Regge-Pole Formalism for Pionic Disintegration of Deuteron*

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We investigate the reaction $\pi^+ + d \to p + p$ within the framework of Regge-pole theory. Reggeization of the amplitudes for this process is carried out in detail, and the asymptotic behavior of the helicity amplitudes at zero momentum transfer is discussed. By assuming that the lowest anomalous threshold singularity in the t channel dominates in the nearby region, we calculate the nucleon-deuteron Regge residue in terms of the pion-nucleon Regge residue. We also discuss the relations to the Regge-pole formalism of various low-energy models used to study this reaction and its inverse.

I. INTRODUCTION

IN the past several years, Regge-pole theory[1] has made tremendous progress toward understanding strong-interaction physics. Recently its applicability has even been extended into low-energy regions via finite-energy sum rules.[2] The intricate relationships among the Regge parameters in the direct and crossed channels, through analyticity, unitarity, and crossing, provide a promising bootstrap framework.[3]

In the present paper, we shall study the reaction $\pi^+ + d \to p + p$ in Regge-pole theory. The purpose of our investigation is threefold.

(1) Experimentally, the high-energy data for this reaction above 3 BeV total c.m. energy are very scant. The data[4] from 2.4 to 3 BeV show that the broad peak near the forward direction (which is symmetric with respect to the backward direction) shrinks and moves toward $\cos\theta = 1$ gradually, evolving into a sharp peak at 3 BeV. In the same energy range, we notice that the total cross section decreases monotonically roughly like $(s - \frac{1}{2}\Sigma)^{-3}$, where s is the square of the total c.m. energy and Σ is the sum of the external masses squared. These facts strongly suggest the presence of Regge behavior with nucleon-trajectory exchange, even at intermediate energies. This reaction, along with backward πN scattering and $N\bar{N} \to \pi\pi$ scattering, provides a place to study the nucleon trajectory and the dip phenomena[5] associated with it. In view of this we believe that a Regge-pole formalism is both necessary and desirable for phenomenological fitting and for stimulating systematic experiments at higher energies in the future.

(2) On the theoretical side, problems of treating the Regge residue functions with unstable or loosely bound particles are extremely interesting, yet these problems attract less attention than they deserve. Udgaonkar and Gell-Mann[6] have employed the semiclassical ray method to study the Pomeranchuk residues for loosely bound particles like nuclei and, in particular, the deuteron. But one expects that these problems should be handled within the framework of analytic S-matrix theory. In S-matrix language, the loosely bound structure of the external particle manifests itself through the existence of anomalous singularities in the residue functions. The reaction that we considered has this feature in that the d-p-n Regge residue function possesses anomalous singularities. We calculate this residue function, to a good approximation, in terms of the π-N-N residue function and the pion-nucleon and deuteron-nucleon coupling constants. The method that we used, which is analogous to the one employed by Cutkosky[7] to study the deuteron form factor, is applicable to many other cases.

(3) Finally, turning to the low-energy region, there have been extensive investigations both theoretically and experimentally for the reaction $\pi^+ + d \to p + p$ and its inverse (we shall call them pionic-disintegration and pion-production reactions). Various models have been proposed to account for the low-energy pion-production data. In nonrelativistic calculations, Rosenfeld[8] and Gell-Mann and Watson[8], have used the model of final state interaction between the two nucleons and obtained a good fit to the data with a small number of parameters. Mandelstam[9] has extended the analysis to higher energies by including πN final-state interaction. The outgoing pion is assumed to be in a resonant $(\frac{3}{2},\frac{3}{2})$ state with one of the nucleons. Both the total and differential

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[2] K. Igi, Phys. Rev. Letters **9**, 76 (1962); R. Dolen, D. Horn, and C. Schmid, *ibid.* **19**, 402 (1967).

[3] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

[4] R. M. Heinz, O. E. Overseth, D. E. Pellett, and M. L. Perl, Phys. Rev. **167**, 1232 (1968); D. Dekkers, B. Jordan, R. Mermod, C. C. Ting, G. Weber, T. R. Willitts, K. Winter, X. DeBonard, and M. Vivargent, Phys. Letters **11**, 161 (1964). Actually it is the inverse reaction $p + p \to \pi + d$ that is studied in these references. References for other experiments can be found in these two papers.

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[6] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962).

[7] R. E. Cutkosky, in *Proceedings of the Tenth Annual Conference on High-Energy Physics at Rochester, 1960*, edited by E. C. G. Sudarshan *et al.* (Interscience Publishers, Inc., New York, 1961), p. 236.

[8] A. H. Rosenfeld, Phys. Rev. **96**, 139 (1954); M. Gell-Mann and K. M. Watson, Ann. Rev. Nucl. Sci. **4**, 219 (1954).

[9] S. Mandelstam, Proc. Roy. Soc. (London) **A244**, 491 (1958).

cross sections can be fitted very well with three parameters.

In relativistic calculations, the one-pion exchange (OPE) model[10] has been employed, in which one nucleon is peripherally scattered and then combines with the other nucleon to form a deuteron. There is also the one-nucleon exchange model.[11] Both models only have moderate success in certain ranges of energy. Dispersive approaches have been carried out by Vasavada,[12] by Chahoud, Russo, and Selleri,[13] and recently by Schiff and Vann[14] to study the pionic-disintegration reaction. The anomalous cut in the cross channel is included together with the one-nucleon Born term. Good agreement with the data is obtained.

While we expect that the high-energy data should be best understood in the framework of Regge theory, even in the low-energy region the Regge-pole theory incorporates in a broad sense the ideas contained in all the models mentioned above. We attempt to understand these models in a coherent picture through the Regge formalism. This has been made possible by using the duality concept that has been emphasized by Chew and Pignotti[15] recently in connection with multiperipheralism and resonance production.

The work is as follows: In Sec. II, the singularity structure for the full and partial-wave amplitudes is studied, with special care for the anomalous singularities. In Sec. III, we consider the construction of t-channel helicity amplitudes and their relationship to a set of invariant amplitudes. The kinematical singularities are extracted. Section IV considers the analytic continuation of the partial-wave amplitudes into the complex J plane. The Regge representation for the amplitudes and the formula for differential cross section suitable for data fitting are obtained. The asymptotic behavior of the helicity amplitudes at $t=0$ is discussed. In Sec. V, we calculate the discontinuity of the d-p-n residue function across the anomalous cut and evaluate it by approximation. For simplicity the treatment is carried out for spinless nucleon and deuteron; we indicate how to handle the actual spin. Section VI contains a discussion of the connection of the Regge formalism to various low-energy models.

II. ANALYTICITY PROPERTIES OF AMPLITUDES

For definiteness we call the process $\pi^+ + d \to p + p$ the s reaction and $\bar{p}+d \to p+\pi^-$ the t reaction. Since we shall spend most of our time working with the t reaction, we denote the four-momenta of the particles[16] in this reaction by $\bar{p}$, d, p, and q, in corresponding order. The Mandelstam variables are defined as $s=(p-\bar{p})^2$, $t=(\bar{p}+d)^2$, and $u=(d-p)^2$, with $s+t+u=2m^2+M^2+\mu^2 \equiv \Sigma$, and where m, M, and μ are the masses of the proton, dueteron, and pion, respectively.

Since the t- and u-channel reactions are identical, so are their singularity structures. It will be sufficient for us merely to study the singularities in s and t. Throughout this paper it is understood that all the results obtained in the t channel apply to the u channel as well.

A. Full Amplitudes

The singularity structure of the full amplitude and the s-channel partial-wave amplitudes for the reaction $\pi^+ + d \to p+p$ have been studied in detail by Vasavada.[12] We shall not consider the normal threshold singularities, except for listing them at the end of this subsection. The formalism for the anomalous threshold singularities utilizes a rather lengthy examination of various graphs. One finds that there is only one anomalous threshold in s, while two are present in t (or u).

The general methods and criteria to determine the singularities associated with Feynman graphs have been developed by Karplus, Sommerfield, and Wichmann and by Landau.[17] To facilitate our discussion, we summarize their conditions and general formulas for the anomalous singularities associated with triangle graphs in Appendix A.

The anomalous singularity in s arises from the graph shown in Fig. 1. Using Eq. (A3), we get

$$s_a = M^2+\mu^2-M^2\mu^2/2m^2 + (M\mu/2m^2)[(4m^2-M^2)(4m^2-\mu^2)]^{1/2} \approx M^2-\mu(\mu-4\bar{\alpha}), \quad (2.1)$$

where $\bar{\alpha}^2 \equiv mB$ and B is the deuteron binding energy.

