The field equations for the ρ field obtained from Eqs. (2.1) read

$$_{\rho}K^{\mu}{}_{\nu}\, v^{\nu}{}_{a}=S^{\mu}{}_{a}\,, \tag{A5}$$

where $_{\rho}K^{\mu}{}_{\nu}$ is the ρ Proca operator and the relevant part of $S^{\mu}{}_{a}$ is given by

$$S^{\mu}{}_{a}=\epsilon_{abc}[g_{\pi\rho A}\,\varphi_b a^{\mu}{}_c+\lambda_{\pi\rho A}\,\varphi_{\lambda b}H^{\mu\lambda}{}_c]-2\epsilon_{abc}\partial_\nu \times[\mu_{\pi\rho A}\,\varphi_b H^{\mu\nu}{}_c-\tfrac{1}{2}\bar{\lambda}_{\pi\rho A}(a^{\mu}{}_b\varphi^{\nu}{}_c-a^{\nu}{}_b\varphi^{\mu}{}_c)]. \tag{A6}$$

The matrix element of Eq. (A4) thus becomes

$$\langle p_2,q|v^{\mu}{}_3(0)|p_1\rangle={}_{\rho}\Delta^{\mu}{}_{\alpha}(k)\langle p_2;q|S^{\alpha}{}_a(0)|p_1\rangle, \tag{A7}$$

where $_{\rho}\Delta^{\mu}{}_{\alpha}(k)$ is the ρ propagator,

$$_{\rho}\Delta^{\mu}{}_{\alpha}\equiv(k^2+m_\rho{}^2)^{-1}(\delta^{\mu}{}_a+k^{\mu}k_{\alpha}m_{\rho}{}^{-2}). \tag{A8}$$

We now make the peripheral approximation by replacing the A_1 fields $a^{\mu}{}_a$ and $H^{\mu\nu}{}_a$ by their free out-fields. They then annihilate the A_1 in the out state yielding

$$S=i(2\pi)^4\delta^4(p_1+k-q-p_2)N_kN_q e g_\rho m_\rho{}^{-2} \times(\lambda_{\pi\rho A}-2\mu_{\pi\rho A}-\bar{\lambda}_{\pi\rho A})\epsilon_\nu{}^{\sigma\prime *}(q)\epsilon_\mu{}^{\sigma}(k) \times(q^\mu k^\nu-kqg^{\mu\nu})\epsilon_{ab3}\langle p_2\sigma_2\tau_2|\,\varphi_b(0)\,|p_1\sigma_1\tau_1\rangle. \tag{A9}$$

We note that Eq. (A9) is gauge-invariant as S vanishes when $\epsilon_\mu{}^{\sigma}(k)$ is replaced by k_μ. (This is in contrast to the photoproduction of the pion where the peripheral diagram itself is *not* gauge-invariant.) The pion matrix element of Eq. (A9) is defined in Ref. 21. The remainder of the calculation of the cross section of Eq. (4.8) is now straightforward.

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Analytic Continuation of an Amplitude*

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A general procedure is given for successively continuing the invariant amplitude to arbitrary regions of the z plane when the partial-wave amplitude is given explicitly by the most general modified Cheng representation. The partial-wave expansion is summed in terms of elementary functions or integrals thereof, and no dispersion or background integrals are required.

I. INTRODUCTION

ATTEMPTS to calculate strongly-interacting particle cross-sections self-consistently, require knowledge of the invariant amplitude in unphysical regions of the dynamical variables of energy squared (s) and momentum transfer squared (t), since only in these regions are the crossing relations nonempty.[1] However, if the invariant amplitude is defined in terms of a partial-wave expansion over various orbital angular-momentum states, as it usually is, such an expansion will in general only converge in a finite region of the s-t plane, and the problem therefore revolves around how one can make analytic continuations of such an expansion.

