu-\omega_L-I\Delta n)/q+\Delta n/q. \tag{A28}$$

Substituting (A28) in (A27), after some algebra, one finds

$$\delta\chi_{33}'=\chi_{33c}'-\tfrac{1}{3}\chi_c'=\tfrac{2}{3}(\omega-\omega_L-I\Delta n)^{-2}\Lambda(\omega),$$

$$\Lambda(\omega)=T\sum_{q,\nu}\Big(I[\chi_0^{-+}(q)-\chi_0(q-\omega)]\{[(\nu-\omega_L-I\Delta n)^2/q^2-1][1-I\chi_0(q-\omega)]^{-1}-[(\nu-\omega_L)^2/q^2-1]$$
$$\times[1-I\chi_0^{-+}(q)]^{-1}\}+(I\Delta n)^2/q^2\Big). \tag{A29}$$

In (A29), we have set $\omega=\omega_L$ for those ω which can be factored outside the (q,ν) sum. $\Lambda(\omega)$ depends on ω_L explicitly and also through χ_0^{-+} [see (A2)]. Since ω_L is small [see (3.16)], we expand $\Lambda(\omega)$ in powers of ω_L

$$\Lambda(\omega)=\Lambda_0(\omega)+\omega_L\Lambda_1(\omega)+\omega_L^2\Lambda_2(\omega)+\cdots. \tag{A30}$$

The explicit expression for $\Lambda_0(\omega)$ is, setting $\omega_L=0$ in (A29),

$$\Lambda_0(\omega) = T\sum_{\vec{q},\nu} I[\chi_0(q)-\chi_0(q-\omega)](\nu^2/q^2-1)\{[1-I\chi_0(q-\omega)]^{-1}-[1-I\chi_0(q)]^{-1}\}$$
$$= 2T\sum_{\vec{q},\nu} I[\chi_0(q-\omega)-\chi_0(q)](\nu^2/q^2-1-\nu\omega/q)[1-I\chi_0(q)]^{-1}. \quad \text{(A31)}$$

We shall not write out the expressions for Λ_1 and Λ_2 here since they are lengthy and will not contribute to the leading terms in which we are interested.

$\Lambda_0(\omega)$ is a symmetric function of ω. After summing over ν and continuing ω analytically to immediately above the real axis, we must have

$$\mathrm{Re}\Lambda_0(\omega+i0^+) = \mathrm{Re}\Lambda_0(-\omega+i0^+), \quad \mathrm{Im}\Lambda_0(\omega+i0^+) = -\mathrm{Im}\Lambda_0(-\omega+i0^+). \quad \text{(A32)}$$

We shall give the relevant details after listing the results on the order of magnitudes. For small ω, one finds

$$\mathrm{Im}\Lambda_0(\omega) = O(T^2\omega\bar{p}_1/K_0^4) + O(T^3\omega/K_0^6) + O(\omega^3 T/K_0^6) + O(\omega^3\bar{p}_1/K_0^4)$$

$$\mathrm{Re}\Lambda_0(\omega) = O(\omega^2) + O(\omega^2 T^2/K_0^2). \quad \text{(A33)}$$

Similarly, $\mathrm{Re}\Lambda_1(\omega)$ is odd and $\mathrm{Im}\Lambda_1(\omega)$ is even. Also, $\mathrm{Re}\Lambda_2(\omega)$ is even and $\mathrm{Im}\Lambda_2(\omega)$ is odd.

We found

$$\mathrm{Im}\Lambda_1(\omega) = O(\omega^2 T^2) + O(\omega^4), \quad \mathrm{Re}\Lambda_1(\omega) = O(\omega/K_0^2) + O(\omega T^2),$$

$$\mathrm{Im}\Lambda_2(\omega) = O(\omega T^2) + O(\omega^3), \quad \mathrm{Re}\Lambda_2(\omega) = O(1/K_0^4) + O(T^2). \quad \text{(A34)}$$

We have not determined the K_0^2 dependence of the T^2 terms in (A34) since these terms are negligible compared to the leading T^2 term in $\mathrm{Im}\Lambda_0(\omega)$ in (A33).

