Relativistic Three-Body Theory with Applications to π-N Scattering*

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Linear relativistic three-body equations for the scattering of a particle from a bound state or correlated pair of the others are constructed by combining the quasiparticle or isobar idea with two- and three-body unitarity as suggested by Blankenbecler and Sugar. After a partial-wave decomposition, the equations turn out to be one-dimensional, and hence are easily solved numerically. Any exchange mechanism and any number of isobars or separable two-body interactions can be included in the equations without violating two- and three-body unitarity and Lorentz invariance. Higher integer-spin separable interactions or isobars are included, in very close analogy to the nonrelativistic case. Applying the equation to the π-N system with pseudoscalar coupling, that is, with only nucleon exchange and no π-π interaction or π-N^* intermediate state, gives a (3,3) resonance but no other interesting structure. That is just what one would expect from such a simple mechanism and encourages us to go on to richer input. Analyzing the answers as a function of nucleon mass shows that the static-model expansion converges very slowly.

I. INTRODUCTION

THE pion-nucleon system is the "classical" domain of strong-interaction physics. The discovery of higher symmetries and the proliferation of hadrons has shown that there is much more to strong interactions than just pions and nucleons, but at low to moderate energy they form a fairly well contained and experimentally rich and well-explored system. The basic form and strength of pion-nucleon coupling have been known for some time. However, how the experimental data emerge, if at all, from the basic features and the extent to which π-π interactions play a role remain basically unanswered. Some of the strongest aspects of the problem, the dominance of p waves and the existence of a 3-3 resonance, have a qualitative explanation in the static model of Chew and Low,[1] but nothing that can be called a genuine dynamical scheme has been successful. In this paper, we present what we hope will be the first tentative step toward such a scheme.

Up to pion kinetic energies of about 1 BeV, substantial single-pion production in π-N collisions is a dominant feature, but multipion production seems to be far less important. Hence any dynamical scheme of π-N scattering in this range must include the effects of the N-π-π states, that is, it must be at least a three-body theory. Recent advances in three-body theory of a formal and of a technical nature have led to significant improvements in our ability to calculate.[2] In this paper, we apply that technology to the relativistic problem. We show how tractable equations can be obtained and how for the simplest version of the π-N problem sensible, but not yet very good, answers are obtained.

There seem to be two avenues approaching dynamical calculations—off-shell and on-shell. The classic example of the former is the Schrödinger equation or, equivalently, the Lippmann-Schwinger equation in nonrelativistic quantum mechanics. Its most common relativistic manifestation is the Bethe-Salpeter equation. We shall present an alternative off-shell form, which we believe suits our problem better. The on-shell approach is that of S-matrix theory. It has had some limited success in the two-body problem, but we know of no tractable method for including the dynamical effects of higher-particle sectors in it. One feature of it, however, we do borrow. That is its emphasis on unitarity. In discussing π-N elastic scattering and π production in π-N collisions in a domain where both are big, it is essential that the constraints of unitarity on these amplitudes be accurately imposed. This is, of course, precisely what the Schrödinger equation or Lippmann-

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[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[2] A review of these developments will be found in K. M. Watson and J. Nuttall, *Topics in Several Particle Dynamics* (Holden-Day, San Francisco, 1967); R. D. Amado, in Lectures on Theoretical Physics, Brandeis Summer School, 1967 (unpublished).

Schwinger equation manage to do (in spite of being linear equations). The Bethe-Salpeter equation in its usual truncated form does not do so well with the multiparticle states,[3] and this is our main reason for replacing it with one more of the Schrödinger or Lippmann-Schwinger spirit. It turns out that this replacement also reduces the dimensionality of the equation, without violating Lorentz invariance, and hence makes the technical problems of obtaining numbers easier.

Our method for obtaining the relativistic equations follows closely the method of Blankenbecler and Sugar (BS)[4] and also that of Freedman, Lovelace, and Namyslowski (FLN).[5] We want the amplitudes for three-body scattering. We begin by assuming that the two-body interaction is dominated by a bound state or quasiparticle or isobar. This leads to a separable two-body interaction. (The procedure is easily generalized to more than one bound state or isobar.) We then write down the amplitude for elastic scattering of one particle from a bound state of the other two by analogy with the equations of this type used in the nonrelativistic problem.[6,7] Except for Lorentz invariance, and for certain assumptions about the variables that they depend on, we do not specify the quantities entering the equation. We wish to find them by imposing unitarity. Unitarity will couple the elastic scattering from the isobar, or bound state, to the breakup or production amplitude. Again led by our nonrelativistic experience, we postulate a form for that amplitude that relates it back to the elastic bound-state amplitude. Imposing unitarity then gives us the discontinuities of the objects entering our equations in terms of known things, like mass-shell δ functions. Assuming that the functions have no further discontinuities than those required by unitarity, we write dispersion integrals for those that are easily done, and obtain their form. This procedure reduces the equations to three-vector equations that after partial-wave decomposition become one-dimensional linear integral equations—and yet remain Lorentz-invariant. The solutions, of course, also automatically satisfy two- and three-body unitarity at *all* energies. The amplitudes obtained do not have the same (and presumably correct) left-hand cut structure as do the Bethe-Salpeter amplitudes, but they do take better account of unitarity and particularly of the multiparticle states. For π-N scattering from threshold to 1 BeV we would guess that this is more important than the left-hand cut that is very far away. We have no objection to treating crossing and the left-hand cut properly as well as unitarity; we simply do not know how to do it.

Having constructed these equations, we apply them to the π-N system. We begin with the simplest π-N dynamics that we can have—namely, only $N \leftrightarrows N+\pi$ pseudoscalar coupling. There is no difficulty in applying the method to this case where one of the particles (N) is both the quasiparticle and one of the constituents. In this simple theory the dynamical mechanism, or Born term, is nucleon exchange. Hence, the problem we are solving is π-N scattering with nucleon exchange including the effects of the π-N and π-π-N intermediate state, but with all π-N interactions in the nucleon (1,1) channel and no π-π interaction. This is a theory very similar in spirit to the Chew-Low static model and it gives very similar results. We find that for reasonable choices of the parameters, the 3,3 phase shift resonates, all other phase shifts are small, but almost always of the sign indicated by experiment, and there is very little inelasticity. These are sensible, if not yet good, answers. They show that the theory is reasonable and yields answers that seem to be what one would expect. They are not yet good enough, but if such minimal assumptions about the mechanisms yielded all the answers, we would be very surprised and very suspicious of the theory. The next step, therefore, will be to enrich the mechanisms. We plan to put in N-π interactions in the 3,3 state and at least the π-π interaction through the ρ. This will add considerably to the technical problems involved in solving the equation, but should also greatly improve agreement with experiment. It will also allow us to study one of the most interesting questions in this subject, namely, how does production go in the presence of resonances between more than one final pair, and how is the pair-resonance information distributed over the final state.

In Sec. II the derivation of the unitary scattering equation is given for the case of three identical spinless particles. The reader interested in only the final answer should look at Eq. (29), which is almost self-evident by analogy with nonrelativistic and static-model results. In Sec. III we show how to include the case of higher spin for the isobars. In keeping with our attempt to follow the nonrelativistic form as closely as possible, we show there that Lorentz-invariant descriptions of higher spin can be given entirely in terms of three-vectors and of Y_{lm}'s of *three* vectors. That section is largely technical in nature. Section IV shows how to write the equations for the π-N system and presents the numerical results. Some conclusions and plans for the future are presented in Sec. V.

