Electromagnetic Interactions of Loosely-Bound Composite Systems*

STANLEY J. BRODSKY AND JOEL R. PRIMACK
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
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Contrary to popular assumption, the interaction of a composite system with an external electromagnetic field is not equal to the sum of the individual Foldy-Wouthyusen interactions of the constituents if the constituents have spin. We give the correct interaction, and note that it is consistent with the Drell-Hearn-Gerasimov sum rule and the low-energy theorem for Compton scattering. We also discuss the validity of additivity of the individual Dirac interactions, and the corrections to this approximation, with particular reference to the atomic Zeeman effect, which is of importance in the fine-structure and Lamb-shift measurements.

1. INTRODUCTION

IT has been assumed almost universally[1,2] in the literature of atomic and nuclear physics that the interaction of a loosely-bound composite system with an external electromagnetic field is given by the sum of the Foldy-Wouthuysen (FW) interactions of the constituents. We have found, on the contrary, that *additivity of the individual* FW *interactions is incorrect even in order* $1/m^2$ *if the constituents have spin.* If one uses such an additive FW Hamiltonian, one finds that the Drell-Hearn[3]-Gerasimov[4] (DHG) sum rule[5] and the low-energy theorem for Compton scattering[6] on the composite system[7] are violated. The crucial error in deriving[1] FW additivity is in neglecting the spin transformation of the composite-state wave function associated with the center-of-mass (c.m.) motion.

The correct nonrelativistic reduction of the interaction Hamiltonian for a composite system of two spin-$\frac{1}{2}$ particles in an external electromagnetic field takes the following form[8]:

$$H_{\mathrm{NR}}{}^{\mathrm{em}}=\sum_{s=a,b}\left[\frac{-\mathbf{p}_s\cdot e_s\mathbf{A}_s}{m_s}+\frac{e_s{}^2\mathbf{A}_s{}^2}{2m_s}+e_sA_s{}^0-\mu_s\boldsymbol{\sigma}_s\cdot\mathbf{B}_s-\left(2\mu_s-\frac{e_s}{2m_s}\right)\boldsymbol{\sigma}_s\cdot\mathbf{E}_s\times\frac{(\mathbf{p}_s-e_s\mathbf{A}_s)}{2m_s}\right]+\frac{1}{4M_T}\left(\frac{\boldsymbol{\sigma}_a}{m_a}-\frac{\boldsymbol{\sigma}_b}{m_b}\right)\cdot[e_b\mathbf{E}_b\times(\mathbf{p}_a-e_a\mathbf{A}_a)-e_a\mathbf{E}_a\times(\mathbf{p}_b-e_b\mathbf{A}_b)]+O(1/m^3). \quad (1)$$

The terms proportional to $(M_Tm_a)^{-1}$ or $(M_Tm_b)^{-1}$ are correction terms to FW additivity. For a uniform electric field, the "spin-orbit" terms combine to

$$\left[\left(\frac{e_T}{2M_T}-2\mu_a\right)\frac{\boldsymbol{\sigma}_a}{2M_T}+(a\to b)\right]\cdot(\mathbf{E}\times\mathbf{P})+\left[\left(\frac{e_a}{2m_a}+\frac{e_T}{2M_T}-2\mu_a\right)\frac{\boldsymbol{\sigma}_a}{2m_a}-(a\to b)\right]\cdot(\mathbf{E}\times\mathbf{p}). \quad (2)$$

The presence of these "spin-orbit" terms is essential in obtaining the correct low-energy limit of the Compton scattering amplitude and the DHG sum rule.

The calculation of Barton and Dombey,[5] which purports to show that if the DHG sum rule holds for nucleons, it must fail for bound states containing a nucleon, was based on the assumption that $H_{\mathrm{NR}}{}^{\mathrm{em}}$ equals the sum of the FW interactions of the constituent particles. If it had been correct, this calculation would have proved that there is an additive constant, sometimes called a "subtraction at infinity," present[9] in the dispersion relation for the spin-flip forward Compton amplitude f_2 for a composite system, even if there is none for the constituents. Such a state of affairs would

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[1] K. W. McVoy and L. Van Hove, Phys. Rev. **125**, 1034 (1962).

[2] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957), Sec. 39.

[3] S. D. Drell and A. C. Hearn, Phys. Rev. Letters **16**, 908 (1966).

[4] S. B. Gerasimov, Yadern. Fiz. **2**, 598 (1965) [English transl.: Soviet J. Nucl. Phys. **2**, 430 (1966)]. Cf. also L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **41**, 1546 (1961) [English transl.: Soviet Phys.—JETP **14**, 1102 (1962)]. An independent derivation was given by M. Hosoda and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) **30**, 425 (1966).