There are two anomalous singularities in t that are associated with the graphs shown in Figs. 2(a) and 2(b). Again using the formula (A3), we obtain

$$t_a = m^2+M^2\mu^2/2m^2 + (M\mu/2m^2)[(4m^2-M^2)(4m^2-\mu^2)]^{1/2} \approx m^2+2\mu(\mu+2\bar{\alpha}) \quad (2.2)$$

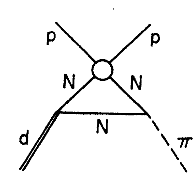

FIG. 1. Graph that gives rise to the anomalous threshold at $s=s_a$.

[10] J. Chahoud and G. Russo, Phys. Rev. Letters **11**, 506 (1963); H. G. Dosch, Phys. Letters **9**, 197 (1964); T. Yao, Phys. Rev. **134**, B454 (1964).

[11] M. L. Perl, L. W. Jones, and C. C. Ting, Phys. Rev. **132**, 1273 (1963); R. M. Heinz, O. E. Overseth, and M. H. Ross, Bull. Am. Phys. Soc. **10**, 19 (1965); J. Mathews and B. Deo, Phys. Rev. **143**, 1340 (1966).

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[13] J. Chahoud, G. Russo, and F. Selleri, Nuovo Cimento **45**, 38 (1966).

[14] D. Schiff and J. T. T. Van, Orsay Report, 1967 (unpublished).

[15] G. F. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968).

[16] We use the metric $p^2=p_0^2-\mathbf{p}^2$; $|\mathbf{p}|$ is also denoted by p.

[17] R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **111**, 1187 (1958); **114**, 375 (1959); L. D. Landau, Nucl. Phys. **13**, 181 (1959).

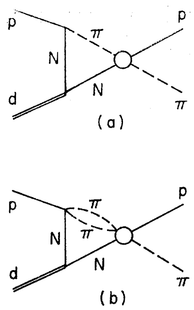

FIG. 2. Graph that gives rise to the anomalous threshold at (a) $t=t_a$ and (b) $t=t_a'$.

and

$$t_a'=m^2+2M^2\mu^2/m^2+(2M\mu/m^2)[(4m^2-M^2)(m^2-\mu^2)]^{1/2}\approx m^2+8\mu(\mu+\bar{\alpha}). \quad (2.3)$$

We summarize the singularities of the full amplitude as follows:

(a) There is an anomalous threshold at $s=s_a$, a normal threshold at $s=4m^2$, a physical threshold at $s=(M+\mu)^2$, and higher physical s thresholds.

(b) There is a pole at $t=m^2$, two anomalous thresholds at $t=t_a$ and $t=t_a'$, a normal threshold at $t=(m+\mu)^2$, a physical threshold at $t=(M+m)^2$, and higher physical t thresholds.

B. Partial-Wave Amplitudes

The right-hand cuts of the s- and t-channel partial-wave amplitudes are the same as described above in (a) and (b), respectively. We shall only discuss some of the important left-hand cuts. The general formulas to obtain them are rederived in a simple way and given in Appendix B.

For the s-channel partial-wave amplitudes we list the following singularities:

(a) The pole at $t=m^2$ gives rise to a finite cut from $s=s_+$ to $s=s_-$, where

$$s_\pm=M^2+\mu^2-M^2\mu^2/2m^2\pm(M\mu/2m^2)[(4m^2-M^2)(4m^2-\mu^2)]^{1/2}. \quad (2.4)$$

(b) The normal threshold cut starting at $t=(m+\mu)^2$ gives rise to a cut that can be conveniently described by the functions

$$S^\pm(t)=\tfrac{1}{2}(\Sigma-t)+(1/2t)(m^2-\mu^2)(M^2-m^2)\pm(1/2t)[\lambda(t,m^2,u^2)\lambda(t,m^2,M^2)]^{1/2}, \quad (2.5)$$

where

$$\lambda(x,y,z)\equiv x^2+y^2+z^2-2xy-2yz-2zx.$$

This cut starts at $s=S^+[(m+\mu)^2]$ and becomes complex and symmetrical about the real axis for $S^+[(M+m)^2]<s<S^+[(m+\mu)^2]$, finally becoming real again for $s\leq S^+[(M+m)^2]$.

(c) The anomalous cut from $t=t_a$ to $t=(m+\mu)^2$ generates a cut from $s=S^+[(m+\mu)^2]$ to $s=S^+(t_a)$.

For the t-channel partial-wave amplitudes we list the following singularities:

(a) a cut from $t=-\infty$ to $t=0$ that is always present if the initial and final channels all have unequal masses;

(b) a cut with branch point at

$$t=t_\pm=m^2+M^2\mu^2/2m^2\pm(M\mu/2m^2)[(4m^2-M^2)(4m^2-\mu^2)]^{1/2}, \quad (2.6)$$

which is generated by the pole at $u=m^2$;

(c) a cut that can be traced by the functions

$$T^\pm(u)=\tfrac{1}{2}(\Sigma-u)-(m^2-\mu^2)(M^2-m^2)/2u\pm(1/2u)[\lambda(u,m^2,\mu^2)\lambda(u,m^2,M^2)]^{1/2},$$

which is generated by the threshold cut at $u=(m+\mu)^2$. This cut is real at $t=T^+[(m+\mu)^2]$, becomes complex and symmetrical about the real axis for $T^+[(m+M)^2]<t<T^+[(m+\mu)^2]$, and becomes real again for $t\leq T^+[(m+M)^2]$;

(d) a cut with branch points from $t=T^+[(m+\mu)^2]$ to $t=T^+(u_a)$ generated by the anomalous cut from $u=u_a$ (numerically the same as t_a) to $u=(m+\mu)^2$; and

(e) a cut starting at $u=\frac{1}{2}(M^2+\mu^2-2m^2)$, which then becomes complex and symmetrical about the real axis until $u\leq(m^2-M\mu)$, and which is generated by the s-channel normal threshold cut at $s=4m^2$.

In passing, we note the very interesting fact that the rightmost left-hand branch point s_+, given by (2.4) for the s-channel partial-wave amplitudes, *coincides* with the lowest right-hand branch point s_a. Similarly, for the t-channel partial-wave amplitudes, t_+ coincides with t_a. This results in the rather unfamiliar situation of having the whole real axis covered by cuts for the partial-wave amplitudes. One may wonder whether this situation always occurs in amplitudes possessing anomalous singularities. It turns out that this is not the case (an example is $\pi\pi\to d\bar{d}$). It is quite easy to show, however, that this situation will occur whenever the external masses and the mass of the cross-channel pole satisfy certain constraints. The proof is given in Appendix B.

III. HELICITY AMPLITUDES AND KINEMATICAL SINGULARITIES

Since we shall Reggeize the t-channel amplitudes, we first construct the helicity amplitudes in this channel and derive their relationship to a set of invariant amplitudes. In the t-channel c.m. system, $\bar{\mathbf{p}}$ is chosen along the positive z axis and $\mathbf{p}$ is taken to lie in the xz plane. The scattering angle is defined by $z\equiv\cos\theta=\mathbf{p}\cdot\bar{\mathbf{p}}/p\bar{p}$. The following expression is very useful later:

$$\cos\theta=(1/4p\bar{p})[(s-u)-(M^2-m^2)(m^2-\mu^2)/t]. \quad (3.1)$$

A. Helicity Amplitudes and Invariant Amplitudes

From Lorentz invariance we can write the amplitudes in the form

$$F_{fi}=\bar{u}(p)M\cdot\epsilon v(\bar{p})\,, \qquad (3.2)$$

where $\bar{u}(p)$ and $v(\bar{p})$ are the Dirac spinors for the proton and antiproton, respectively. The 4×4 matrix $M\cdot\epsilon$ is a Lorentz scalar constructed out of particle momenta and Dirac matrices[18] and is linear in the polarization vector of the deuteron,[19] denoted by ϵ. With the consideration of parity conservation and a little manipulation of the Dirac matrices, it can be shown that there are six linearly independent invariants. We shall use a set of invariants for $M\cdot\epsilon$ as follows:

$$\begin{aligned}M\cdot\epsilon&=\sum_{i=1}^{6}A_iI_i\\&=A_1(\epsilon\cdot\bar{p})+A_2(\epsilon\cdot p)+A_3(\gamma\cdot\epsilon)+A_4(\gamma\cdot d)(\gamma\cdot\epsilon)\\&\quad+A_5(\gamma\cdot d)(\epsilon\cdot\bar{p})+A_6(\gamma\cdot d)(\epsilon\cdot p)\,, \qquad (3.3)\end{aligned}$$

where the A_i are functions of the invariant variables s, t, and u. The function M_μ is essentially the same as the M function introduced by Stapp.[20] The basic analyticity assumption is that the M functions are analytic functions of the components of the momentum vectors, except at dynamical singularities. It follows from the work of Hepp and of Williams[21] that the invariant functions A_i are free of kinematical singularities, except for the possibility of simple poles at $\phi(s,t)=0$, where ϕ is the Kibble boundary function.[22]