II. EQUATIONS

We now use the methods developed by Blankenbecler and Sugar[3] to obtain a set of Lorentz-invariant, linear integral equations that describe the scattering from a bound state and breakup or production. Usually the

[3] For an interesting attempt to improve this in the Bethe-Salpeter context, see M. J. Levine, J. Wright, and J. A. Tjon, Phys. Rev. **157**, 1416 (1967).

[4] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

[5] D. Freedman, C. Lovelace, and J. Namyslowski, Nuovo Cimento **43**, 258 (1966). An exhaustive set of references will be found here.

[6] R. D. Amado, Phys. Rev. **132**, 485 (1963).

[7] R. Aaron, Phys. Rev. **151**, 1293 (1966).

Bethe-Salpeter equations are taken to form such a set, but in their usual truncated form (i.e., ladder approximation) they do not satisfy unitarity at all energies, since they always contain some multiparticle contributions, but not all. We shall require that our equations satisfy two- and three-body unitarity at all energies. This requirement will yield, as an added bonus, three-dimensional equations, hence greatly simplifying the numerical problems, but will still preserve Lorentz invariance.

Since unitarity will be our strongest tool, we first review our notation for it. In the succeeding discussion, we shall use the following expression of unitarity:

$$T_{fi}-T_{fi}^{\dagger}=i\sum_n d\Omega_n T_{fn}T_{ni}^{\dagger}=i\sum_n d\Omega_n T_{fn}^{\dagger}T_{ni}, \quad (1)$$

where

$$d\Omega_n=(2\pi)^4\delta^4(P_f-\sum_{i=1}^n q_i)\prod_{i=1}^n\left(\frac{d^4q_i}{(2\pi)^4}2\pi\delta^+(q_i^2-m_1^2)\right) \quad (2)$$

is n-body phase space. The tranition (T) matrix is defined in terms of the S matrix by[8]

$$S_{fi}=\delta_{fi}+(2\pi)^4i\delta^4(P_f-P_i)T_{fi}. \quad (3)$$

To demonstrate the BS[4] techniques, we first apply them to a two-body equation of the Bethe-Salpeter type which we write in the form

$$T_{pq}(s)=V_{pq}+\frac{1}{(2\pi)^4}\int d^4k\, V_{pk}G_k(s)T_{kq}(s), \quad (4)$$

where one usually writes

$$G_k(s)=[(k_1^2-m_1^2)(k_2^2-m_2^2)]^{-1}, \quad (5)$$

but we take $G_k(s)$ arbitrary for the moment. The variables are chosen so that

$$k_1+k_2=P,\quad k_1-k_2=2k,\quad P=(W,0,0,0). \quad (6)$$

We take q on the energy shell and $s=W^2$ is the square of the total energy in the c.m. system. V_{pq} may be thought of as the usual ladder potential, but, in fact, any real symmetric V_{pq} will do. Using the fact that V_{pq} is real and symmetric and that $T_{pq}^*(s^+)=T_{qp}(s^-)$, we obtain from Eq. (4)

$$T_{pq}(s^+)-T_{pq}(s^-)=\frac{1}{(2\pi)^4}\int d^4k\, T_{pk}(s^+)[G_k(s^+)-G_k(s^-)]T_{kq}(s^-). \quad (7)$$

If we allow only two-body intermediate states in the unitarity relation, Eq. (1), we obtain a similar equation, i.e.,

$$T_{pq}(s^+)-T_{pq}(s^-)=\frac{i}{(2\pi)^4}\int d^4k\, T_{pk}(s^+)\times[(2\pi)^2\delta^+(k_1^2-m_1^2)\delta^+(k_2^2-m_2^2)]T_{kq}(s^-). \quad (8)$$

From direct comparison of Eq. (7) and Eq. (8) with p on the energy shell, it is clear that the choice

$$G_k(s^+)-G_k(s^-)=i(2\pi)^2\delta^+(k_1^2-m_1^2)\delta^+(k_2^2-m_2^2) \quad (9)$$

will give an integral equation that satisfies two-body unitarity for all energies. To obtain a Green's function $G_k(s)$, we then write the dispersion relation

$$G_k(s)=\frac{1}{2\pi i}\int_{(m_1+m_2)^2}^{\infty} ds'\frac{\mathrm{disc}G_k(s')}{s'-s}, \quad (10)$$

where

$$\mathrm{disc}G_k(s)=G_k(s^+)-G_k(s^-).$$

The integral is easily done to yield

$$G_k(s)=\frac{\pi}{\omega_1\omega_2}\delta(k^0-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)\frac{\omega_1+\omega_2}{(\omega_1+\omega_2)^2-s}, \quad (11)$$

with

$$\omega_1=(\mathbf{k}^2+m_1^2)^{1/2},\quad \omega_2=(\mathbf{k}^2+m_2^2)^{1/2}. \quad (12)$$

The remaining δ function in Eq. (11) allows one to evaluate the integral in Eq. (4) so that one ends up with a three-dimensional integral equation, but since the steps leading to it have been covariant, so is the equation. It need hardly be mentioned that the above prescription for obtaining $G_k(s)$ is not unique. In particular, further cuts can be added to $G_k(s)$. We have taken the simplest choice within the constraints of unitarity and Lorentz invariance. The final equation that we obtain by substituting Eq. (11) in Eq. (4) is

$$T_{pq}(s)=V_{pq}+\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_1\omega_2}V_{pk}\frac{(\omega_1+\omega_2)}{(\omega_1+\omega_2)^2-s}T_{kq}. \quad (13)$$

This equation may be thought of as the relativistic analog of the Lippmann-Schwinger equation. It has many faults. Some of them, like not treating crossing correctly, it shares with the two-body Bethe-Salpeter equation; others, like improper treatment of the left-hand cuts, are special to it. Its main forte is that it does not take into account only parts of the multiparticle states, and hence does not violate unitarity. This seems an obvious advantage in the scattering region. Even below a multiparticle threshold the inconsistent treatment of virtual states by the Bethe-Salpeter equation may lead to incorrect results. Moreover, Eq. (13) has the technical advantage of being only a three-dimensional equation, and one with rather good convergence properties at that.

We now use techniques similar to those described above to obtain a set of relativistic three-body equations. We consider, for simplicity, the case of three identical spinless particles and consider bound-state scattering, that is, the elastic scattering of one particle from a bound state of the other two. Just as in the two-body case, we start by assuming a form for the

[8] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964).

equation, which we take as

$$\langle p|T(s)|q\rangle=\langle p|B(s)|q\rangle+\frac{1}{(2\pi)^4}\int d^4k\,\langle p|B(s)|k\rangle \times\tau(\sigma_k)\langle k|T(s)|q!, \quad (14)$$

with

$$\sigma_k=(P-k)^2. \quad (15)$$

Equation (14) is a straightforward generalization of the nonrelativistic three-body equation originally proposed by one of us.[6] Its essential ingredient is the assumption that the two-body interaction proceeds via a quasiparticle, or separable interaction, or, in the more usual relativistic language, via an isobar. The function $\tau(\sigma_k)$ is the propagator of that isobar. The external bound states are also that isobar, so that the Born term B is just particle exchange between the external isobars. Equation (14) may be represented diagrammatically as shown in Fig. 1, where the variables are also defined. To use unitarity, we shall also need to know the production (two-body $\rightarrow$ three-body) amplitude $\langle p|T(s)|q_1q_2q_3\rangle$. In this quasiparticle formulation it has the form[9]