[5] G. Barton and N. Dombey, Phys. Rev. **162**, 1520 (1967).

[6] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).

[7] We wish to thank Dr. H. R. Pagels for suggesting that the low-energy theorem might be violated if the sum of FW Hamiltonians is used. This was also independently discovered by G. Barton (unpublished).

[8] Here e_a, m_a, μ_a, and $\frac{1}{2}\boldsymbol{\sigma}_a$ are the charge, mass, total magnetic moment, and spin of fermion a. Note that we take $\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}_a+\boldsymbol{\sigma}_b)$ to be the total spin in the c.m. frame. The relative and total four-momentum are given $M_Tp=m_bp_a-m_ap_b$, $P=p_a+p_b$, with $M_T=m_a+m_b$. As indicated in Eq. (3) below, the wave function $\phi(\mathbf{x}_a',\mathbf{x}_b')$ to be used for evaluating matrix elements of Eq. (1) must include the Lorentz contraction $\mathbf{x}'=\Lambda\mathbf{x}$. This is important for evaluating the DHG sum rule and low-energy theorem for bound states with $l\geq 1$. Equation (1) includes only terms involving the external field $A_s{}^\mu\equiv(A_s{}^0,\mathbf{A}_s)\equiv A^\mu(x_s)$; there are consequently no Darwin terms. The Hamiltonian for the atom in zero external field is assumed to be known. Binding correction factors of order $(1+W/m)$ are neglected here as well as cross terms in the binding potential U and the external field such as $\boldsymbol{\sigma}\cdot\nabla U\times\mathbf{A}$. If required, the binding corrections corresponding to the relativistic Hamiltonian (8) can be readily obtained from Eq. (7). Equation (1) can be generalized to composite systems of more than two constituents since it does not depend on the binding interaction.

[9] H. D. I. Abarbanel and M. L. Goldberger [Phys. Rev. **165**, 1594 (1968)] have shown that such an additive constant in the DHG sum rule would correspond to a fixed pole at $J=1$ in the complex angular momentum plane. There is no experimental evidence for this singularity, but it cannot be ruled out *a priori.* It would consequently be extremely interesting to have an experimental test of the DHG sum rule for the proton.

be physically most unreasonable, since a subtraction at infinity is associated with the asymptotic behavior of $f_2(\omega)$ for $|\omega| \to \infty$, and the asymptotic behavior of the Compton amplitude for the composite system should be no worse than that of the sum of the amplitudes of the constituents.

With the inclusion of the terms arising from the spin transformation of the wave function, we are able to verify explicitly both the DHG sum rule and the low-energy theorem for Compton scattering.[10] Thus we have shown that there is nothing in the treatment of loosely-bound composite systems which introduces into the dispersion relation an additive constant. After our calculations were completed, we learned that the DHG sum rule and the threshold theorem for Compton scattering have also been verified independently by Osborn,[11] using different methods.

II. CORRECTIONS TO FW ADDITIVITY

Let us now trace the origin of the correct spin-orbit terms. Since momentum is transferred, the matrix element of the external potential requires knowledge of the bound-state wave function at different total momenta. As is well known, the wave function for a moving system is determined from the c.m. wave function by application of the Lorentz boost operator. For the homogeneous Lorentz transformation $x'=\Lambda x$, $(E,\mathbf{P})=\Lambda(\mathfrak{M},\mathbf{0})$, corresponding to a boost of a two-fermion bound state (of mass $\mathfrak{M}$) to velocity $\mathbf{V}=\mathbf{P}/E$, the required transformation law for the corresponding Bethe-Salpeter amplitude[12] is

$$\chi_{E,\mathbf{P}}{}^{\alpha'\beta'}(x_a',x_b')=S_a{}^{\alpha'\alpha}(\Lambda)S_b{}^{\beta'\beta}(\Lambda)\chi_{\mathfrak{M},\mathbf{0}}{}^{\alpha\beta}(x_a,x_b)\,. \quad (3)$$

The spin-$\frac{1}{2}$ transformation matrix is

$$S_a(\Lambda)=\exp(\tfrac{1}{2}\boldsymbol{\alpha}_a\cdot\hat{V}\tanh^{-1}V)=\left(\frac{E+\mathfrak{M}}{2\mathfrak{M}}\right)^{1/2}\left(1+\frac{\boldsymbol{\alpha}_a\cdot\mathbf{P}}{E+\mathfrak{M}}\right). \quad (4)$$