Substituting (A33) and (A34) in (A30), the leading terms are

$$\mathrm{Im}\Lambda(\omega_L) = O(T^2\omega_L\bar{p}_1/K_0^4) + O(T^3\omega_L/K_0^6) + O(\omega_L^3 T/K_0^6) + O(\omega_L^3\bar{p}_1/K_0^4), \quad \mathrm{Re}\Lambda(\omega_L) = O(\omega_L^2/K_0^4). \quad \text{(A35)}$$

Substituting (A35) in (A29), the leading terms are

$$\mathrm{Im}\,\delta\chi_{33}'(0,\omega_L) = O(T^2\bar{p}_1/\omega_L) + O(T^3/\omega_L K_0^2) + O(\omega_L T/K_0^2) + O(\omega_L\bar{p}_1), \quad \text{(A36)}$$

$$\mathrm{Re}\,\delta\chi_{33}'(0,\omega_L) = O(1), \quad \text{(A37)}$$

since $(\omega-\omega_L-I\Delta n)^{-2} \approx K_0^4/\omega_L^2$ for $\omega = \omega_L$.

To determine the coefficients in (A36), we only have to consider $\Lambda_0(\omega)$. Let

$$J(\nu) \equiv -2I(1+\nu\omega/q-\nu^2/q^2)[1-I\chi_0(q)]^{-1}, \quad K(\nu) = \chi_0(q). \quad \text{(A38)}$$

Then (A31) becomes

$$\Lambda_0(\omega) = T\sum_{\vec{q},\nu} J(\nu)[K(\nu-\omega)-K(\nu)] = \sum_{\vec{q}} \int_{-\infty}^{\infty} (d\nu/\pi)(e^{\nu/T}-1)^{-1}\{\mathrm{Im}J(\nu)[K(\nu-\omega-i0^+)-K(\nu-i0^+)]$$
$$+\mathrm{Im}K(\nu)[J(\nu+\omega)-J(\nu)]\}. \quad \text{(A39)}$$

An additional $i0^+$ is implied for all the real frequency variables unless $-i0^+$ is written out explicitly. Now write

$$K(\nu-\omega-i0^+)-K(\nu-i0^+) = -\omega\frac{\partial K(\nu-i0^+)}{\partial\nu} + \frac{\omega^2}{2}\frac{\partial^2}{\partial\nu^2}K(\nu-i0^+) - \frac{\omega^3}{6}\frac{\partial^3}{\partial\nu^3}K(\nu-i0^+) + \cdots \quad \text{(A40)}$$

and a similar expression for $J(\nu+\omega)-J(\nu)$. Thus (A39) is expanded in powers of ω. Consider the $O(\omega)$ term of $\mathrm{Im}\Lambda_0(\omega)$. Since $\mathrm{Im}K(\nu-i0^+) = -\mathrm{Im}K(\nu)$, we have

$$\mathrm{Im}\Lambda_0(\omega) = \sum_{\vec{q}} \int_0^{\infty} (d\nu/\pi)[\mathrm{Im}J(\nu)(\partial/\partial\nu)\,\mathrm{Im}K(\nu) + \mathrm{Im}K(\nu)(\partial/\partial\nu)\,\mathrm{Im}J(\nu)]\,(e^{\nu/T}-1)^{-1} + O(\omega^3) \quad \text{(A41)}$$

$$= 2\omega \sum_{\vec{q}} \int_0^\infty (d\nu/\pi)(e^{\nu/T}-1)^{-1}(\partial/\partial\nu)[\mathrm{Im}J(\nu)\,\mathrm{Im}K(\nu)] = 2\omega(2\pi)^{-3}2I \int_0^{\bar{p}_1} 4\pi q^2 dq \int (dx/\pi)(e^x-1)^{-1}$$
$$\times (d/dx)\tfrac{1}{4}\pi^2 N(0) s^2[(K_0{}^2+s^2)^2+\tfrac{1}{4}\pi^2 s^2]^{-1} + O(\omega^3)\,, \tag{A42}$$

where we have used the approximation

$$\chi_0(q) \approx N(0)(1-s^2+\tfrac{1}{2}\pi i s), \quad s \equiv \nu/q \equiv Tx/q\,. \tag{A43}$$

The leading term is of $O(T^2)$, obtained by setting $T=0$ in the denominator of (A42). The next term would be an $O(T^4)$ term were it not for the linear divergence of the q integral, which forces the T-dependence to $O(T^3)$. The algebra is easy. One has, neglecting $o(T^3)$ terms,

$$\mathrm{Im}\Lambda_0(\omega) = -\omega\pi^2 T^2(\bar{p}_1 - 9\zeta(3)T/2K_0{}^2)/6K_0{}^4 + O(\omega^3)\,, \tag{A44}$$

where $\zeta(3) = \frac{1}{2}\int_0^\infty x^2 dx/(e^x-1) = 1.202$.