$$\langle p|T(s)|q_1q_2q_3\rangle=\frac{1}{\sqrt{3}!}\sum_{n=1}^{3}\langle p|T(s)|q_n\rangle S(\sigma_{q_n})v(p_n^2), \quad (16)$$

where, for example, $p_1^2=(q_2-q_3)^2$, and where v is the vertex for quasiparticle disassociation. S is a propagator function whose relation to τ will be determined. The diagrammatic interpretation of Eq. (16) is given in Fig. 2. With p and q on the bound-state energy shell, FLN have shown by methods very similar to that needed to get Eq. (7) that for equations of the form of (14), the discontinuity of T satisfies the relation[5]

$$\langle p|T(s^+)|q\rangle-\langle p|T(s^-)|q\rangle =\frac{1}{(2\pi)^4}\int d^4k\,\langle p|T(s^+)|k\rangle[\tau(\sigma_k^+)-\tau(\sigma_k^-)] \times\langle k|T(s^-)|q\rangle+\frac{1}{(2\pi)^8}\int d^4k\int d^4k'\,\langle p|T(s^+)|k\rangle \times\tau(\sigma_k^+)[\langle k|B(s^+)|k'\rangle-\langle k|B(s^-)|k'\rangle] \times\tau(\sigma_{k'}^-)\langle k'|T(s^-)|q\rangle. \quad (17)$$

This equation is represented diagrammatically in Fig. 3. To obtain an equation that satisfies two- and three-body unitarity, we compare Eq. (17) with Eq. (1),

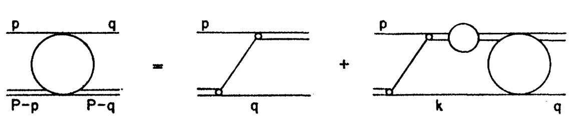

FIG. 1. Diagrammatic representation of Eq. (14).

[9] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

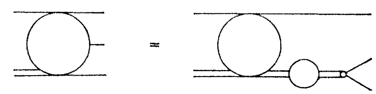

FIG. 2. Diagrammatic representation of Eq. (16).

using Eq. (16) for the breakup amplitude. This isobar ansatz for the production is pivotal to the analysis, since it relates the $2\rightarrow 3$ amplitude back to the $2\rightarrow 2$ and gives a closed set of equations. The unitarity relation gives

$$\langle p|T(s^+)|q\rangle-\langle p|T(s^-)|q\rangle =\frac{i}{(2\pi)^4}\int d^4k\,\langle p|T(s^+)|k\rangle\langle k|T(s^-)|q\rangle(2\pi)^2 \times\delta^+(\sigma_k-\mu^2)\delta^+(k^2-m^2)+\frac{i}{(2\pi)^5}\int\delta^4(P-k_1-k_2-k_3) \times\delta^+(k_1^2-m^2)\delta^+(k_2^2-m^2)\delta^+(k_3^2-m^2)d^4k_1d^4k_2d^4k_3 \times\frac{1}{3!}\sum_{n,m=1}^{3}\langle p|T(s^+)|k_n\rangle S(\sigma_{k_n}^+)v(p_n^2) \times v(p_m^2)S(\sigma_{k_m}^-)\langle k_m|T(s^-)|q\rangle. \quad (18)$$

The first term is the contribution to unitarity of the elastic bound-state scattering, μ is the bound-state mass, and m is the mass of one of the identical particles. From Eq. (17) the first term will clearly contribute to the discontinuity of τ. The second term comes from the breakup. It makes two kinds of contributions. Those for which $m=n$ will contribute to the discontinuity of τ; they come from cutting the propagator "bubble." Those for which $m\neq n$ involve the exchange of a particle between the bound states and contribute to the discontinuity of B. Comparison of Eqs. (18) and (17) gives for the discontinuities,

$$\tau(\sigma_k^+)[\langle k|B(s^+)|k'\rangle-\langle k|B(s^-)|k'\rangle]\tau(\sigma_{k'}^-) =iv((P-k-2k')^2)S(\sigma_k^+)(2\pi)^3\delta^+(k^2-m^2)\delta^+(k'^2-m^2) \times\delta^+((P-k-k')^2-m^2)S(\sigma_{k'}^-)v((P-2k-k')^2) \quad (19)$$

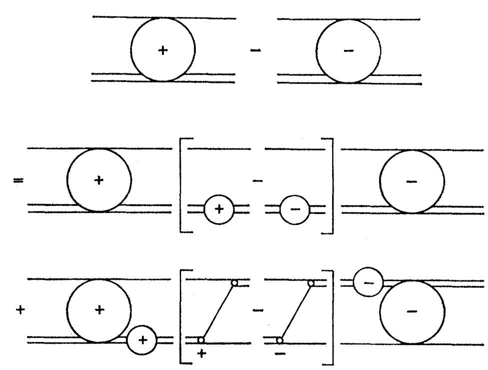

FIG. 3. Diagrammatic representation of Eq. (17).

and

$$\tau(\sigma_k^+)-\tau(\sigma_k^-)=i(2\pi)^2\delta^+(k^2-m^2)\delta^+(\sigma_k-\mu^2)-i\frac{\delta^+(k^2-m^2)}{2(2\pi)^3}\times S(\sigma_k^+)S(\sigma_k^-)\int d^4p_{12}\,v^2(4p_{12}^2)(2\pi)^2\times\delta^+(p_1^2-m^2)\delta^+(p_2^2-m^2)\,,\quad(20)$$

where

$$P=k+p_1+p_2\,,\quad p_{12}=\tfrac{1}{2}(p_1-p_2)\,.\quad(21)$$

From Eqs. (19) and (20) we see that the relation of $S(\sigma)$ and $\tau(\sigma)$ should be

$$\tau(\sigma_k)=2\pi\delta^+(k^2-m^2)S(\sigma_k)\,.\quad(22)$$

From this and Eq. (19) we find

$$\langle k|B(s^+)|k'\rangle-\langle k|B(s^-)|k'\rangle=iv((P-k-2k')^2)2\pi\delta^+((P-k-k')^2-m^2)\times v((P-2k-k')^2)\,.\quad(23)$$

Subject to the constraint

$$k^2=m^2\,,\quad k'^2=m^2\,,\quad(24)$$

which comes from the δ function, we have factored off. To obtain B from Eq. (23), we write a dispersion relation in s and assume no cut contribution from v. We obtain in the three-body c.m. system

$$\langle k|B(s)|k'\rangle=v((P-k-2k')^2)\bar{W}v((P-2k-k')^2)/\omega_{k+k'}(\bar{W}^2-s)\,,$$
$$\bar{W}=\omega_k+\omega_{k'}+\omega_{k+k'}\,,\quad(25)$$
$$\omega_k=(\mathbf{k}+m^2)^{1/2}\,.$$

In BS[4] there is some discussion of the alternative forms and additional ambiguities that arise when imposing three-body unitarity. These are related to the possible ways of choosing the energies for which the discontinuities are taken. We have already made these choices in Eq. (14). It is clear that all three particles contribute equally to the discontinuity of B; hence its discontinuity is in the total-energy variable s, whereas the discontinuity of τ or S comes from the pair interacting in τ. Hence S depends on $\sigma_k=(P-k)^2$. By making this choice, we assure ourselves the correct cluster-decomposition properties for the intermediate states.