In several previous reports, a representation for the two-body, single-channel, partial-wave S-matrix element was studied and found to withstand, quite well, a number of tests to which it was subjected.[2–6] It has also been written for multichannel reactions, although no numerical comparisons have yet been made in this case.[7] It should be noted that while a comparison of a given conjectured partial-wave S-matrix with exact potential-theory results is negative, in the sense that a favorable comparison would clearly not necessarily imply the representation to be a valid relativistic one, an unfavorable comparison can at least be used to exclude many representations, since intuitively we expect any conjectured relativistic S-matrix element to also be valid in a "correspondence principle" or nonrelativistic limit.

Although the Regge method,[8] of rewriting the partial-wave expansion as a background integral plus pole terms in the angular-momentum plane, provides an analytic continuation of the invariant amplitude to all t in principle, in practice if one wishes to satisfy the crossing relations in threshold and intermediate regions of the s-t plane, it then becomes necessary to evaluate the background integral explicitly, and this is a formidable task because of the poor convergence properties of this integral.

The purpose of this report is to show that it is possible, however, to avoid the above difficulties with the background integral when one uses explicitly the modi-

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[3] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **141**, 1513 (1966).
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[5] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **154**, 1515 (1967).
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[8] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).

fied Cheng representation for the partial-wave S matrix. In the next section we outline a general procedure whereby the partial-wave expansion can be analytically continued to successively larger regions of the s-t plane, with any order being obtainable in principle. In Sec. III we give the mathematical details and derive several relevant formulas in the Appendix.

II. OUTLINE OF PROCEDURE

The modified Cheng representation for the partial-wave S-matrix may be written in its most general form as[3]

$$\ln S(l,\nu)\equiv 2i\delta_l(\nu)=\sum_{n=1}^{\infty}\left(\int_{\alpha_n}^{\alpha_n^*}\frac{\exp[(l'-l)\bar{\xi}]}{l'-l}dl' -\frac{ig^2}{\nu^p}\frac{\exp[-(l-c+n)\bar{\xi}]}{l-c+n}\int_\mu^\infty P_{n-1}(\cosh\xi')\sigma(\mu')d\mu'\right) +\frac{ig^2}{\nu^p}\int_\mu^\infty Q_{l-c}(\cosh\xi')\sigma(\mu')d\mu', \quad (1)$$

where $\sigma(\mu')$ arises from the potential being written as a superposition of Yukawas

$$rV(r)=g^2\int_\mu^\infty \sigma(\mu')e^{-\mu' r}d\mu', \quad (2)$$

ξ and $\bar{\xi}$ are defined by

$$\cosh\xi=1+(\mu^2/2\nu) \quad (3a)$$

$$\cosh\bar{\xi}=1+(2\mu)^2/2\nu. \quad (3b)$$

The parameters p and c arise from assuming that the trajectories have the asymptotic form

$$\alpha_n(\nu)\underset{\nu\to\infty}{\longrightarrow} -n+c+\frac{ig^2}{2\nu^p}\int_\mu^\infty P_{n-1}(\cosh\xi')\sigma(\mu')d\mu' \qquad n=1,2,3,\cdots. \quad (4)$$

As discussed in Ref. 3, the threshold behavior of the phase shift will still be $\nu^{l+\frac{1}{2}}$ if the trajectory parameters p and c satisfy

$$p\leqslant \tfrac{1}{2}-c. \quad (5)$$

If the trajectories turn around, we expect $p>0$ and therefore $c<\frac{1}{2}$. In potential theory, for a simple Yukawa potential, $p=\frac{1}{2}$ and $c=0$.

Finally, in (1) l is the usual angular momentum and ν is the center-of-mass momentum squared; the $\alpha_n(\nu)$ are the Regge poles. The representation (1) has the important feature that

$$S(l,\nu)S^*(l,\nu)=1 \quad (6)$$

independent of the number of terms retained in the sum over trajectories.