Now if we put the helicity spinors $u_{\lambda_p}(p)$, $v_{\lambda_{\bar{p}}}(\bar{p})$, and ϵ^{λ_d} into (3.2) and (3.3), the $F_{\lambda_p,\lambda_{\bar{p}}\lambda_d}$ obtained are just the helicity amplitudes defined by Jacob and Wick, apart from normalization.[23] After some algebra we have (we use $\pm$ to stand for $\lambda_p=\pm\frac{1}{2}$ or $\lambda_{\bar{p}}=\pm\frac{1}{2}$)

$$\begin{aligned}F_{\pm,+1}=&-\tfrac{1}{2}\sin\theta(1\pm\cos\theta)^{1/2}p\\&\times[(\pm G_1-G_2)A_2+(\pm G_5+G_6)A_6]\\&+(1\mp\cos\theta)^{1/2}[(\mp G_3+G_4)A_3\\&+(\pm G_7-G_8)A_4]\,, \qquad (3.4a)\end{aligned}$$

$$\begin{aligned}F_{\pm,+0}=&(1/\sqrt{2}M)(1\pm\cos\theta)^{1/2}\\&\times\{(G_1\mp G_2)[\bar{p}wA_1+(\bar{p}p_0+d_0p\cos\theta)A_2]\\&+\sqrt{2}(G_7\pm G_8)A_3-\sqrt{2}(G_3\mp G_4)A_4+(\pm G_5+G_6)\\&\times[\bar{p}wA_5+(\bar{p}p_0+d_0p\cos\theta)A_6]\}\,, \qquad (3.4b)\end{aligned}$$

$$\begin{aligned}F_{\pm,+-1}=&\tfrac{1}{2}\sqrt{2}\sin\theta p\\&\times[(\pm G_1-G_2)A_2+(\pm G_5+G_6)A_6]\,, \qquad (3.4c)\end{aligned}$$

where $w=\sqrt{t}$ and the G's are given by

$$\begin{aligned}G_{1,2}&=(1/2m)[(\bar{p}_0\mp m)(p_0\pm m)]^{1/2}\,,\\G_{3,4}&=(1/2m)[(\bar{p}_0\mp m)(p_0\mp m)]^{1/2}\,,\qquad (3.5)\\G_{4+i}&=G_i[w-(-1)^im]\,,\quad i=1,2,3,4.\end{aligned}$$

The other six helicity amplitudes are related to the above set by parity conservation, i.e.,

$$F_{-\lambda_p,-\lambda_{\bar{p}}-\lambda_d}(s,t)=\eta(-1)^{\lambda_p-\lambda_{\bar{p}}+\lambda_d}F_{\lambda_p,\lambda_{\bar{p}}\lambda_d}(s,t)\,, \qquad (3.6)$$

where $\eta\equiv\eta_\pi\eta_p\eta_{\bar{p}}\eta_d(-1)^{s_\pi+s_p-s_{\bar{p}}-s_d}=-1$ and $\eta_i(s_i)$ is the intrinsic parity (spin) of the particle i.

The partial-wave expansions[20] for the helicity amplitudes are

$$F_{\lambda_p,\lambda_{\bar{p}}\lambda_d}(s,t)=\sum_J(2J+1)F_{\lambda_p,\lambda_{\bar{p}}\lambda_d}{}^J(w)d_{\lambda\mu}{}^J(\theta)\,, \qquad (3.7)$$

where $\lambda\equiv\lambda_p-\lambda_\pi$, $\mu\equiv\lambda_{\bar{p}}-\lambda_d$, and $d_{\lambda\mu}{}^J$ is the d function for the rotation group. The reason that we use w rather than t as the variable for F^J is due to the existence of the MacDowell symmetry[24]; w is the proper variable for studying F^J.

B. Kinematical Singularities

We follow the general method for Reggeization of amplitude with spin developed by Gell-Mann *et al.*[25] to form the parity-conserving amplitudes. These amplitudes have the advantage that they are asymptotically dominated by a Regge pole of definite parity. They are defined by the rule

$$\begin{aligned}f_{\lambda_p,\lambda_{\bar{p}}\lambda_d}{}^{\pm}\equiv&(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}F_{\lambda_p,\lambda_{\bar{p}}\lambda_d}\\&\pm(-1)^{\lambda+\lambda_m+1}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda+\mu|}\\&\times(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda-\mu|}F_{-\lambda_p,\lambda_{\bar{p}}\lambda_d}\,, \qquad (3.8)\end{aligned}$$

where $\lambda_m\equiv\max(|\lambda|,|\mu|)$. Thus we have

$$\begin{aligned}f_1{}^+=&-A_2p(G_1\cos\theta-G_2)\\&-2A_3G_4-2A_4G_8-A_6p(G_5\cos\theta+G_6)\,, \qquad (3.9a)\end{aligned}$$

$$\begin{aligned}f_1{}^-=&-A_2p(G_1-G_2\cos\theta)\\&-2A_3G_3-2A_4G_7-A_6p(G_5+G_6\cos\theta)\,, \qquad (3.9b)\end{aligned}$$

$$\begin{aligned}f_0{}^+=&(\sqrt{2}/M)pw(A_1G_1+A_5G_5)+A_3G_8+M^2A_4G_4\\&+(\bar{p}p_0+d_0p\cos\theta)(A_2G_1+A_6G_5)\,, \qquad (3.9c)\end{aligned}$$

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18 The Dirac matrices used here are the same as those used by S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Co., New York, 1961).

19 In the rest frame of the deuteron, the spin states can be characterized by $\bar{\varepsilon}^{\lambda_d}$, where $\bar{\varepsilon}^{\lambda_d}=(\bar{\boldsymbol{\varepsilon}}^{\lambda_d},0)$ with $\bar{\boldsymbol{\varepsilon}}^{\pm1}=\pm(\hat{x}\pm i\hat{y})/\sqrt{2}$ and $\bar{\boldsymbol{\varepsilon}}^0=\hat{z}$. The helicity states in the t-channel c.m. system are given by $\epsilon^{\lambda_d}=R[e^{i\pi J_y}]B(-\mathbf{d})\bar{\varepsilon}^{\lambda_d}$, where B is a boost taking the rest deuteron to a state with momentum $-\mathbf{d}$ and R is a rotation.

20 H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

21 Klaus Hepp, Helv. Phys. Acta **37**, 55 (1964); D. N. Williams, University of California Lawrence Radiation Laboratory Report No. UCRL-11113, 1963 (unpublished).

22 T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

23 M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959). The amplitude F that we used is related to the S matrix by

$$\begin{aligned}S_{cd,ab}(s,t)-\delta_{cd,ab}\\=(2\pi)^4i\delta(p_c+p_d-p_a-p_b)(p_{a0}p_{b0}p_{c0}p_{d0})^{-1/2}F_{cd,ab}(s,t).\end{aligned}$$

24 S. W. MacDowell, Phys. Rev. **116**, 774 (1958); J. D. Stack, Ph.D. thesis, University of California, Berkeley, 1965 (unpublished).

25 M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964). Note that there is a misprint in the formula for $e_{\frac{1}{2}\frac{1}{2}}{}^{J-}$. The correct expression is given in (4.3).

$$f_0^- = (\sqrt{2}/M)pw(-A_1G_2+A_5G_6)+A_3G_7-M^2A_4G_3 + (\bar{p}p_0+d_0p\cos\theta)(-A_2G_2+A_6G_6)\,, \quad (3.9d)$$

$$f_{-1}^+ = A_2pG_1+A_6pG_5\,, \quad (3.9e)$$

$$f_{-1}^- = -A_2pG_2+A_6pG_6 \quad (3.9f)$$

where we define $f_{\lambda_d}^\pm \equiv f_{+,+\lambda_d}^\pm$. As functions of w, the G's have the following properties:

$$G_i(w) = -G_{i+1}(-w)\,, \quad i=1,3$$
$$G_i(w) = G_{i+1}(-w)\,, \quad i=5,7.$$

Hence we see explicitly that the amplitudes $f_{\lambda_d}^\pm$ have the MacDowell symmetry, as expected:

$$f_{\lambda_d}^\pm(s,w) = (-1)^{\lambda_d} f_{\lambda_d}^\mp(s,-w) \quad (3.10)$$

The implication of this symmetry relation on Regge trajectories and residue functions will be discussed later.