To obtain τ we use Eqs. (20) and (22) to get

$$S(\sigma_k^+)-S(\sigma_k^-)=2\pi i\delta^+(\sigma_k-\mu^2)+\frac{S(\sigma_k^+)S(\sigma_k^-)}{2(2\pi)^4}i\int d^4p_{12}\,v^2(4p_{12}^2)\times(2\pi)^2\delta^+(p_1^2-m^2)\delta^2(p_2^2-m^2)\,.\quad(26)$$

It is clear that the inverse of S will be more easily obtained than S from Eq. (26), so that we write

$$S(\sigma)=-D^{-1}(\sigma)\,,\quad(27)$$

and then one easily finds that Eq. (26) is satisfied for a form

$$D(\sigma)=(\sigma-\mu^2)\left(1-\frac{\sigma-\mu^2}{2(2\pi)^3}\int\frac{d^3k\,v^2}{\omega_k(\sigma-\bar{\sigma})(\bar{\sigma}-\mu^2)^2}\right),\quad(28)$$

where

$$\bar{\sigma}=4(\mathbf{k}^2+m^2)$$

and the argument of v^2 is appropriate to c.m. momentum k. The factor $\sigma-\mu^2$ in front makes s have the δ-function discontinuity imposed by two-body unitarity. If the two-body state were not stable, there would be only the continuum contribution to D from three-body unitarity and there would be no zeros of D (or, what is the same thing, no subtractions).

Putting together the pieces of Eqs. (22), (25), (27), and (28) in Eq. (14) and changing T to $-T$ to conform with the static-model conventions,[7] we get, finally,

$$\langle p|T(s)|q\rangle=\frac{v(\omega_p+\omega_q+\omega_{p+q})v}{\omega_{p+q}[s-(\omega_p+\omega_q+\omega_{p+q})^2]}+\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\times\frac{v(\omega_p+\omega_k+\omega_{p+k})v}{\omega_{p+k}[s-(\omega_p+\omega_k+\omega_{p+k})^2]}\frac{\langle k|T(s)|q\rangle}{D(\sigma_k)}\,,\quad(29)$$

where we have suppressed the arguments of the vertex function v, but they may be determined from Eqs. (14) and (25). Equation (29) ends our quest. It is a linear, three-dimensional, Lorentz-invariant integral equation for the elastic scattering of one particle from the bound state of two. Its solutions are constructed to satisfy two- and three-body unitarity and to have no higher-particle contributions at all. Furthermore, from a knowledge of the solution of (29) one can construct the production or breakup amplitude by using Eq. (16). No new equations need be solved. It is clear that, just as in the nonrelativistic case, if one wishes to put in more bound states, separable interactions, or isobars, one will just get a coupled set of such equations. As discussed after Eq. (28), unstable quasiparticles are handled on the same footing as stable ones. In subsequent sections, we shall show how to include spin and fermions, but effectively this will only change numerators. The denominators, which are the real seat of unitarity, will remain as in Eq. (29).

III. HIGHER INTEGER-SPIN QUASIPARTICLE

Implicit in our discussion of Sec. II was that the quasiparticle or isobar had spin zero as did the particles that made it. We now consider the case of higher integer spin for the quasiparticle. Let us begin with spin 1. (We obviously must relax the requirement that the constituent particles be identical.) The problem is to find a separable interaction, or separable t matrix that scatters only in p waves. This is in analogy with the work of Sec. II which can be thought of as defining a separable s-wave two-body t matrix for scattering from relative four-momentum p to relative four-momentum q with c.m. four-momentum K so that $s=K^2$. This t matrix is of the form

$$\langle p|T_{sep}^{(0)}(s)|q\rangle = v(p^2)v(q^2)/D(s). \tag{30}$$

From such a construction we read off $v(p^2)$ as the vertex for forming the s-wave quasiparticle. For higher spin we need a corresponding expression and from it we shall be able to read off the higher-spin vertex. For spin 1, or for pure p-wave scattering, the t matrix must have the form in the two-body c.m. system,

$$\langle p|T_{sep}^{(1)}(s)|q\rangle = f(p^2)\mathbf{p}\cdot\mathbf{q}f(q^2)/D(s). \tag{31}$$

The factor $\mathbf{p}\cdot\mathbf{q}$ ensures pure p-wave scattering. To make the theory relativistic, we must find the four-vector dot product that reduces to $\mathbf{p}\cdot\mathbf{q}$ in the c.m. system. (This requirement is unique.) To construct this, define the four-vectors

$$P=p-p\cdot KK/K^2,\quad Q=q-q\cdot KK/K^2. \tag{32}$$

Since in the c.m. system K has only a fourth component, P and Q reduce to $\mathbf{p}$ and $\mathbf{q}$ in that frame. Therefore the required Lorentz scalar is given by $-P\cdot Q$, which is $\mathbf{p}\cdot\mathbf{q}$ for $\mathbf{K}=0$.

We shall show that $-P\cdot Q$ can be written as a three-dimensional dot product in all frames.[10] This fact will greatly simplify the spin analysis, since it will reduce it to doing essentially what one does nonrelativistically, but Lorentz invariance is maintained. The fact that one can do this is related to the well-known fact that these are only three independent components of spin 1. In our terms the subsidiary condition takes the form of the identity

$$K\cdot P=K\cdot Q=0. \tag{33}$$

Therefore we have

$$P_0=\mathbf{K}\cdot\mathbf{P}/K_0,\quad Q_0=\mathbf{K}\cdot\mathbf{Q}/K_0. \tag{34}$$

Hence

$$\begin{aligned} P\cdot Q &= P_0Q_0-\mathbf{P}\cdot\mathbf{Q} \\ &= \mathbf{K}\cdot\mathbf{P}\mathbf{Q}\cdot\mathbf{K}/K_0^2-\mathbf{P}\cdot\mathbf{Q}. \end{aligned} \tag{35}$$

This can be written

$$P\cdot Q=-(\mathbf{P}-\alpha^{-1}\mathbf{P}\cdot\mathbf{K}\mathbf{K})\cdot(\mathbf{Q}-\alpha^{-1}\mathbf{Q}\cdot\mathbf{K}\mathbf{K}), \tag{36}$$

with α determined so that Eqs. (35) and (36) agree. This gives

$$\alpha=K_0(K_0+W),\quad W=\sqrt{s}. \tag{37}$$

Thus we have

$$-P\cdot Q=\mathbf{v}_p\cdot\mathbf{v}_q, \tag{38}$$

with

$$\mathbf{v}_p=\mathbf{P}-\mathbf{P}\cdot\mathbf{K}\mathbf{K}/K_0(K_0+W). \tag{39}$$

The expression $\mathbf{v}_p\cdot\mathbf{v}_q$ is a three-dimensional dot product; it reduces to $\mathbf{p}\cdot\mathbf{q}$ in the c.m. system, but is *Lorentz-invariant.* Thus we can write for Eq. (31)

$$\langle p|T_{sep}^{(1)}(s)|q\rangle = f(p^2)\mathbf{v}_p\cdot\mathbf{v}_qf(q^2)/D(s) \tag{40}$$

in *all* frames.

The formalism described above for spin 1 is easily generalized to spin l. The separable potential scattering only in the lth wave is given by

$$\langle p|T_{sep}^{(l)}(s)|q\rangle = \sum_m \frac{f(p^2)Y_{lm}^*(\hat{v}_p)Y_{lm}(\hat{v}_q)f(q^2)}{D(s)}. \tag{41}$$

Since $\mathbf{v}_p$ and $\mathbf{v}_q$ reduce to $\mathbf{p}$ and $\mathbf{q}$ at $\mathbf{K}=0$, this clearly gives only l waves in the c.m. system; it remains only to show that it is Lorentz-invariant. From the addition theorem we have

$$P_l(\hat{v}_p\cdot\hat{v}_q)=\frac{4\pi}{2l+1}\sum_m Y_{lm}^*(\hat{v}_p)Y_{lm}(\hat{v}_q). \tag{42}$$

We also have from Eq. (38) that $\hat{v}_p\cdot\hat{v}_q=P\cdot Qv_pv_q$. It is easily seen that $\mathbf{v}_p\cdot\mathbf{v}_q=-P^2$ and $\mathbf{v}_q\cdot\mathbf{v}_q=-Q^2$, so that v_p and v_q are Lorentz scalars and therefore $\hat{v}_p\cdot\hat{v}_q$ is Lorentz-invariant.