Thus if the bound-state wave function in the c.m. system has the Dirac structure[13]

$$\begin{Bmatrix}1\\ \dfrac{1}{2m_a+k_a}\boldsymbol{\sigma}_a\cdot\mathbf{p}\end{Bmatrix}\otimes\begin{Bmatrix}1\\ \dfrac{1}{2m_b+k_b}\boldsymbol{\sigma}_b\cdot(-\mathbf{p})\end{Bmatrix}, \quad (5)$$

then, as shown by McGee[14] for the case of an unbound system, the wave function in the moving frame must have the structure

$$\begin{Bmatrix}1+\dfrac{\boldsymbol{\sigma}_a\cdot\mathbf{P}}{\mathfrak{M}+E}\dfrac{\boldsymbol{\sigma}_a\cdot\mathbf{p}}{2m_a+k_a}\\ \boldsymbol{\sigma}_a\cdot\left(\dfrac{\mathbf{P}}{\mathfrak{M}+E}+\dfrac{\mathbf{p}}{2m_a+k_a}\right)\end{Bmatrix}\otimes\begin{Bmatrix}1-\dfrac{\boldsymbol{\sigma}_b\cdot\mathbf{P}}{\mathfrak{M}+E}\dfrac{\boldsymbol{\sigma}_b\cdot\mathbf{p}}{2m_b+k_b}\\ \boldsymbol{\sigma}_b\cdot\left(\dfrac{\mathbf{P}}{\mathfrak{M}+E}-\dfrac{\mathbf{p}}{2m_b+k_b}\right)\end{Bmatrix}. \quad (6)$$

Wave-packet representations of the moving composite system can be constructed from a superposition of such wave functions. The possibly unexpected feature of (6) is the appearance of extra terms in the large components. Physically, they correspond to the fact that a spin-triplet wave function in the c.m. frame appears partially as a spin singlet in the moving frame.[14] It is just these terms which are ignored in the usual FW analysis.

It is worth noting that there is nothing wrong in principle in using the FW transformation to eliminate "odd" operators in the relativistic Hamiltonian. What is incorrect is to assume that this reduces the bound-state *wave function* to a simple Pauli form. Inserting a FW unitary operator U in the matrix element, one obtains

$$\begin{aligned}\langle f,P_f|H|i,P_i\rangle&=\langle f,P_f|U^{-1}(UHU^{-1})U|i,P_i\rangle\\&=\langle f,\mathbf{0}|S(P_f)^\dagger U^{-1}H_{\mathrm{FW}}US(P_i)|i,\mathbf{0}\rangle.\end{aligned} \quad (7)$$

The presence of the Lorentz boost operator S introduces the extra terms into the matrix element which appear in (1).[15] These terms can also be obtained by the usual large component reduction method.

As we demonstrate in Ref. 10, one can in fact avoid entirely the use of a nonrelativistic electromagnetic Hamiltonian in calculating such expressions as the integral in the DHG sum rule. By a judicious use of such identities as $\boldsymbol{\alpha}_a=i[H_0,\mathbf{r}_a]$, where H_0, the unperturbed Hamiltonian, has the Breit form

$$H_0=\boldsymbol{\alpha}_a\cdot\mathbf{p}_a+\beta_a m_a+\boldsymbol{\alpha}_b\cdot\mathbf{p}_b+\beta_b m_b+U(\mathbf{r}_a-\mathbf{r}_b)\,, \quad (8)$$

and a proper treatment of the Lorentz transformation of the wave functions, one can reduce the integral in the DHG sum rule to a form in which the superconvergent nature of the sum rule is especially clear.

III. RELATIVISTIC ADDITIVITY

Implicit in our derivation of the nonrelativistic interaction Hamiltonian Eq. (1) is the assumption that the

[10] S. J. Brodsky and J. R. Primack (to be published).

[11] H. Osborn, Phys. Rev. (to be published).

[12] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951). We have only utilized the Bethe-Salpeter amplitude in order to determine rigorously the Lorentz-transformation properties of the bound-state wave function. The resulting correction terms to FW additivity in Eq. (1) are kinematic in origin and do not depend on finding dynamical solutions to the Bethe-Salpeter equation.

[13] This is the structure of the Dirac spinors for the eigensolutions corresponding to the model Hamiltonian given in Eq. (8). We have defined "kinetic energy" operators $k_{a,b}=-(U+W)(m_{b,a}/M_T)$, where $W=M_T-\mathfrak{M}$ is the binding energy of the state. We discuss in detail such relativistic two-body wave functions and their relationship to the solutions of the Bethe-Salpeter equation in Ref. 10.

[14] Ian J. McGee, Phys. Rev. **158**, 1500 (1967). We wish to thank Professor L. Durand for calling this work to our attention.