The amplitudes $f_{\lambda_d}^\pm$ do not have any kinematical singularities in s, by construction. Their kinematical singularities in w are almost explicitly displayed in (3.9). Before going any further, we remark that because of the symmetry (3.10), any kinematical factor $K(w)$ contained in $f^+(s,w)$ implies a factor $K(-w)$ in $f^-(s,w)$, and vice versa. This observation serves as a useful check on the general results giving kinematical singularities for boson-fermion amplitudes obtained by indirect methods. Indeed, we have found that the kinematical-singularity factors for a general $BF\to BF$ amplitude obtained by Hara and by Wang[26] are incorrect, since their prescriptions do not satisfy this criterion. The kinematical singularities of $f_{\lambda_d}^\pm$ are extracted as follows:

$$f_{\lambda_d}^\pm(s,w) \equiv K_{\lambda_d}^\pm(w)\bar{f}_{\lambda_d}^\pm(s,w)\,, \quad (3.11)$$

where

$$K_1^\pm(w) = K_0^\pm(w) = (1/w)[(w\pm m)^2-M^2]^{-1/2} \times[(w\pm m)^2-\mu^2]^{1/2}, \quad (3.12)$$

$$K_{-1}^\pm(w) = (p/w)[(w\mp m)^2-M^2]^{1/2} \times[(w\pm m)^2-\mu^2]^{1/2}, \quad (3.13)$$

the functions $\bar{f}_{\lambda_d}^\pm(s,w)$ being free of kinematical singularities in both s and w. Our results (3.12) and (3.13) agree with those obtained using the general formulas worked out by Cohen-Tannoudji, Morel, and Navelet and by Jackson and Hite.[27]

IV. REGGE REPRESENTATION

A. Analytic Continuation in Angular Momentum

To proceed toward the Regge representation of the amplitudes $f^\pm$, we start with their partial-wave expansions,[25] which are readily obtained from (3.7) and (3.8),

$$f_{\lambda_d}^\pm = \sum_J (2J+1)[e_{\frac12,\frac12-\lambda_d}^{J+}(z)F_{\lambda_d}^{J\pm}(w) + e_{\frac12,\frac12-\lambda_d}^{J-}(z)F_{\lambda_d}^{J\mp}(w)]. \quad (4.1)$$

The parity-conserving partial-wave amplitudes $F_{\lambda_d}^{J\pm}(w)$, which connect states of parity $\pm(-1)^{J-1/2}$, are given by

$$F_{\lambda_d}^{J\pm} = F_{+,+\lambda_d}^{J} \mp F_{-,+\lambda_d}^{J}\,. \quad (4.2)$$

The functions $e_{\lambda\mu}^{J\pm}(z)$ are defined in Ref. 25; for our purpose we only need to know the following relations to Legendre polynomials:

$$e_{\frac12-\frac12}^{J\pm} = \mp e_{\frac12\frac12}^{J\pm} = -P_{J\pm\frac12}'(z)/\sqrt{2}(J+\tfrac12)\,,$$
$$e_{\frac12\frac32}^{J\pm} = \pm P_{J\pm\frac12}''(z)/\sqrt{2}(J+\tfrac12)[(J-\tfrac12)(J+\tfrac32)]^{1/2}. \quad (4.3)$$

From (4.1) it is evident that $F_{\lambda_d}^{J\pm}(w)$ possesses the same MacDowell symmetry as does $f_{\lambda_d}^\pm(s,w)$,

$$F_{\lambda_d}^{J\pm}(w) = (-1)^{\lambda_d}F_{\lambda_d}^{J\mp}(-w)\,. \quad (4.4)$$

The inversion formulas of (4.1) are[25]

$$F_i^{J\pm}(w) = \frac12\int_{-1}^1 \frac{dz}{\sqrt2}[(-1)^i P_{J\mp\frac12}(z)f_i^\mp(s,w) + P_{J\pm\frac12}(z)f_i^\mp(s,w)], \quad i=1,0 \quad (4.5)$$

$$F_{-1}^{J\pm}(w) = \frac{[l(l+2)]^{1/2}}{2\sqrt2(2l-1)}\int_{-1}^1 dz \times[P_{l-1}(z)-P_{l+1}(z)]f_{-1}^\pm(s,w) + \frac{[l(l+2)]^{1/2}}{2\sqrt2(2l+3)}\int_{-1}^1 dz \times[P_l(z)-P_{l+2}(z)]f_{-1}^\mp(s,w)\,, \quad (4.6)$$

where for convenience we define $l\equiv J-\frac12$. Equations (4.5) and (4.6) can be readily continued analytically into the complex J plane by the Froissart-Gribov procedure if we assume that the functions $\bar{f}_{\lambda_d}^\pm$ satisfy the Mandelstam representation with a finite number of subtractions. As usual we must continue the even and odd $(J-\frac12)$ partial waves separately in order to allow the Sommerfeld-Watson transformation to be made. Let $A_{\lambda_d,s}^\pm$ and $A_{\lambda_d,u}^\pm$ be the spectral functions for $\bar{f}_{\lambda_d}^\pm$, so that

$$\bar{f}_{\lambda_d}^\pm = \frac1\pi\int_{s_a}^\infty \frac{ds'\,A_{\lambda_d,s}^\pm(s',w)}{s'-s} + \frac1\pi\int_{u_a}^\infty \frac{du'\,A_{\lambda_d,u}^\pm(u',w)}{u'-u}\,. \quad (4.7)$$

Putting (4.7) into (4.5) and (4.6), we obtain the J-continued partial-wave amplitudes of definite

[26] Y. Hara, Phys. Rev. **136**, B507 (1964); L. C. Wang, *ibid.* **142**, 1187 (1966).

[27] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Saclay Report (unpublished); J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968).

signature σ,

$$F_i^{J\pm,\sigma}(w)=K_i^{\pm}(w)(-1)^{\lambda_d}f_i^{l\pm,\sigma}(w)+K_i^{\mp}(w)f_i^{(l+1)\mp,-\sigma}(w)\,,\quad i=1,0 \quad (4.8)$$

$$F_{-1}^{J\pm,\sigma}(w)=K_{-1}^{\pm}(w)\frac{[l(l+2)]^{1/2}}{2l+1}f_{-1}^{(l-1)\pm,-\sigma}(w)+K_{-1}^{\mp}(w)\frac{[l(l+2)]^{1/2}}{2l+3}f_{-1}^{l\mp,\sigma}(w)\,,\quad (4.9)$$

where $f_{\lambda_d}^{l\pm,\sigma}$ is defined by

$$f_{\lambda_d}^{l\pm,\sigma}=\frac{1}{2\pi p\bar{p}}\int_{s_a}^{\infty}ds'\,A_{\lambda_d,s}^{\pm}(s',w)\tfrac{1}{2}\sqrt{2}\times[Q_l(z)-\delta_{-1,\lambda_d}Q_{l+2}(z)]+\sigma\frac{1}{2\pi p\bar{p}}\int_{u_a}^{\infty}du'\,A_{\lambda_d,u}^{\pm}(u',w)\tfrac{1}{2}\sqrt{2}\times[Q_l(z)-\delta_{-1,\lambda_d}Q_{l+2}(z)]\,. \quad (4.10)$$

The amplitude $F_{\lambda_d}^{J\pm,\sigma}$ agrees with the physical amplitudes at values of J given by $J=2n+\sigma^{1/2}$, with n an integer.