Equation (41) allows us to construct the vertex $\langle q|\Gamma|K, lm\rangle$ for two spinless particles of relative four-momentum q and total momentum K forming a quasiparticle or isobar or bound state of spin l, with z component m, viz.,

$$\langle q|\Gamma|K,lm\rangle = f_l(q^2)Y_{lm}^*(\hat{v}_q). \tag{43}$$

If the quantization axis is taken along K, then m is the helicity. The vertex (43) can now be used directly in the Born term, D function, etc., in place of the scalar vertex v used in Sec. II. The unitary arguments connected with propagators, etc., given there remain the same. Thus the Born term corresponding to Eq. (25) for a particle of moment k' incident on a spin-l' quasiparticle with component m' and momentum $P-k'$ going to a particle of momentum k and a spin-l quasiparticle with projection m and momentum $P-k$ is

$$\langle k, P-k, lm|B(s)|k', P-k', l'm'\rangle = \frac{\langle P-k, lm|\Gamma|P-k-2k'\rangle\bar{W}\langle P-2k-k'|\Gamma|P-k', l'm'\rangle}{\omega_{k+k'}(\bar{W}^2-s)}. \tag{44}$$

In the three-body c.m. system ($\mathbf{P}=0$), one can project the initial and final orbital angular momenta $\lambda\mu$ and $\lambda'\mu'$, for example, by multiplying by $\langle\lambda\mu|\hat{k}\rangle$ and integrating over Ω_k. This will give the Born term in the representation

$$\langle k,\lambda\mu,lm|B(s)|k',\lambda'\mu',l'm'\rangle. \tag{45}$$

With Clebsch-Gordan coefficients one can pass to a

[10] A similar approach to higher spin has been given by C. Zemach, Phys. Rev. **140**, B97 (1965).

representation in which total J is specified:

$$\langle k,\lambda lJM|B(s)|k',\lambda'l'JM\rangle. \quad (46)$$

B will be diagonal in J and M and, of course, will not depend on M. In the J representation, and assuming that there is more than one quasiparticle isobar or bound state present, but only one in each state l, Eq. (29) can be written schematically as the following set of coupled equations:

$$\langle k,\lambda lJ|T(s)|k',\lambda'l'J\rangle = \langle k,\lambda lJ|B(s)|k',\lambda'l'J\rangle + \sum_{\lambda''l''}\int\frac{d^3k''}{(2\pi)^3} \times\frac{\langle k,\lambda lJ|B(s)|k'',\lambda''l''J\rangle\langle k'',\lambda''l''J|T(s)|k',\lambda'l'J\rangle}{2\omega_{k''}D_{l''}(\sigma_{k''})}. \quad (47)$$

IV. π-N SCATTERING

We now turn to the π-N problem. To do so, we must include spin-$\frac{1}{2}$ particles in our formalism. There is no essential difference from our previous spinless or integer-spin discussion. The prescription is to rationalize all Feynman denominators appearing and use the BS procedure discussed in Sec. II on the remaining scalar propagators. The Dirac matrices appearing in the numerators can be absorbed into the form factors and the Dirac algebra can be done independently of the BS procedure. In this first encounter with the π-N system we shall only study the pseudoscalar coupling of pions to nucleons and no π-π interaction. To obtain the $N\pi\rightleftarrows N$ vertices, consider the Feynman diagram of Fig. 4 corresponding to the amplitude[8]

$$b_{r,r'}(kp;k'p')=\bar{u}_r(p)\gamma_5 i(\mathbf{K}-M)^{-1}\gamma_5 u_{r'}(p'), \quad (48)$$

where

$$K=k+p=k'+p'$$

and r and r' are spinor indices. For the moment we suppress isotopic spin. We now insert a complete set of *positive*-energy spinors into Eq. (48). Including negative-energy states would not be commensurate with our earlier treatment of unitarity, i.e., the negative-energy intermediate states contain more than three particles. We then get for Eq. (48)

$$b_{r,r'}(kp;k'p') = \sum_t \frac{\bar{u}_r(p)\gamma_5 u_t(K)2Mi\bar{u}_t(K)\gamma_5 u_{r'}(p')}{s-M^2}, \quad (49)$$

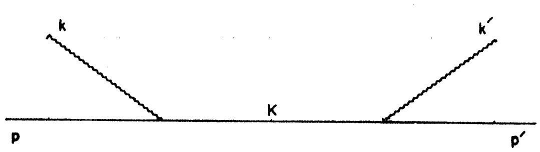

FIG. 4. Direct-nucleon-pole Feynman diagram for π-N scattering.

with $s=K^2$. It is easy to see that in the two-body c.m. system the amplitude defined above has only a $P_{1/2}$ projection. It is the "separable potential" for the π-N system. As for the case of integer spin in Sec. III, we now read off from it the rules for π-N scattering in our formalism:

(a) For a nucleon propagator we write

$$2Mi/(K^2-M^2), \quad (50)$$

where K is the nucleon four-momentum and M is the nucleon mass. Since $(K^2-M^2)^{-1}$ is the usual propagator for scalar particles, the work of Sec. II is changed for the propagator only by the appending of $2Mi$ to every nucleon line.

(b) At a vertex, shown schematically in Fig. 5, we write

$$\gamma\bar{u}_r(p+k)\gamma_5 u_s(p)v((p-k)^2), \quad (51)$$

where γ is the pion-nucleon coupling constant and v is a scalar vertex or cutoff function. γ^2 is related to the usual pion-nucleon pseudovector coupling constant f^2 by

$$f^2/4\pi=\gamma^2/48\pi M^2=0.0785. \quad (52)$$

(We take units in which $\hbar=c=$ pion mass$=1$ throughout.) The role of the vertex is to provide convergence

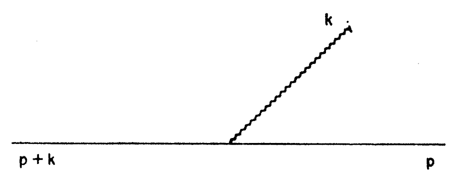

FIG. 5. Diagrammatic representation of the $\pi+N\rightleftarrows N$ vertex.

and to represent the structure of the $\pi N\rightleftarrows N$ vertex. It is related to the wave function of nonrelativistic mechanics. v must be normalized so that it is 1 at the nucleon pole, i.e., at $(p+k)^2=M^2$. Using the fact that we always have mass-shell δ functions in our equations, so that $p^2=M^2$ and $k^2=1$, we obtain

$$(p-k)^2=2+2M^2-s, \quad s=(p+k)^2. \quad (53)$$

Hence the normalization is $v(2+M^2)=1$. In treating the vertex it is convenient to introduce the quantities

$$t'=(t^2-M^2)/(1+M^2-2t), \quad t=\tfrac{1}{2}[(p-k)^2-M^2-1]. \quad (54)$$

t' reduces to $\mathbf{q}^2$ the relative three-momentum squared in the c.m. system. When $s=M^2$, $t'=(1/4M^2)-1$. We use two functional forms for the vertex: a Gaussian form

$$v_G((p-k)^2)=\exp[(-t'-1+1/4M^2)/2\beta^2] \quad (55)$$

and a Yamaguchi[11] form

$$v_Y((p-k)^2)=(\beta^2-1+1/4M^2)/(\beta^2+t'^2). \quad (56)$$

In each case β is a cutoff parameter.