The partial-wave expansion then yields the scattering amplitude

$$(\sqrt{\nu})f(\nu,z)=\sum_{l=0}^{\infty}(2l+1)\left[\frac{S(l,\nu)-1}{2i}\right]P_l(z), \quad (7)$$

where $z=\cos\vartheta=1+t/2\nu$, ϑ being the c.m. scattering angle and t the momentum transfer squared. Now the S matrix given by the representation (1) has the asymptotic behavior in the right-half l plane

$$S(l,\nu)\underset{\substack{l\to\infty\\ \mathrm{Re}\,l>-\frac{1}{2}}}{\longrightarrow} 1+O(e^{-l\xi}/\sqrt{l}). \quad (8)$$

Since the Legendre function appearing in the expansion (7) behaves as

$$P_l\left(1+\frac{t}{2\nu}\right)\underset{l\to\infty}{\longrightarrow} O(e^{+l\eta}/\sqrt{l}), \quad (9)$$

where $\cosh\eta=1+t/2\nu$, we see that the expansion (7) converges for $t<t_0=\mu^2$ or, in terms of ϑ, $2\nu(\cos\vartheta-1)<\mu^2$; in the ϑ plane this is the region

$$-\pi<\mathrm{Re}\vartheta<\pi \quad |\mathrm{Im}\vartheta|<\xi, \quad (10)$$

which maps into the Lehmann ellipse[9] in the z plane:

$$\left(\frac{\mathrm{Re}z}{\cosh\xi}\right)^2+\left(\frac{\mathrm{Im}z}{\sinh\xi}\right)^2=1. \quad (11)$$

However, the scattering amplitude $f(\nu,z)$ in (7) may be analytically continued to successively larger regions in t by the following procedure; suppose we add and subtract to the expansion (7) N terms of the exponential expansion of the S matrix in powers of $2i\delta_l(\nu)$

$$(\sqrt{\nu})f(\nu,z)=\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\left\{[S_l(\nu)-1]-\left[2i\delta_l+\frac{(2i\delta_l)^2}{2!}+\cdots+\frac{(2i\delta_l)^N}{N!}\right]\right\}P_l(z)+\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1) \times\left(2i\delta_l+\frac{(2i\delta_l)^2}{2!}+\cdots+\frac{(2i\delta_l)^N}{N!}\right)P_l(z) \quad (12a)$$

$$=\sqrt{\nu}\sum_{l=0}^{\infty}(2l+1)\bar{f}_l^N(\nu)P_l(z) +\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\left(2i\delta_l+\frac{(2i\delta_l)^2}{2!}+\cdots+\frac{(2i\delta_l)^N}{N!}\right)P_l(z), \quad (12b)$$

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[9] H. Lehmann, Nuovo Cimento **10**, 579 (1958).

where

$$\bar{f}_l^N(\nu)=f_l(\nu)-\frac{1}{2i\sqrt{\nu}}\times\left(2i\delta_l+\frac{(2i\delta_l)^2}{2!}+\cdots+\frac{(2i\delta_l)^N}{N!}\right), \quad (13)$$

and $N=1, 2, 3, \cdots$. As we shall see explicitly in the next section, when the phase shift $\delta_l(\nu)$ is given by the modified Cheng representation (1), the sums over l of successively higher powers $(2i\delta_l)^N/N!$ in the second sum of (12b) can be performed for all z to any order N, so that the region of validity for $f(\nu,z)$ is then determined by the region of convergence of the first sum of (12b); namely, by

$$\sum_{l=0}^{\infty}(2l+1)\bar{f}_l^N(\nu)P_l(z). \quad (14)$$

However, for successively higher values of $N=1, 2, 3, \cdots$, the asymptotic behavior of the Legendre function (9) combined with the explicit behavior of the modified Cheng representation from (1), then yields larger regions of convergence for (14). For example, for $N=1$,

$$\bar{f}_l^{N=1}(\nu)\underset{l\to\infty}{\longrightarrow}e^{-2l\xi} \quad (15)$$

and (14) is valid for

$$t<t_1=4\mu^2+(\mu^4/\nu). \quad (16)$$

For $N=2$,

$$\bar{f}_l^{N=2}(\nu)\underset{l\to\infty}{\longrightarrow}e^{-3l\xi} \quad (17)$$

and (14) is valid for

$$t<t_2=9\mu^2+\frac{6\mu^4}{\nu}+\frac{\mu^6}{\nu^2}, \quad (18)$$

etc. In this way it will be possible to work one's way steadily into larger regions of the s-t plane.