B. Asymptotic Form of Amplitudes

Following the method of Mandelstam,[28] we rewrite the expansion (4.1) as

$$\begin{aligned}f_{\lambda_d}^{\pm}=\sum_{J=-\infty}^{\infty}(J+\tfrac{1}{2})\\ \times\{[E_{\lambda\mu}^{J+}(z)+(-1)^{|\mu|-1/2}E_{\lambda\mu}^{J+}(-z)]F_{\lambda_d}^{J\pm,+}(w)\\ +[E_{\lambda\mu}^{J+}(z)-(-1)^{|\mu|-1/2}E_{\lambda\mu}^{J+}(-z)]F_{\lambda_d}^{J\pm,-}(w)\\ +[E_{\lambda\mu}^{J-}(z)-(-1)^{|\mu|-1/2}E_{\lambda\mu}^{J-}(-z)]F_{\lambda_d}^{J\mp,+}(w)\\ +[E_{\lambda\mu}^{J-}(z)+(-1)^{|\mu|-1/2}E_{\lambda\mu}^{J-}(-z)]F_{\lambda_d}^{J\mp,-}(w)\}\,,\end{aligned} \quad (4.11)$$

where $\lambda=\frac{1}{2}$, $\mu=\frac{1}{2}-\lambda_d$, and $E_{\lambda\mu}^{J\pm}(z)$ is the function obtained from $e_{\lambda\mu}^{J\pm}(z)$ by replacing P_l with $\mathcal{P}_l\equiv-(1/\pi)\times\tan l\pi\,Q_{-l-1}$. The summation in J runs over all half-integers. In obtaining (4.11), we have used the fact[25] that $E_{\frac{1}{2},\frac{1}{2}-\lambda_d}^{J\pm}=0$ for negative half-integral J and $E_{\frac{1}{2},\frac{1}{2}-\lambda_d}^{J\pm}=e_{\frac{1}{2},\frac{1}{2}-\lambda_d}^{J\pm}$ for positive half-integral J. Now we convert (4.11) into a contour integral by replacing

$$\sum_{J=-\infty}^{\infty}\quad\text{with}\quad -\tfrac{1}{2}i\int_C\frac{dJ}{\cos\pi J}\,,$$

with the contour C enclosing the whole real axis. The Regge poles are contained in $F_{\lambda_d}^{J\pm,\sigma}$, with quantum numbers $I=\frac{1}{2}$, $B=1$, parity $\pm$, and signature σ. The leading trajectories with σ positive are N_α and N_β, and with σ negative they are N_γ and N_δ. In the Chew-Frautschi plot N_γ is lower than N_α by about $\frac{1}{2}\hbar$; hence in practice we neglect the contributions from N_γ and N_δ at high energies. In fact, good fits[5,29] have been obtained for the backward πN scattering by retaining only the nucleon trajectory for $I=\frac{1}{2}$ exchange. Furthermore, for large z, E^{J+} dominates over E^{J-} asymptotically by a factor z, so that the latter is also negligible. Thus, as we expand the contour C to infinity, throwing away everything but the moving poles at the far left, only the first term of the curly bracket in (4.11) remains:

$$f_{\lambda_d}^{\pm}(s,w)=\pi\frac{2\alpha_\pm+1}{\cos\pi\alpha_\pm}\eta(\alpha_\pm)\beta_{\lambda_d}^{\pm}E_{\frac{1}{2},\frac{1}{2}-\lambda_d}^{\alpha\pm,+}(z)\,. \quad (4.12)$$

Here the $\alpha_\pm(w)$ denote the trajectory functions of N_α and N_β, respectively, and the $\beta_{\lambda_d}^{\pm}(w)$ are the residue functions defined by

$$\beta_{\lambda_d}^{\pm}(w)\equiv\lim_{J\to\alpha\pm(w)}[J-\alpha_\pm(w)]F_{\lambda_d}^{J\pm,+}(w)\,, \quad (4.13)$$

with

$$\eta(\alpha_\pm)\equiv\tfrac{1}{2}(1+e^{-i\pi(\alpha_\pm-1/2)})\,.$$

The MacDowell symmetry (3.10) implies that

$$\alpha_\pm(w)=\alpha_\mp(-w) \quad (4.14)$$

and

$$\beta_{\lambda_d}^{\pm}(w)=(-1)^{\lambda_d}\beta_{\lambda_d}^{\mp}(-w)\,. \quad (4.15)$$

From (4.8)–(4.10) we see that the residue functions have the following factors of α and kinematical singularities:

$$\beta_i^{\pm}(w)\propto K_i^{\pm}(w)(p\bar{p})^{\alpha\pm-1/2}\,,\quad i=1,0$$

$$\beta_{-1}^{\pm}(w)\propto K_{-1}^{\pm}(w)\times[(\alpha_\pm-\tfrac{1}{2})(\alpha_\pm+\tfrac{3}{2})]^{1/2}(p\bar{p})^{\alpha\pm-3/2}\,. \quad (4.16)$$

Furthermore, since we consider $\alpha_\pm$ to be the leading trajectory, $\beta_{\lambda_d}^{\pm}(w)$ must vanish[3,28] at the point when $\alpha_\pm(w)$ passes through a negative integer. Hence we can define reduced residue functions that are suitable for parametrization in data fitting as

$$\gamma_i^{\pm}(w)=(-1)^i(2/\pi)^{1/2}\beta_i^{\pm}(w)\Gamma(\alpha_\pm+1)/K_i^{\pm}(w)(p\bar{p})^{\alpha\pm-1/2}\,,\quad i=1,0$$

$$\gamma_{-1}^{\pm}(w)=(2/\pi)^{1/2}\beta_{-1}^{\pm}(w)\Gamma(\alpha_\pm+1)\times[(\alpha_\pm-\tfrac{1}{2})(\alpha_\pm+\tfrac{3}{2})]^{1/2}/K_{-1}^{\pm}(w)(p\bar{p})^{\alpha\pm-3/2}\,. \quad (4.17)$$

Putting the asymptotic form[30] of the E functions and (4.17) into (4.12), we obtain the Regge representation

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[28] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).

[29] V. Barger and D. Cline, Phys. Rev. Letters **19**, 1504 (1967).

[30] The asymptotic form of the E functions are

$$E_{\frac{1}{2},\pm\frac{1}{2}}^{\alpha,+}(z)\sim\pm\left(\frac{2}{\pi}\right)^{1/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}(2z)^{\alpha-1/2},$$

$$E_{\frac{1}{2},\frac{3}{2}}^{\alpha,+}(z)\sim2\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\alpha-\frac{1}{2}}{\alpha+\frac{3}{2}}\right)^{1/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}(2z)^{\alpha-3/2}.$$

for the amplitudes

$$f_i^{\pm}(s,w)=\frac{\eta(\alpha_\pm)}{\Gamma(\alpha_\pm+\frac{1}{2})\cos\pi\alpha_\pm}\times K_i^{\pm}(w)\gamma_i^{\pm}(w)s^{\alpha_\pm-1/2},\quad i=1,0 \qquad (4.18)$$

$$f_{-1}^{\pm}(s,w)=\frac{\eta(\alpha_\pm)(2\alpha_\pm-1)}{\Gamma(\alpha_\pm+\frac{1}{2})\cos\pi\alpha_\pm}\times K_{-1}^{\pm}(w)\gamma_{-1}^{\pm}(w)s^{\alpha_\pm-3/2}. \qquad (4.19)$$

The factor $2\alpha_\pm-1$ in (4.19) is due to $J=\frac{1}{2}$ being a sense-nonsense transition point for these amplitudes; hence the nucleon pole is not present. The factor $\eta(\alpha)[\Gamma(\alpha+\frac{1}{2})\cos\pi\alpha]^{-1}$ vanishes at nonsense wrong-signature points ($\alpha=-\frac{1}{2},-\frac{5}{2},-\frac{9}{2},\cdots$) in the same way that gives the dip phenomenon in backward πN scattering. It has been argued by Mandelstam and Wang[31] that, in general, because of the existence of a fixed pole at a nonsense wrong-signature point, one does not expect to observe dips unless the third double spectral function is small. However, from the experimental fact that the dip in backward πN scattering is very prominent, factorization leads us to believe that an observable dip should be present in our reaction.

Using the Trueman-Wick crossing matrix,[32] we obtain the formula for the differential cross section of the $\pi^++d\to p+p$ reaction:

$$\frac{d\sigma}{dt}=\frac{\pi}{2p_ip_f}\left(\frac{d\sigma}{d\Omega}\right)=\frac{1}{3}\frac{1}{8\pi sp_i^2}\sum_{\text{all }\lambda}|f_{\lambda_p,\lambda_{\bar p}\lambda_d}|^2$$
$$=(1/6\pi sp_i^2)(|\sin\tfrac{1}{2}\theta|^2+|\cos\tfrac{1}{2}\theta|^2)[|f_{+,++}{}^+|^2+|f_{+,++}{}^-|^2+|f_{+,+0}{}^+|^2+|f_{+,+0}{}^-|^2+|\sin\theta|^2(|f_{+,+-}{}^+|^2+|f_{+,+-}{}^-|^2)]$$
$$+(1/3\pi sp_i^2)(|\cos\tfrac{1}{2}\theta|^2-|\sin\tfrac{1}{2}\theta|^2)\,\mathrm{Re}(f_{+,++}{}^+f_{+,++}{}^{-*}+f_{+,+0}{}^+f_{+,+0}{}^{-*}-|\sin\theta|^2f_{+,+-}{}^+f_{+,+-}{}^{-*}), \qquad (4.20)$$

where p_i (p_f) is the initial (final) momentum in the s-channel c.m. system.