For the $O(\omega^3)$ term of $\mathrm{Im}\Lambda_0(\omega)$, we shall only consider the $T=0$ case. It turns out that the $\nu\omega/q$ term in $J(\nu)$ [see (A38)] does not contribute to this order. By (A40) and (A39), we have

$$-\frac{\omega^3}{6}\sum_{\vec{q}} \int_{-\infty}^0 \frac{d\nu}{\pi}\left[\mathrm{Im}J(\nu)\frac{\partial^3}{\partial\nu^3}\,\mathrm{Im}K(\nu) + \mathrm{Im}K(\nu)\frac{\partial^3}{\partial\nu^3}\,\mathrm{Im}J(\nu)\right]$$
$$= \frac{\omega^3}{6}\frac{1}{\pi}\sum_{\vec{q}}\left[\frac{\partial}{\partial\nu}\,\mathrm{Im}K(\nu)\frac{\partial}{\partial\nu}\,\mathrm{Im}J(\nu)\right]_{\nu=0} = -\omega^3\bar{p}_1/24\pi K_0{}^4\,. \tag{A45}$$

For finite T, the leading correction is of $O(T)$. Combining (A41) and (A45), we have

$$\mathrm{Im}\Lambda_0(\omega) = -\omega\pi\bar{p}_1\, T^2/6K_0{}^4 + 9\omega\pi^2\zeta(3)T^3/2K_0{}^6 - \omega^3 b/4\pi^2 K_0{}^2, \quad b = \pi\bar{p}_1/6K_0{}^2 + O(T/K_0{}^4)\,. \tag{A46}$$

The detailed analysis of (A34) and the $O(T)$ term of (A46) will not be given here since we are not interested in the coefficients of these terms.

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[1]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 33, 1227 (1957) [English transl.: Soviet Phys.–JETP 6, 945 (1958); D. R. Fredkin and J. M. Goodkind, in Abstracts of the Tenth International Conference on Low Temperature Physics, Moscow, 1966 (Viniti Publishing House, Moscow, 1967), p. 44; P. M. Platzman and P. A. Wolff, Phys. Rev. Letters 18, 280 (1967); D. R. Fredkin and A. R. Wilson, Phys. Rev. Letters, to be published.

[2]N. F. Berk and J. R. Schrieffer, Phys. Rev. Letters 17, 433 (1966); S. Doniach and S. Engelsberg, Phys. Rev. Letters 17, 750 (1966).

[3]The ambiguities of the paramagnon model has been criticized, for example, by S. Misawa, Progr. Theoret. Phys. 38, 1207 (1967); some of the criticisms seem, however, to be not well founded. The fact that the paramagnon model is not at all as ambiguous as regarded by these criticisms seems to be evident, based on the arguments of Schrieffer and Berk (see Ref. 7).

[4]J. C. Wheatley, in Quantum Fluid, edited by A. F. Brewer (North-Holland Publishing Company, Inc., Amsterdam, 1966).

[5]This identity is a generalization of that given by P. Nozieres, Interacting Fermi Systems (W. A. Benjamin, New York, 1964), Eq. (6-176), p. 285.

[6]More precisely, the correction to the quantity $\bar{I}$.

[7]See, for example, J. R. Schrieffer and N. F. Berk, Phys. Letters 24A, 604 (1967).

[8]M. T. Béal-Monod, S. Ma, and D. R. Fredkin, to be published.

[9]Note that $\int_0^\infty d\nu d\nu_1 \cdots d\nu_n \delta(\omega-\nu-\nu_1\ldots\nu_n)\nu\nu_1\ldots\nu_n = \omega^{2n+1}/(2n+1)!$.

[10]For a detailed discussion of the Padé approximation and its applications, see G. A. Baker, in Advances in Theoretical Physics, edited by K. A. Brueckner (Academic Press Inc., New York, 1965), Vol. I.

[11]Using a transport theory, M. J. Rice [Phys. Rev. 159, 153 (1961); 163, 206 (1967)] obtained an expression for D similar to (5.3) but with a $K_0{}^4$ ($\sim T^{*2}$) dependence, since his cutoff $\bar{Q}$ is practically independent of pressure. In a more recent analysis of the transport equations, Betts and Rice [D. S. Betts and M. J. Rice, Phys. Rev. 166, 159 (1068)] obtained a more complicated T^* dependence.

[12]C. T. Pethick, "Transport Coefficients of a Normal Fermi Liquid at Finite Temperatures" (to be published).