There are a number of advantages to this procedure:

(i) If the partial-wave S matrix is exact, the imaginary part of the amplitude across the t axis is exact in successively larger regions of t, depending on the order of the approximation. For example, $\mathrm{Im}_t f(\nu,t)$ is exact in

$$t_0<t<t_1 \quad \text{for} \quad N=1 \quad (19)$$

and

$$t_0<t<t_2 \quad \text{for} \quad N=2 \quad (20)$$

etc.

(ii) One may attempt to satisfy the crossing relations in threshold regions of the s-t plane where the representation (1) is believed to be most reliable. An iterative procedure is suggested wherein one optimally satisfies crossing in a small region, thereby using the parameters thus obtained as input for the next iteration, and so on.

(iii) We avoid having to perform the difficult background integral along the line $\mathrm{Re}l=-\frac{1}{2}$.

(iv) The procedure can be carried out (in principle) to any desired order.

The mathematical details will now be presented in Sec. III.

III. MATHEMATICAL DETAILS

The essential ideas can be illustrated by a one-trajectory approximation of (1)

$$2i\delta_l(\nu)=\int_\alpha^{\alpha^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'-\frac{i\bar{g}^2}{\nu^p}\frac{\exp[-(l-c+1)\xi]}{l-c+1}+\frac{ig^2}{\nu^p}\int_\mu^\infty Q_{l-c}(\cosh\xi')\sigma(\mu')d\mu', \quad (21)$$

where

$$\bar{g}^2=g^2\int_\mu^\infty\sigma(\mu')d\mu'. \quad (22)$$

We will require the generating function for Legendre polynomials[10]

$$\frac{1}{(\cosh x-z)^{1/2}}=\sqrt{2}\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})x}P_n(z), \quad (23)$$

the integral representation for the Q function[11]

$$Q_{l-c}(\cosh\xi)=\frac{1}{\sqrt{2}}\int_\xi^\infty\frac{e^{-(l-c+\frac{1}{2})y}}{(\cosh y-\cosh\xi)^{1/2}}dy; \quad (24)$$

the addition formula for the Q functions[12]

$$Q_{l-c}(y_1)Q_{l-c}(y_2)=\int_0^\infty Q_{l-c}\times[y_1y_2+(y_1^2-1)^{1/2}(y_2^2-1)^{1/2}\cosh\varphi]d\varphi, \quad (25)$$

along with the following two sums which will be performed explicitly in the Appendix. First,

$$U(c,z_0,z)\equiv\sum_{l=0}^{\infty}(2l+1)Q_{l-c}(z_0)P_l(z)=\frac{e^{c\xi}}{z_0-z}-\frac{c}{z_0-z}\int_\xi^\infty e^{cy}\left[\left(\frac{\cosh y-z_0}{\cosh y-z}\right)^{1/2}-1\right]dy, \quad (26\text{a})$$

[10] *Higher Transcendental Functions*, Bateman Manuscript Project, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 154, Eq. (33).

[11] *Higher Transcendental Functions*, Bateman Manuscript Project, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 155, Eq. (4).

[12] V. DeAlfaro, T. Regge, and C. Rossetti, Nuovo Cimento **26**, 1029 (1962).

where $z_0=\cosh\xi$, and second,

$$R(\alpha,\xi,z)\equiv\sum_{l=0}^{\infty}(2l+1)\frac{e^{-(l-\alpha)\xi}}{l-\alpha}P_l(z)$$
$$=\frac{\sqrt{2}e^{[\alpha+(1/2)]\xi}}{(\cosh\xi-z)^{1/2}}+(2\alpha+1)T(\alpha+1,\xi,z)\,, \tag{26b}$$

where, in (26b), $T(\alpha,\xi,z)$ is defined by

$$T(\alpha,\xi,z)=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{e^{[\alpha-(1/2)]x}}{(\cosh x-z)^{1/2}}dx,\quad \mathrm{Re}\alpha<1\,. \tag{27}$$