C. Asymptotic Behavior at $t=0$

The asymptotic behavior of the helicity amplitudes for large s at $t=0$ is worth a few words. Let us first invert the relation (3.8); we get

$$F_{\pm,++}=(1\mp\cos\theta)^{1/2}(f_1^+\pm f_1^-), \qquad (4.21a)$$

$$F_{\pm,+0}=(1\pm\cos\theta)^{1/2}(f_0^-\pm f_0^+), \qquad (4.21b)$$

$$F_{\pm,+-}=\sin\theta(1\pm\cos\theta)^{1/2}(f_{-1}^-\pm f_{-1}^+). \qquad (4.21c)$$

As is well known, in the narrow strip between $\cos\theta_s=1$ (θ_s is the s-channel scattering angle) and $t=0$, the quantity $|\cos\theta|$ is bounded by unity no matter how large s is. From the work of Freedman and Wang,[33] we know that the power behavior of the amplitudes $f^\pm$ in (4.18) and (4.19) is preserved even at $t=0$. One is tempted to conclude that the asymptotic behavior of the helicity amplitudes at $t=0$ will at most go like

$$F_{\pm,+\lambda_d}\sim s^{\alpha-|1/2-\lambda_d|}. \qquad (4.22)$$

A careful examination shows that this is not correct. From (3.1) it is seen that as $t\to0$ for large s, one has

$$\cos\tfrac{1}{2}\theta\approx[st/(M^2-m^2)(m^2-\mu^2)]^{1/2},\quad \sin\tfrac{1}{2}\theta\approx1. \qquad (4.23)$$

On the other hand, from (3.11) we know that the functions $wf_1^{\pm}(s,w)$, $wf_0^{\pm}(s,w)$, and $w^2f_{-1}^{\pm}(s,w)$ can be expanded in Taylor series at $w=0$ for a fixed s. Taking $f_1^\pm$, for example, using (3.10) and (3.12), we can write

$$f_1^{\pm}(s,w)=\frac{1}{w}\sum_{n=0}^{\infty}f_1^{(n)}(s)w^n(\pm1)^{n+1}. \qquad (4.24)$$

Using the Regge representation to calculate the right-hand side of (4.24) and putting the result and (4.23) into (4.21a), we obtain the following in the limit $w\to0$:

$$F_{+,++}(s,0)=2f_1^{(1)}(s)\sim(C_1+C_2\alpha'\ln s)s^{\alpha-1/2}, \qquad (4.25a)$$

$$F_{-,++}(s,0)\sim-2[s/(M^2-m^2)(m^2-\mu^2)]^{1/2}\times f_1^{(0)}(s)\sim s^\alpha. \qquad (4.25b)$$

Similarly, we obtain

$$F_{+,+0}(s,0)\sim s^\alpha, \qquad (4.25c)$$

$$F_{-,+0}(s,0)\sim(C_3+C_4\alpha'\ln s)s^{\alpha-1/2}, \qquad (4.25d)$$

$$F_{+,+-}(s,0)\sim s^{\alpha-1/2}, \qquad (4.25e)$$

$$F_{-,+-}(s,0)\sim(C_5+C_6\alpha'\ln s)s^{\alpha-1} \qquad (4.25f)$$

where the C's are constants, $\alpha\equiv\alpha(0)$, and $\alpha'\equiv d\alpha(0)/dw$.

Note that the power in s for each amplitude is exactly equal to $\alpha-\frac{1}{2}|\lambda-\mu|$. It is easy to show that the s-channel helicity amplitudes, denoted by $G_{\lambda_1\lambda_2,\lambda_\pi\lambda_d}$, have the asymptotic behavior like $s^{\alpha-\frac{1}{2}|\lambda'-\mu'|}$, with $\lambda'\equiv\lambda_1-\lambda_2$ and $\mu'=\lambda_\pi-\lambda_d$.

V. NUCLEON-DEUTERON REGGE RESIDUE

From factorization we can decompose the residue function $\beta_{\lambda\mu}$ as

$$\beta_{\lambda\mu}=\beta_\lambda{}^\pi\beta_\mu{}^d \qquad (5.1)$$

[31] S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).
[32] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).
[33] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

where β_λ^π and β_μ^d are the residue functions for the π-N-N and d-p-n vertices, respectively. Since the binding energy of the deuteron is extremely small, the impulse approximation will be very accurate. In this approximation, the dominant reaction mechanism is that in which the incident pion interacts only with one of the nucleons within the deuteron. In S-matrix calculations, this is equivalent to assuming that the anomalous cuts are the dominating singularities in the nearby region. In the following, we calculate the d-p-n residue function in terms of the π-N-N residue function by assuming that the anomalous singularity associated with Fig. 2(a) is the dominating one. The anomalous singularity associated with the graph shown in Fig. 2(b) is presumably very weak because the π-π scattering amplitude is small near threshold.

We shall ignore the spins of the nucleon and deuteron, which merely complicate the algebra. The mass difference between the proton and neutron is also neglected. At the end we shall outline how to treat the actual spin case.

Referring to Fig. 3, we apply the Cutkosky rules[34] to write the discontinuity of the amplitude across the anomalous cut, denoted by $[F(s,t)]_{\text{a.c.}}$:

$$[F(s,t)]_{\text{a.c.}}=\frac{gG}{2\pi i}\int d^4q_2 \times\delta(q_1{}^2-\mu^2)\delta(q_2{}^2-m^2)\delta(q_3{}^2-m^2)F_{\pi N}(\bar{s},t)\,, \quad (5.2)$$

where $F_{\pi N}$ is the pion-nucleon scattering amplitude and $\bar{s}=(q_3-q)^2$. Here g and G are the π-N-N and d-p-n coupling constants, respectively. We change variables by using

$$d^4q_2=dq_1{}^2dq_2{}^2dq_3{}^2d\bar{s}\;J(s,\bar{s},t)\theta(J)\,, \quad (5.3)$$

where $J^{-2}=\det|2q_i\cdot q_j|$, with $i,j=1,2,3,4$ and $q_4\equiv q_3-q$. We obtain

$$[F(s,t)]_{\text{a.c.}}=\frac{gG(4\pi i)^{-1}}{[-\lambda(t,M^2,m^2)]^{1/2}}\int_{\bar{s}_-}^{\bar{s}_+}d\bar{s} \times\frac{F_{\pi N}(\bar{s},t)}{[(\bar{s}_+-\bar{s})(\bar{s}-\bar{s}_-)]^{1/2}}\,, \quad (5.4)$$

where $\bar{s}_\pm$ are the two roots for $\bar{s}$ of the equation $\det|q_i\cdot q_j|=0$. We are only interested in the asymptotic expression for $\bar{s}_\pm$ at large s such that $s\gg M^2$ and $s\gg t$, which takes the form

$$\bar{s}_\pm\approx s(1-b/a) \pm(s/a)[b^2+4atm^2-a(M^2+\mu^2-2m^2)^2]^{1/2}\,, \quad (5.5)$$

where

$$a(t)=\lambda(t,M^2,m^2)$$

and

$$b(t)=(M^2-m^2+t)(\mu^2-2m^2)+M^2(M^2-m^2-t)\,.$$

[34] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

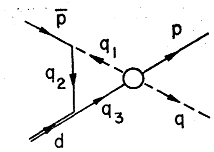


FIG. 3. Diagram for calculating the discontinuity.

For large s and fixed t, $\bar{s}_\pm$ increases linearly with s. Hence we can use the Regge representation for both $F(s,t)$ and $F_{\pi N}(\bar{s},t)$ in (5.4).

Following the method employed by Cutkosky[7] in calculating the deuteron form factor, we can write (5.4) approximately as

$$F(s,t)\approx-\frac{1}{8\pi}gGf_d(t)\int_{\bar{s}_-}^{\bar{s}_+}d\bar{s}\frac{F_{\pi N}(\bar{s},t)}{[(\bar{s}_+-\bar{s})(\bar{s}-\bar{s}_-)]^{1/2}}\,, \quad (5.6)$$

where

$$f_d(t)\equiv\frac{1}{\pi}\int_{t_a}^{t_n\equiv(m+\mu)}dt'\,[-\lambda(t',M^2,m^2)]^{-1/2}\frac{1}{t'-t} =f(t,t_n)-f(t,t_a)\,, \quad (5.7)$$

with

$$f(t,t')=[a(t)]^{-1/2}\times\sin^{-1}\left[\frac{a(t)+(t-M^2-m^2)(t'-t)}{2Mm(t-t')}\right] \quad \text{if} \quad a(t)\neq0$$
$$=a(t')/(t-M^2-m^2)(t-t') \qquad \text{if} \quad a(t)=0.$$

In (5.6) we have ignored those terms arising from the pinching of the $\bar{s}$ integration between $\bar{s}_\pm$ and the singularities of $F_{\pi N}(\bar{s},t)$ in $\bar{s}$. With the amplitude $F_{\pi N}$ approximated by its Regge asymptotic form (legitimate for large s), those terms do not arise.