[11] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

We are now in a position to write the equation for π-N scattering, which is represented schematically in Fig. 6, where momenta are also defined. This equation, in analogy with Eq. (29), is

$$\langle k,r|T^{(T)}(s)|k',r'\rangle = i\langle k,r|B^{(T)}(s)|k',r'\rangle + \sum_{r''}\int\frac{d^3q}{(2\pi)^3} \times\frac{\langle k,r|B^{(T)}(s)|q,r''\rangle\langle q,r''|T^{(T)}(s)|k',r'\rangle}{2\omega_q D(\sigma_q)}. \quad (57)$$

The superscript T is the isotopic spin that can be $\frac{3}{2}$ or $\frac{1}{2}$. The r's are spinor indices. First, we shall show how to construct the Born term B and return later to the propagator. We have

$$\langle k,r|B^{(T)}(s)|k',r'\rangle = \sum_t C_T\gamma^2 v((P-k-2k')^2)\bar{u}_r(P-k)\gamma_5 u_t(P-k-k') \times J(k,k',s)\bar{u}_t(P-k-k') \times\gamma_5 u_r(P-k')v((P-k'-2k)^2), \quad (58)$$

where C_T is an isotopic-spin factor, $C_{3/2}=\frac{2}{3}$, $C_{1/2}=-\frac{1}{3}$.

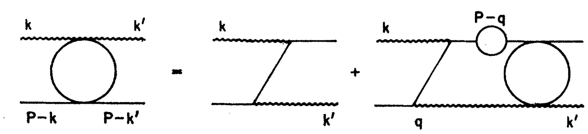

FIG. 6. Diagrammatic representation of Eq. (57).

These factors are obtained in this case just as in the static-model calculation,[7] for example, by Clebsch-Gordan algebra. In the three-body c.m. system the propagator J is defined by

$$J(k,q,s)=\frac{2Mi(E_{\mathbf{k}+\mathbf{q}}+\omega_k+\omega_q)}{E_{\mathbf{k}+\mathbf{q}}[s-(E_{\mathbf{k}+\mathbf{q}}+\omega_k+\omega_q)^2]}, \quad (59)$$

where

$$E_{\mathbf{k}+\mathbf{q}}=[(\mathbf{k}+\mathbf{q})^2+M^2]^{1/2},\quad \omega_k=(\mathbf{k}^2+1)^{1/2}.$$

We can perform the spin sum in Eq. (58) and reduce it to an expression between two-spinors χ. We get in the three-body c.m. system

$$\sum_t \bar{u}_r(P-k)\gamma_5 u_t(P-k-k')\bar{u}_t(P-k-k')\gamma_s u_{r'}(P-k') = \chi^\dagger\mathcal{O}\chi_{r'}, \quad (60)$$

with

$$\mathcal{O}=-N_kN_{\mathbf{k}+\mathbf{k}'}{}^2N_{k'}\left(\frac{\sigma\cdot(\mathbf{k}+\mathbf{k}')}{E_{\mathbf{k}+\mathbf{k}'}+M}-\frac{\sigma\cdot\mathbf{k}}{E_k+M}\right) \times\left(\frac{\sigma\cdot(\mathbf{k}+\mathbf{k}')}{E_{\mathbf{k}+\mathbf{k}'}+M}-\frac{\sigma\cdot\mathbf{k}'}{E_{k'}+M}\right), \quad (61)$$

where the spinor normalizations are, for example,

$$N_k=[(E_k+M)/2M]^{1/2}.$$

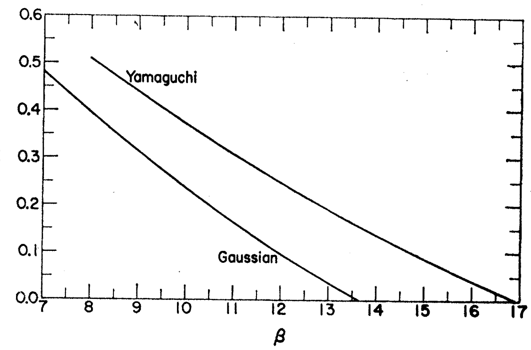

FIG. 7. Wave-function renormalization constant Z as a function of the cutoff parameter β for the two cutoff forms, Gaussian and Yamaguchi.

From Eq. (61) it is straightforward to put B in the standard form in the three-body c.m. system:

$$i\langle k,r|B^{(T)}(s)|k',r'\rangle=\chi_r^\dagger[E^{(T)}(\mathbf{k},\mathbf{k}',s) + i\sigma\cdot\mathbf{k}\times\mathbf{k}'F^{(T)}(\mathbf{k},\mathbf{k}',s)]\chi_{r'}. \quad (62)$$

The propagator in Eq. (57) takes the form

$$D(\sigma)=\frac{\sigma-M^2}{2Mi}\left(1-(\sigma-M^2)\frac{M^2\gamma^2}{2\pi^2} \times\int_0^\infty\frac{dk\,\rho(k)}{(\sigma-x)(M^2-x)^2}\right), \quad (63)$$

where

$$x=(E_k+\omega_k)^2$$

and

$$\rho(k)=k^4v^2\sqrt{x}/E_k\omega_k(E_k+M).$$

The argument of the vertex v corresponds to k being the magnitude of the c.m. three-momentum of each of the particles in the self-energy loop. It can be constructed from Eq. (54). The condition that the propatator of Eq. (63) have no ghost zeros is

$$Z=1-\frac{M\gamma^2}{2\pi^2}\int_0^\infty\frac{dk\,\rho(k)}{(M^2-x)^2}\geq 0. \quad (64)$$

This defines the nucleon wave-function renormalization constant Z in the "bubble" approximation. For fixed γ^2 the constraint on Z is a condition on the cutoff. For the forms given in Eqs. (55) and (56), this is a constraint on the parameter β. In Fig. 7 we plot Z versus β for the Gaussian and Yamaguchi cutoff forms. We shall be working near $Z=0$, and in both cases we see that this gives a β considerably larger than the masses in the problem, which presumably is a good thing. We shall find that Z is a rather good reflection of the strength of the interaction and the two forms seem to give about the same answers for the same Z.