Although (27) is valid only for $\mathrm{Re}\alpha<1$, following a standard procedure given in the Appendix, it can be extended to

$$\mathrm{Re}\alpha<m+2,\quad m=0,\,1,\,2,\,\cdots. \tag{28}$$

The important point is that the right-hand sides of (26a) and (26b) can be continued to $\mathrm{Re}z>\cosh\xi$. The sum (26a) reduces to the usual pole $(z_0-z)^{-1}$ when $c=0$; the integral in (26a) converges for $c<1$ which will be adequate for our purposes since we only require $c<\frac{1}{2}$ as discussed above.

We therefore want to show that the sums

$$(\mathrm{Sum})_N=\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\times\left(2i\delta_l+\frac{(2i\delta_l)^2}{2!}+\cdots+\frac{(2i\delta_l)^N}{N!}\right)P_l(z) \tag{29}$$

for $N=1,\,2,\,3,\,\cdots$ can be performed for all z when the phase shift δ_l is defined by (21). Since there are three separate terms in $2i\delta_l$, there will be 3 sums to perform when $N=1$, 6 more when $N=2$, 10 more when $N=3$, etc., so we adopt a notation $(\mathrm{Sum})_{Nj}$ where N refers to the power of $2i\delta_l$ and j is the term index. In other words we write

$$(\mathrm{Sum})_N=\sum_{j=1}^{3}(\mathrm{Sum})_{1j}+\sum_{j=1}^{6}(\mathrm{Sum})_{2j}+\cdots. \tag{30}$$

For $N=1$ we have

$$(\mathrm{Sum})_{11}=\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\times\left(\int_{\alpha}^{\alpha^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'\right)P_l(z)\,. \tag{31}$$

Using (26b), (31) may be written

$$(\mathrm{Sum})_{11}=-\frac{1}{2i}\int_{\alpha}^{\alpha^*}R(l',\xi,z)dl'\,. \tag{32}$$

The second sum ($j=2$) for $N=1$ is easily performed similarly:

$$(\mathrm{Sum})_{12}=\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\left(-\frac{i\bar{g}^2}{\nu^p}\frac{\exp[-(l-c+1)\xi]}{l-c+1}\right)P_l(z)$$
$$=-\frac{\bar{g}^2}{2\nu^p}R(c-1,\,\xi,\,z)\,. \tag{33}$$

Finally, the third sum with $N=1$ is performed with the help of (26a);

$$(\mathrm{Sum})_{13}=\frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\times\left[\frac{ig^2}{\nu^p}\int_{\mu}^{\infty}Q_{l-c}(\cosh\xi')\sigma(\mu')d\mu'\right]P_l(z)$$
$$=\frac{g^2}{2\nu^p}\int_{\mu}^{\infty}U(c,\cosh\xi',z)\sigma(\mu')d\mu'\,, \tag{34}$$

where $\cosh\xi'=1+(\mu')^2/2\nu$.

For $N=2$, the expressions become more complicated, but can be performed in the same way

$$(\mathrm{Sum})_{21}=\frac{1}{2i}\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\left[\int_{\alpha}^{\alpha^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'\right]\times\left[\int_{\alpha}^{\alpha^*}\frac{\exp[(l''-l)\xi]}{l''-l}dl''\right]P_l(z)\,. \tag{35}$$

Writing

$$\frac{1}{l-l'}\frac{1}{l-l''}=\frac{1}{l''-l'}\left[\frac{1}{l-l''}-\frac{1}{l-l'}\right], \tag{36}$$

Eq. (35) may be written

$$(\mathrm{Sum})_{21}=\frac{1}{4i}\int_{\alpha}^{\alpha^*}dl''\int_{\alpha}^{\alpha^*}dl'\left(\frac{\exp[(l''-l')\xi]R(l',2\xi,z)}{l'-l''}-\frac{\exp[-(l''-l')\xi]R(l'',2\xi,z)}{l'-l''}\right)\,. \tag{37}$$