We write the Regge representations for the amplitudes F and $F_{\pi N}$ as follows:

$$F(s,t)=\frac{1-e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}\beta^\pi(t)\beta^d(t)s^{\alpha(t)}\,,$$
$$F_{\pi N}(s,t)=\frac{1-e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}[\beta^\pi(t)]^2s^{\alpha(t)}\,. \quad (5.8)$$

Putting (5.8) into (5.6), we obtain (with $x\equiv\bar{s}/s$ and $x_\pm=\bar{s}_\pm/s$)

$$\beta^d(t)=-\frac{1}{8\pi}gGf_d(t)\beta^\pi(t)\int_{x_-}^{x_+}dx \times\frac{x^{\alpha(t)}}{[(x_+-x)(x-x_-)]^{1/2}}\,. \quad (5.9)$$

The integral in (5.9) can be calculated numerically if $\alpha(t)$ is given. Physically the most interesting region is for small and negative t. In this region we see from (5.5) that $|\bar{s}_+-\bar{s}_-|\ll|\bar{s}_++\bar{s}_-|$; hence we can approximate the integral in (5.9) by replacing x^α with $(1-b/a)^\alpha$. Finally, we obtain

$$\beta^d(t)=-\tfrac{1}{8}gGf_d(t)(1-b/a)^{\alpha(t)}\beta^\pi(t). \quad (5.10)$$

We outline briefly the procedures to treat the spin: With spin included, the amplitudes corresponding to the graph shown in Fig. 3 can be written

$$F(s,t)=\frac{g}{(2\pi)^4}\bar{u}(p)\left[\int d^4q_2\, F_{\pi N}{}^{(1/2)}(\bar{s},t)\frac{\boldsymbol{q}_3+m}{q_1{}^2-\mu^2+i\epsilon}\times\frac{(G_S\gamma\cdot\epsilon-G_Dq_2\cdot\epsilon)(-\boldsymbol{q}_2+m)}{(q_2{}^2-m^2+i\epsilon)(q_3{}^2-m^2+i\epsilon)}\right]\gamma_5 v(\bar{p}), \quad (5.11)$$

where G_S and G_D are the S- and D-wave d-p-n coupling constants,[35] and $F_{\pi N}{}^{(1/2)}(\bar{s},t)$ is the pion-nucleon scattering amplitude with isospin $I=\frac{1}{2}$ in the t channel. Applying the Cutkosky rules to (5.11), we have

$$[F(s,t)]_{\text{a.c.}}=\frac{g}{2\pi i}\bar{u}(p)\left[\int d^4q_2\, F_{\pi N}{}^{(1/2)}(\bar{s},t)\times(\boldsymbol{q}_3+m)(G_S\gamma\cdot\epsilon-G_Dq_2\cdot\epsilon)(-\boldsymbol{q}_2+m)\times\delta(q_1{}^2-\mu^2)\delta(q_2{}^2-m^2)\delta(q_3{}^2-m^2)\right]\gamma_5 v(\bar{p}). \quad (5.12)$$

The δ functions ensure that only the on-shell values of $F_{\pi N}$ will be needed. In terms of the usual A and B amplitudes, $F_{\pi N}$ is given by

$$F_{\pi N}{}^{(1/2)}(\bar{s},t)=A^{(1/2)}(\bar{s},t)+\tfrac{1}{2}(\boldsymbol{q}+\gamma\cdot\bar{p}-\boldsymbol{q}_2)B^{(1/2)}(\bar{s},t). \quad (5.13)$$

Putting (5.13) into (5.12), we have only three types of integral to carry out, namely,

$$I=\int d^4q_2\, DR(\bar{s},t), \quad (5.14a)$$

$$I_\mu=\int d^4q_2\, DR(\bar{s},t)q_{2\mu}, \quad (5.14b)$$

$$I_{\mu\nu}=\int d^4q_2\, DR(\bar{s},t)q_{2\mu}q_{2\nu}, \quad (5.14c)$$

where $D\equiv\delta(q_1{}^2-\mu^2)\delta(q_2{}^2-m^2)\delta(q_3{}^2-m^2)$ and R can be either A or B. Using their Lorentz-covariant properties, it is easy to reduce the last two types of integral into the first one. Taking I_μ, for example, since the three 4-vectors q, d, and $\bar{p}$ are coplanar, we can write

$$I_\mu=J_1d_\mu+J_2\bar{p}_\mu. \quad (5.15)$$

Reducing (5.15) to $I\cdot\bar{p}=J_1(d\cdot\bar{p})+J_2m^2$ and $I\cdot d=J_1M^2+J_2(d\cdot\bar{p})$, we solve for J_1 and J_2:

$$J_1=\frac{(2m^2-\mu^2)(t-m^2-M^2)+2m^2M^2}{\lambda(t,M^2,m^2)}I,$$

$$J_2=-\frac{t+3m^2-M^2-2\mu^2}{\lambda(t,M^2,m^2)}I.$$

We can write $I_{\mu\nu}$ as

$$I_{\mu\nu}=J_3g_{\mu\nu}+J_4d_\mu d_\nu+J_5\bar{p}_\mu\bar{p}_\nu+J_6(\bar{p}_\mu d_\nu+d_\mu\bar{p}_\nu)$$

and solve for $J_3,\cdots,J_6$ in terms of I in a similar way, which we shall not elaborate. Now the integral I is of the same type as that in (5.2) and hence can be handled just as in the spinless case.

For the remaining job, we must form the parity-conserving amplitudes for $F(s,t)$ in (5.12); they in turn will depend only on the right parity-conserving πN amplitudes. Thus we can obtain the expressions for the Regge residue functions $\beta_\mu{}^\pm$ separately for the trajectories $\alpha_\pm$.

VI. LOW-ENERGY MODELS

Through the recent development of finite-energy sum rules, it has been recognized that the Regge representation, when extrapolated down to the low-energy region, still describes the actual amplitude in an average sense.[2] On the other hand, amplitudes at low energies in general can be represented very well by the direct-channel resonances using the Breit-Wigner form. Thus there exists a connection between the high- and low-energy parameters. With the duality concept introduced by Chew and Pignotti,[15] one can see how a smooth transition is achieved from the single Regge-pole representation to the multiple Regge-pole representation and conversely. We shall employ this duality concept to relate the various low-energy models, used previously to study the reactions $\pi^++d\leftrightarrows p+p$, to the Regge model.

Consider the double Regge-pole representation for the amplitude of the reaction $N+N\to N+N+\pi$ as shown in Fig. 4. In addition to the pion trajectory, any meson trajectory that communicates with the $N\bar{N}$ system can also be exchanged. However, since we are interested in the region with small subenergy s_{NN} of the two outgoing nucleons, pion-trajectory exchange will probably dominate because of the small mass of the physical pion that enhances the Regge residue. The left-hand portion of the diagram shown in Fig. 4 (including the middle vertex) can be viewed as either the NN amplitude with one external line Reggeized or

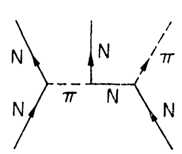


FIG. 4. Double Regge-pole exchange diagram.

[35] R. Blankenbecler, M. L. Goldberger, and F. R. Halpern, Nucl. Phys. 12, 629 (1959).

roughly as the d-p-n Regge residue if $s_{NN}=M^2$. A similar statement can be made for the right-hand portion of the diagram. Hence the final-state interaction between the two nucleons can be described approximately by the pion-trajectory exchange, whereas the final-state interaction between the pion and the "adjacent" nucleon can be described by the nucleon-trajectory exchange. Indeed, a recent study[36] of the pion-nucleon amplitudes using the finite-energy sum rules with nucleon exchange confirms this point of view. Thus we see that the final-state interaction models proposed by Rosenfeld, Gell-Mann, and Watson[8] and by Mandelstam[9] are consistent with the Regge formalism.