[15] In particular, including in H only the external potential A_0 yields the terms linear in the electric field in (1). If there is an additional vector potential $\mathbf{A}(\mathbf{x})$ present, then instead of using straightforward perturbation theory one can replace, in the c.m. equation of motion and in the boost operator (4), the canonical momenta by the mechanical momenta $\boldsymbol{\pi}_s=\mathbf{p}_s-e_s\mathbf{A}_s$ $(s=a,b)$. $\boldsymbol{\Pi}=\boldsymbol{\pi}_a+\boldsymbol{\pi}_b$, $M_T\boldsymbol{\pi}=m_b\boldsymbol{\pi}_a-m_a\boldsymbol{\pi}_b$. Equation (1) then follows.

relativistic interaction is equal to the sum of the Dirac interactions of the constituents ("impulse approximation"). Specifically, for spin-$\frac{1}{2}$ constituents, the interaction of the composite system is assumed to be

$$H_R^{\mathrm{em}} = -e_a \boldsymbol{\alpha}_a \cdot \mathbf{A}_a + e_a A_a{}^0 - e_b \boldsymbol{\alpha}_b \cdot \mathbf{A}_b + e_b A_b{}^0 + (\text{anomalous moment contributions}). \quad (9)$$

We have examined the validity of this approximation. Starting from Lagrangian field theory, and expressing the matrix elements of the current in terms of Bethe-Salpeter (BS) amplitudes,[16] we find that when the BS interaction kernel is replaced by a neutral instantaneous kernel (i.e., potential) in ladder approximation, a relativistic interaction Hamiltonian emerges. For the instantaneous kernel, it is also possible to derive an extended form of Salpeter's[17] equation which includes interactions with an external static or adiabatic field. In fact, one finds that the Breit Hamiltonian H_0 extended to include external electromagnetic interactions by the addition of H_R^{em}, leads to the same results as the BS approach up to terms of relative order $\langle U^2/m_a m_b \rangle$.[10]

Using these procedures, the corrections to the impulse approximation can then be readily traced. We have applied these results to the analysis of the Zeeman spectrum in hydrogenlike atoms, in order to obtain estimates of radiative and reduced-mass corrections not already included in standard calculations.[18] We emphasize that the comparison of theory with experimental measurements of the Lamb shift and fine-structure intervals in H and D require a precise theoretical extrapolation of the experimental results to zero magnetic field. Thus care in the calculation of the Zeeman effect is as essential as it is in the calculation of the zero-field energy levels themselves.

The application to the Zeeman effect is as follows: The relevant kernels of the BS equation which are needed to describe the H atom to the accuracy sufficient for comparison with present experiments are known. Using the techniques of Mandelstam,[16] the corresponding contributions to the electromagnetic current of the atom may be computed. In particular, the kernel corresponding to instantaneous photon exchange in ladder approximation yields the usual relativistic Zeeman interaction Hamiltonian for two Dirac particles, $-e_a \boldsymbol{\alpha}_a \cdot \mathbf{A}_a - e_b \boldsymbol{\alpha}_b \cdot \mathbf{A}_b$, if terms of relative order $(Z\alpha)^4 m_e/M_p$ are neglected. In fact, to this order, one can use the Breit formalism. The self-energy kernels in lowest approximation yield the expected anomalous magnetic moment contributions. The neglected kernels and other approximations which are made correspond to radiative and higher-order reduced-mass corrections to the Zeeman spectrum. The corrections can be readily estimated; their effects on the determination of the Lamb shift and fine structure from zero-field extrapolation in present experiments are less than 1 ppm.[10,19]

[16] S. Mandelstam, Proc. Roy. Soc. (London) **A233**, 248 (1955).

[17] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

[18] W. E. Lamb, Jr., Phys. Rev. **85**, 259 (1952); R. T. Robiscoe, *ibid.* **138**, A22 (1964); S. J. Brodsky and R. G. Parsons, *ibid.* **163**, 134 (1967); R. T. Robiscoe, *ibid.* **168**, 4 (1968).

IV. CONCLUSION

Some essential points about the response of a composite system to an external electromagnetic field have thus been clarified. Contrary to what has been previously assumed, the interaction of a composite system with an external field is not represented by an interaction Hamiltonian of the Foldy-Wouthuysen type which is additive in terms of the properties of the individual particles, but in fact contains extra spin terms of kinematic origin associated with the c.m. motion. The extra terms are precisely what is required to explicitly verify the low-energy theorem and DHG sum rule for a bound state—considered as a composite system of definite total mass, charge, magnetic moment, and spin. On the other hand, we find that additivity of Dirac interaction Hamiltonians as an approximation to the interaction of a loosely-bound system with an external electromagnetic field can be justified; the corrections are calculable from quantum electrodynamics, and are very small for the H atom.

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[19] The terms in Eq. (1) which arise from the spin transformation of the composite wave function are numerically unimportant in these experiments.