To solve Eq. (57) we make a partial-wave decomposition. Since the nucleon has spin $\frac{1}{2}$ and parity is conserved, the orbital angular momentum is a good quantum number, and hence it is convenient to do the analysis in an L-S scheme as outlined in Sec. III rather than using helicity. To do this we must decompose the Born term. We begin with the first term of Eq. (62). Taking the spin matrix elements, we have

$$\langle \mathbf{k},r|E^{(T)}(\mathbf{k},\mathbf{k}',s)|\mathbf{k}',r'\rangle=\delta_{r,r'}E^{(T)}(\mathbf{k},\mathbf{k}',s). \quad (65)$$

Since E is a scalar function, we can write

$$E^{(T)}(\mathbf{k},\mathbf{k}',s)=\sum_{\lambda,\mu} E_\lambda{}^{(T)}(k,k',s)Y_{\lambda\mu}(\hat{k})Y_{\lambda\mu}{}^*(\hat{k}'). \quad (66)$$

Projecting the orbital angular momentum L with component M, we have

$$\langle k,LMr|E^{(T)}(\mathbf{k},\mathbf{k}',s)|k',L'M'r'\rangle =\delta_{r,r'}\delta_{L,L'}\delta_{M,M'}E_L{}^{(T)}(k,k',s). \quad (67)$$

Now we use Clebsch-Gordan coefficients to construct a state of definite J and obtain

$$\langle k,LJM_J|E^{(T)}(\mathbf{k},\mathbf{k}',s)|k',L'J'M_J'\rangle =\delta_{L,L'}\delta_{J,J'}E_L{}^{(T)}(k,k',s). \quad (68)$$

This is really trivial, but we present it to define notation and procedure. Now we do the same for the F term. The Racah algebra is more complicated, but the spirit is the same. We obtain, finally,

$$\langle k, LJM_J|i\sigma\cdot\mathbf{k}\times\mathbf{k}'F^{(T)}(\mathbf{k},\mathbf{k}',s)|k', L'J'M_J'\rangle =\delta_{J,J'}\delta_{M_J,M_J'}\delta_{L,L'}\sum_\lambda 6kk'F_\lambda{}^{(T)}(k,k',s)(2\lambda+1) \times|\langle 10\lambda0|1\lambda L0\rangle|^2(-1)^{L+\frac{1}{2}-J}W(LL11;1\lambda) \times W(\tfrac{1}{2}\tfrac{1}{2}LL;1J), \quad (69)$$

where F_λ is defined like E_λ in Eq. (66), W is the usual Racah coefficient, and $\langle \mathbf{j}_1m_1\mathbf{j}_2m_2|\mathbf{j}_1\mathbf{j}_2\,\mathbf{j}_1+\mathbf{j}_2\,m_1+m_2\rangle$ is a Clebsch-Gordan coefficient. With the projection made, the equation to be solved becomes

$$\langle k, LJ|T^{(T)}(s)|k', LJ\rangle =\langle k, LJ|B^{(T)}(s)|k', LJ\rangle+\int \frac{q^2dq}{(2\pi)^3} \times\frac{\langle k, LJ|B(s)|q, LJ\rangle\langle q, LJ|T^{(T)}(s)|k', LJ\rangle}{2\omega_qD(\sigma_q)}, \quad (70)$$

where $\langle k,LJ|B^{(T)}(s)|k',LJ\rangle$ is just the sum of Eqs. (68) and (69).

The integral equation (70) can now be turned into a matrix equation by writing the integral as a sum. To avoid difficulties introduced by singularities of the kernel, we use the contour-deformation technique of Hetherington and Schick.[12] In this technique one makes an analytic continuation of the equation onto a path in the complex plane far from the kernel singularities, obtains the amplitude along this path, and then uses the equation with only the path of integration deformed to obtain answers for real momenta. This last step involves making some hopeful assumptions about analytic properties of off-shell amplitudes that seem to be correct but have not been proven.

We now present the results. The most striking feature of the low-energy π-N system is the resonance in the (3,3) channel; hence we begin there. Our calculated phase shift for the (3,3) channel plotted against the total c.m. energy is shown in Figs. 8(a) and

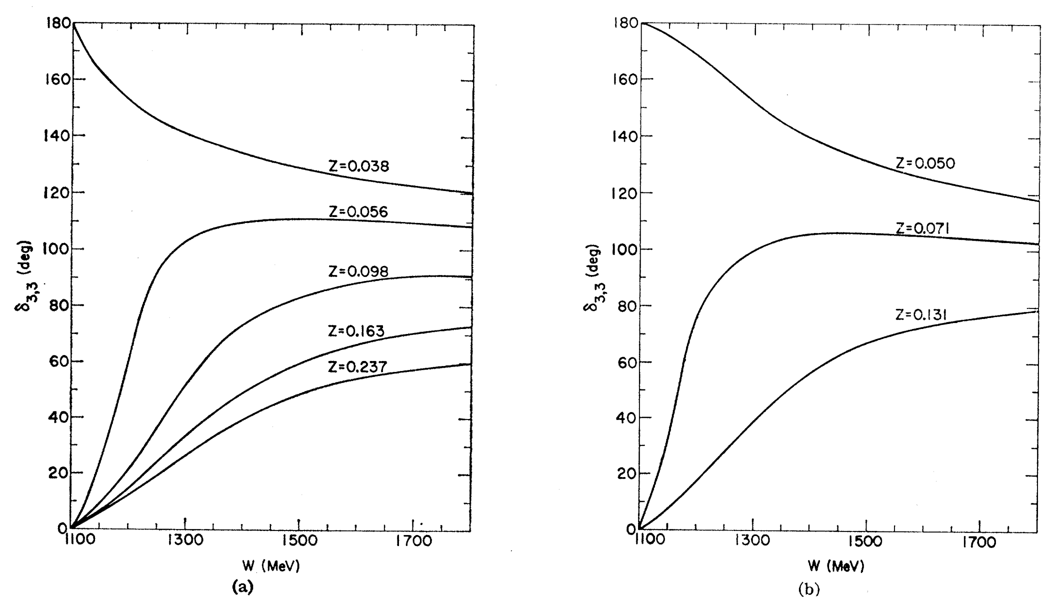

FIG. 8. The $P_{3,3}$ π-N phase shift plotted against total c.m. energy in MeV for various choices of Z, (a) with a Gaussian cutoff, (b) with a Yamaguchi cutoff.

[12] J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935 (1965).

8(b). The first set refers to the Gaussian cutoff form and the second to the Yamaguchi form. Various choices of Z are shown, which via Fig. 7 can be converted to the cutoff parameter β. Figure 8 shows that it is certainly possible to get a resonance around the right place (1238 MeV). On the other hand, the resonant phase shift turns over too fast. This is a common failing of most calculations that produce the (3,3). We also see that the position and very existence of a resonance is a sensitive function of Z, and hence of the cutoff parameter, but *not* of the cutoff form. This is reassuring and indicates that existence of the resonance depends strongly on the "strength of the force," but not critically on its detailed functional form. Note that increasing the coupling past the point required for resonance produces a (3,3) bound state. The inelasticity in this channel, and in all other channels that we have studied, is extremely small. η [$\equiv \exp(-2\ \mathrm{Im}\delta)$] is never less than 0.95 and usually even closer to 1. This is not too surprising. In a theory with Z near zero, the nucleon is nearly a pure π-N bound state. The only way to break this state up in the absence of a π-π interaction is to knock the nucleon out from under the π, but this is difficult to do, since the nucleon is so massive. Introducing a π-π force and other channels will certainly increase the inelasticity. What effect these additions will have on the 3-3 elastic phase remains to be seen.

The other important phase shifts are shown in Figs. 9–11. They are calculated for $Z=0.056$, which gives a reasonable (3,3) resonance, and for a Gaussian cutoff form. Presumably the form of the cutoff is again not important. The phase shifts are all small and uninteresting. This is a reflection of the fact that the nucleon-exchange Born term is only really large in the (3,3) channel. Experimentally there is considerable structure in the $P_{1,1}$, $D_{1,3}$, and $F_{3,5}$ channels.[13] For the other phases we have generally the correct sign, although certainly not the magnitude. The sign probably arises from the fact that nucleon exchange is the longest-range component of the π-N "force." We do not expect the $P_{1,1}$ channel to come out right, since we have left out the nucleon direct pole as well as other mechanisms. In principle, the inclusion of the direct pole in a consistent and unitary way does not constitute a problem.