If we fix the subenergy at $s_{NN}=M^2$ and integrate over the momentum transfer carried by the π trajectory, the result corresponds to a Reggeized version of the OPE model (see Fig. 5).[10] As for the one-nucleon-exchange model,[11] it can be considered as an approximation (though it may not always be an adequate one) to the single Regge-pole exchange formalism.

Several authors[12–14] who study the pionic-disintegration reaction via dispersion relations have also attempted to calculate the anomalous-cut contribution. A practical problem that arises in this approach is the approximation of the pion-nucleon amplitudes that

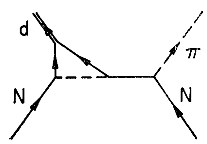


FIG. 5. Double Regge-pole exchange with the two outgoing nucleons combining to form a deuteron.

appear in the discontinuity formula. We would like to point out that the calculation is much simplified by using the Regge asymptotic form for the πN amplitudes, as is done in Sec. V.

To summarize our work, we have set up a Regge-pole formalism for the pionic-disintegration and pion-production reactions. We hope that this will facilitate data fitting at high energies for these reactions. We are able to calculate the d-p-n Regge residue in terms of the π-N-N Regge residue by assuming that the lowest anomalous singularity in the t channel dominates the small-t region. The method that we employed can also be applied to many other reactions involving loosely bound particles, such as elastic πd scattering.

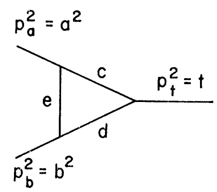
ACKNOWLEDGMENTS

I would like to express my deep appreciation to Professor Geoffrey F. Chew for suggesting this investigation, and for his constant guidance and many useful suggestions throughout the course of this work. I also wish to thank Professor Stanley Mandelstam for valuable discussions.

APPENDIX A

Anomalous threshold singularities are the singularities that lie below the lowest normal threshold on the physical sheet. We summarize here the conditions that

FIG. 6. A general triangle graph.



must be satisfied for the existence of anomalous singularities of a triangle graph shown in Fig. 6. Let

$$y_a=\frac{c^2+e^2-a^2}{2ce},\quad y_b=\frac{d^2+e^2-b^2}{2de},\quad y_t=\frac{c^2+d^2-t}{2cd}.$$

Then, if the conditions

$$|y_a|<1,\quad |y_b|<1, \tag{A1}$$

$$\cos^{-1}y_a+\cos^{-1}y_b>\pi, \tag{A2}$$

are satisfied, an anomalous singularity in t exists and is given by

$$\cos^{-1}y_t=\cos^{-1}y_a+\cos^{-1}y_b$$

or

$$y_t=y_ay_b-[(1-y_a^2)^2(1-y_b^2)]^{1/2}. \tag{A3}$$

APPENDIX B

The methods used to obtain the left-hand cut singularities of partial-wave amplitudes are well known. Special cases have been studied by many authors, and a general treatment has been given by Kennedy and Spearman.[37] We shall present here a somewhat different method that reproduces the general results easily.

Consider the partial-wave projection for the amplitude of the reaction $a+b\rightarrow c+d$:

$$a_l(s)=\int_{-1}^{1}A(s,z_s(s,t))P_l(z_s)dz_s$$

$$=\int_{-1}^{1}A(s,z_s(s,u))P_l(z_s)dz_s.$$

The so-called left-hand singularities of a a_l arise from the pinching of the singularities of A in t or u with the endpoints of integration. Hence these singularities can be obtained from the following equations:

$$z_s(s,t_p)=\pm1, \tag{B1}$$

$$z_s(s,u_p)=\pm1, \tag{B2}$$

[36] H. Lee (to be published).

[37] J. Kennedy and T. D. Spearman, Phys. Rev. **126**, 1596 (1962).

where t_p (u_p) are the positions of either poles or branch points in the t channel (u channel). In general, it is rather awkward to solve for s directly from Eqs. (B1) and (B2). Instead, we employ the following observation: The three manifolds defined by $z_x^2=1$ $(x=s,t,u)$ all contain the physical boundary surface defined by setting the Kibble function $\phi(s,t,u)=0$,[22] and they possibly differ from one another only by surfaces of the form $x=0$, depending on the mass assignments. It is evident that $s=0$ is a solution of (B1) and (B2) if and only if $(a^2-b^2)(c^2-d^2)\neq 0$. Hence we can replace (B1) and (B2) with two equivalent sets of equations:

$$z_t(s,t_p)=\pm 1 \tag{B3}$$

plus $s=0$ if $(a^2-b^2)(c^2-d^2)\neq 0$;

$$z_u(s,u_p)=\pm 1 \tag{B4}$$

plus $s=0$ if $(a^2-b^2)(c^2-d^2)\neq 0$.

Taking (B3) as an example, with the t-channel c.m. scattering angle given by

$$4p_{ac}p_{bd}\cos\theta_t=(s-u)+(a^2-c^2)(b^2-d^2)/t\,,$$

we immediately get

$$s=\tfrac{1}{2}[(a^2+b^2+c^2+d^2)-t_p]-(a^2-c^2)(b^2-d^2)/2t_p \pm(1/2t_p)[\lambda(t_p,a^2,c^2)\lambda(t_p,b^2,d^2)]^{1/2}. \tag{B5}$$

Now we are ready to give the proof promised at the end of Sec. II. Consider the s reaction $a+b\rightarrow c+d$, and let e be a pole in the t reaction $a+\bar{c}\rightarrow \bar{b}+d$. If the masses of these five particles satisfy (A1) and (A2), then the anomalous singularity in s obtained from (A3) will coincide with the highest left-hand branch point generated by e in the s-channel partial wave. A direct application of (A3) gives

$$S_a=c^2+d^2+(1/2e^2)[\lambda(e^2,a^2,c^2)\lambda(e^2,b^2,d^2)]^{1/2} -(1/2e^2)(c^2+e^2-a^2)(d^2+e^2-b^2)\,.$$

But this is identical to the larger one of the two solutions given in (B5) with $t_p=e^2$.

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Extension of Axiomatic Analyticity Properties for Particles with Spin, and Proof of Superconvergence Relations*

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It is shown that any regularized helicity amplitude that is known from axiomatic local field theory to satisfy dispersion relations for $-t_0\leq t\leq 0$ is in fact analytic in the quasi-topological product $|t|<R\times s$ in the cut plane with cuts $s=C+\lambda$, $s=-t-\mu+C'$, where $\lambda,\mu\geq 0$ and R is a fixed number. This is the extension to the scattering of nonzero-spin particles of a result obtained in the scalar case. As a first consequence, the Froissart limits are extended to all helicity amplitudes. Furthermore, it is shown that for $-t_0\leq t\leq 0$ and s going to infinity, the regularized helicity amplitudes in the t channel, with initial (final) helicities λ_1 and λ_2 (μ_2 and μ_2), are bounded by $Cs^{1-\max(|\lambda|,|\mu|)}(\ln s)^2$ if $\lambda+\mu$ is even, or by $Cs^{1-\max(|\lambda|,|\mu|)}(\ln s)^3$ if $\lambda+\mu$ is odd, where $\lambda=\lambda_1-\lambda_2$ and $\mu=\mu_1-\mu_2$. This gives superconvergent amplitudes as soon as one of the spins is larger than 1. The case of spin-0–spin-1 scattering is marginal, and in the absence of any detailed dynamical information, one cannot obtain a superconvergent amplitude in that case.

I. INTRODUCTION

SINCE the discovery of superconvergence relations,[1] it has appeared more and more clearly that these relations had a more general character than what was indicated by their original derivation. In particular, it appeared that current algebra had the only role to supply a locality condition. Later on, superconvergence relations were obtained without current algebra, but in the framework of a Regge model of high-energy scattering. Even this seems to be too restrictive. What we want to do here is to try to see what survives if one starts from axiomatic field theory and unitarity only. The reason why it is not *a priori* hopeless to do this is that in the case of scalar particles it has been shown by one of us (A.M.)[2] that local field theory and unitarity are indeed sufficient to derive a bound on the forward scattering amplitude $|F(s,\cos\theta=1)|<s(\ln s)^2$ that was originally derived by Froissart in the framework of Mandelstam representation.[3]

We shall see that the main difficulty in this problem is to prove the existence of a sufficiently big analyticity

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[1] S. Fubini and G. Segré, Nuovo Cimento **45**, 641 (1966); V. de Alfero, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966); Ann. Phys. (N. Y.) **44**, 165 (1967).

[2] A. Martin, Nuovo Cimento **42**, 930 (1966).

[3] M. Froissart, Phys. Rev. **123**, 1053 (1961).