Defining the $N=2$, $j=2$ term to be

$$(\mathrm{Sum})_{22}=\frac{1}{2i}\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\times\left(-\frac{\bar{g}^4}{\nu^{2p}}\frac{\exp[-2(l-c+1)\xi]}{(l-c+1)^2}\right)P_l(z) \tag{38}$$

and following a procedure similar to that for the (21) sum, we have

$$(\mathrm{Sum})_{22}=-\frac{\bar{g}^4}{4i\nu^{2p}}\left(\frac{\partial R(x,2\xi,z)}{\partial x}-2\xi R(x,2\xi,z)\right)_{x=c-1}\,. \tag{39}$$

The remaining $N=2$ "diagonal" sum then becomes

$$(\mathrm{Sum})_{23}=\frac{1}{2i}\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\left(-\frac{g^4}{\nu^{2p}}\int_\mu^\infty d\mu'\int_\mu^\infty d\mu''\sigma(\mu')\sigma(\mu'')\times Q_{l-c}(\cosh\xi')Q_{l-c}(\cosh\xi'')\right)P_l(z). \quad (40)$$

With the help of (25) and (26a) this becomes

$$(\mathrm{Sum})_{23}=-\frac{g^4}{4i\nu^{2p}}\int_\mu^\infty d\mu''\int_\mu^\infty d\mu'\sigma(\mu'')\sigma(\mu')\times\left(\int_0^\infty d\varphi U(c,\cosh\xi''\cosh\xi'+\sinh\xi''\sinh\xi'\cosh\varphi,z)\right), \quad (41)$$

where, as above, $\cosh\xi'=1+(\mu')^2/2\nu$, $\cosh\xi''=1+(\mu'')^2/2\nu$.

The "off-diagonal" sums may be performed as follows:

$$(\mathrm{Sum})_{24}=(2)\left(\frac{1}{2i}\right)\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\left[\int_\alpha^{\alpha^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'\right]\times\left[-\frac{i\bar{g}^2}{\nu^p}\frac{\exp[-(l-c+1)\xi]}{l-c+1}\right]P_l(z)=\frac{\bar{g}^2}{2\nu^p}\int_\alpha^{\alpha^*}dl'\times\sum_{l=0}^{\infty}(2l+1)\frac{\exp[-(2l-l'-c+1)\xi]}{(l-l')(l-c+1)}P_l(z). \quad (42)$$

Again, using the split-up into partial fractions (36), Eq. (42) becomes

$$(\mathrm{Sum})_{24}=-\frac{\bar{g}^2}{2\nu^p}\int_\alpha^{\alpha^*}dl'\times\left[\frac{\exp[(l'-c+1)\xi]R(-1+c,2\xi,z)}{l'-c+1}-\frac{\exp[-(l'-c+1)\xi]R(l',2\xi,z)}{l'-c+1}\right]. \quad (43)$$

The next off-diagonal sum is

$$(\mathrm{Sum})_{25}=(2)\left(\frac{1}{2i}\right)\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\left\{\left[\int_\alpha^{\alpha^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'\right]\times\left[\frac{ig^2}{\nu^p}\int_\mu^\infty Q_{l-c}(\cosh\xi')\sigma(\mu')d\mu'\right]\right\}P_l(z). \quad (44)$$

Using the integral representation for the Q function given in (24), (44) becomes

$$(\mathrm{Sum})_{25}=-\frac{g^2}{2\sqrt{2}\nu^p}\int_\alpha^{\alpha^*}dl'\int_\mu^\infty d\mu'\sigma(\mu')\times\left[\int_{\xi'}^\infty dy\frac{e^{[c-(1/2)-l']y}R(l',\xi+y,z)}{[\cosh y-\cosh\xi']^{1/2}}\right] \quad (45)$$