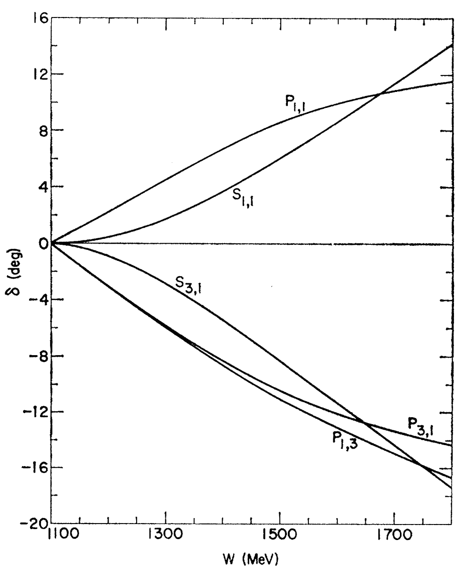

FIG. 9. The S- and P-wave π-N phase shifts plotted against total c.m. energy in MeV for $Z=0.056$ and Gaussian cutoff form. (Note phase scale.)

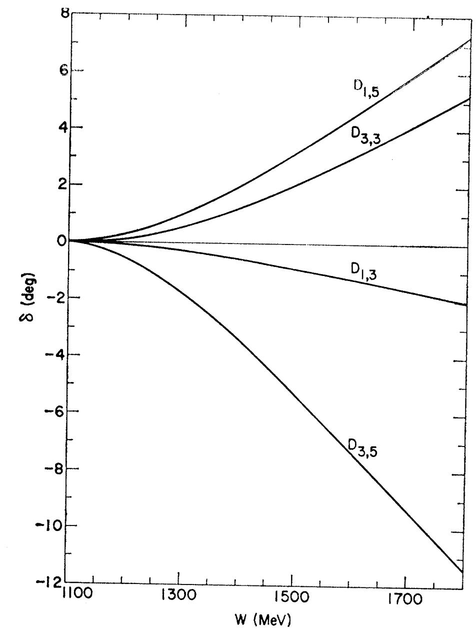

FIG. 10. The D-wave π-N phase shifts plotted against total c.m. energy in MeV for $Z=0.056$ and Gaussian cutoff form. (Note phase scale.)

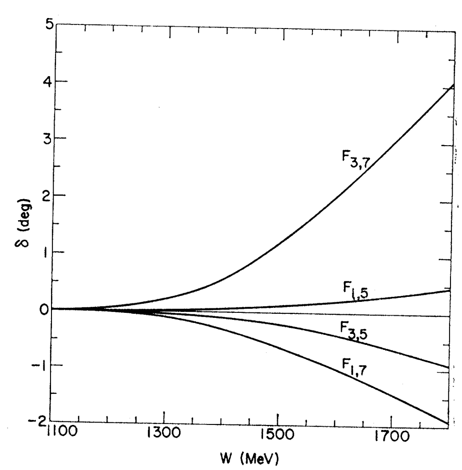

FIG. 11. The F-wave π-N phase shifts plotted against total c.m. energy in MeV for $Z=0.056$ and Gaussian cutoff form. (Note phase scale.)

[13] Cf. A Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Letters **26B**, 161 (1968).

TABLE I. The (3,3) amplitude and phase shift as a function of nucleon mass for meson energy $\omega=2.5$ and a Gaussian cutoff form with $\beta^2=31.5$.

M (muon mass units)	ReT	ImT	δ (deg)
70	−6.40	−10.3	58.3
200	−5.39	−11.4	64.8
500	−4.97	−11.8	67.1
1000	−4.86	−11.9	67.7
∞	−4.70	−11.9	68.5

As mentioned above, the inelasticity of all the channels calculated comes out very small. It is clear from the nature of the nucleon-exchange Born term that any interesting behavior in any channel except (3,3) requires more mechanisms. Whether the ones that we plan to include will be enough remains to be seen.

Both in order to check our calculation against the previous static calculation by one of us[7] and to get insight into the rate of convergence to the static result, we calculated for one fixed set of parameters as a function of the nucleon mass. First, note that the static answer is qualitatively different from the nonstatic. As we saw, we can get a (3,3) resonance and can, in fact, get a bound state with $Z>0$. In the static calculation no value of Z gave a (3,3) resonance and it was necessary to introduce more mechanisms to obtain it. Presumably those same mechanisms will be attractive in our nonstatic case and will allow us to reduce the coupling and to keep the resonance fixed. We see then that the introduction of recoil somehow increases the "force." An investigation of the rate of convergence of the static model is shown in Table I. There we show the real and imaginary parts of the T matrix, and the phase shift for the (3,3) channel with meson energy $\omega=2.5$ and a Gaussian cutoff for $\beta=5.61$. This gives $Z=0$ for the static limit and Z slightly greater than zero for finite but very large M. We see that the convergence is amazingly slow. The usual notion is that the static limit is good to order (meson mass)/(nucleon mass). This number is 1.4% for $M=70$, yet the static answers differ by more than that.

V. DISCUSSION

We have seen that by combining the isobar or quasiparticle idea with unitarity and Lorentz invariance, it is possible to construct a set of linear integral equations in one variable for the partial waves in the scattering of one particle from a correlated pair or bound state. The equation treats two- and three-body unitarity fully and does not contain any partial contributions from higher multiparticle sectors. In that sense it is a kind of relativistic Lippmann-Schwinger equation. It may be that this treatment of the many-particle states is preferable to the more usual Bethe-Salpeter equation with truncated kernel in which some parts of the higher multiparticle states appear but not others. Such a treatment violates unitarity above the threshold for these states. Even below these thresholds the incorrect virtual effects of these states as given by the Bethe-Salpeter equation may be misleading. In any case, the fact that our equation can be reduced to one dimension is certainly a technical advantage over the Bethe-Salpeter equation.

Integral higher-spin quasiparticles are easily introduced in the formalism in nearly precise analogy with the nonrelativistic methods. Lorentz-invariant separable interactions scattering in only one partial wave are constructed in terms of a certain three-vector and Y_{lm}'s of that three-vector. That makes the partial-wave decomposition of the three-body equations particularly simple. The equations are easily extended and solved for the π-N system with pseudoscalar $N+\pi\rightleftarrows N$ coupling. They then become equations for π-N scattering with nucleon exchange and π-π-N intermediate states, but with no π-π interactions and only a π-N interaction in the nucleon or (1,1) channel. As one would expect, this leads to a resonant (3,3) phase with all other phases small. It also gives almost no inelasticity. This is disappointing in terms of the rich and complex physics of the actual π-N system but reassuring in terms of the meager physics of the input assumptions. It gives us confidence that the equations respond in a reasonable way to the input, and therefore we can hope that by adding new mechanisms we can test the consequences of them and hopefully even fit the physics. Among the most important missing features that we hope to include are π-π interactions, at least in the ρ channel and perhaps in the s wave, and the π-N^* intermediate states. That is, we hope to include a $\pi+N\rightleftarrows N^*$ quasiparticle. In addition, in the (1,1) channel we must put in the direct nucleon pole. These added mechanisms should go a long way toward improving the fit. They will certainly greatly increase the inelasticity, and therefore we shall be able to study π production. In particular, we shall be able to study the effects on $\pi\pi N$ final states of both a π-N and π-π resonant final-state interaction. This subject of overlapping resonances is very poorly understood theoretically and phenomenologically. We believe that any light that a soluble *unitary* example can shed on the question will be valuable, even if the theory may not be good enough to account precisely for the data.