valid for all z. The final sum for $N=2$ is

$$(\mathrm{Sum})_{26}=2\left(\frac{1}{2i}\right)\sum_{l=0}^{\infty}\frac{(2l+1)}{2!}\left[\bar{g}^2\frac{\exp[-(l-c+1)\xi]}{l-c+1}\right]\times\left[g^2\int_\mu^\infty\sigma(\mu')Q_{l-c}(\cosh\xi')d\mu'\right]P_l(z) \quad (46)$$

and after following a procedure as in deriving (45) we have

$$(\mathrm{Sum})_{26}=\frac{g^2\bar{g}^2}{2\sqrt{2}\nu^{2p}}\int_\mu^\infty d\mu'\sigma(\mu')\times\left[\int_{\xi'}^\infty dy\frac{e^{y/2}R(-1+c,\xi+y,z)}{(\cosh y-\cosh\xi')^{1/2}}\right]. \quad (47)$$

The expressions for $(\mathrm{Sum})_{Nj}$ above were obtained by freely interchanging the order of summation and integration in Eqs. (31), (34), (35), (37), (40), (42), (44), and (46). As is well known, the mathematical requirement for the validity of this operation is that the series must converge uniformly with respect to the integration variable.[13]

Let us consider (31) as an example. We need to prove that the series

$$F(l')=\sum_{l=0}^{\infty}u_l(l') \quad (48a)$$

with

$$u_l(l')=\frac{1}{2i}(2l+1)\frac{\exp[(l'-l)\xi]}{l'-l}P_l(z) \quad (48b)$$

converges uniformly with respect to l' with $l'\in[\alpha,\alpha^*]$. If we take

$$M_l=\frac{1}{2i}(2l+1)\frac{\exp[(\mathrm{Re}\alpha-l)\xi]}{\mathrm{Re}\alpha-l}P_l(z), \quad (49)$$

then

$$|u_l(l')|\leq M_l \quad (50)$$

for each $l'\in[\alpha,\alpha^*]$, and therefore by the Weierstrass comparison test[14]

$$\int_\alpha^{\alpha^*}F(l')dl'=\sum_{l=0}^{\infty}\int_\alpha^{\alpha^*}u_l(l')dl', \quad (51)$$

[13] For a discussion of this theorem, see for example, R. Creighton Buck, in *Advanced Calculus* (McGraw-Hill Book Co., New York, 1956), p. 133.

[14] R. Creighton Buck, in *Advanced Calculus* (McGraw-Hill Book Co., New York, 1956), p. 130.

since $\sum_{l=0}^{\infty} M_l$ converges. In a similar manner, one may justify the interchange of summation and integration in the remaining equations; however, the weight function $\sigma(\mu')$ must be chosen such that all those series above involving it, converge uniformly with respect to the integration variable μ'.

A further requirement on the weight function $\sigma(\mu')$ must be imposed since we require the improper integrals (34), (41), (45), and (47) to possess analytic continuations in the variable $z=\cos\vartheta$, and therefore by a similar theorem to that of Ref. 13, the weight function $\sigma(\mu')$ must be chosen such that these integrals converge uniformly with respect to z.[15]

While the demonstration so far has been made in a one-trajectory approximation to the modified Cheng representation (1), it is clear that if more trajectories are retained, similar expressions to those above result, and one may formally continue the procedure. The rate of convergence of the representation (1) in terms of trajectories has been studied fairly extensively by now.[2–6] In Refs. 2 and 6, the representation (1) was studied in a one-trajectory approximation and shown to compare favorably with the exact partial-wave S matrix resulting from a direct integration of the Schrödinger equation. In Ref. 3 a very good comparison of the total scattering amplitude in a one-trajectory approximation of (1) was observed, in the physical region $-1\leq z\leq 1$, where exact results are known. Moreover, even a one-trajectory approximation of (1) contains contributions from all the infinitely many trajectories through the appearance of the Q function in (1); the interested reader is referred to Refs. 2 and 3 and others cited there for details of how this mechanism arises.

These facts, combined with the further test of the representation in π-N phenomenology in Ref. 4, and the approximate calculation of the ρ-meson in Ref. 5, where a procedure identical to that described here (with $N=1$) was used, lead us to believe that the relatively simple procedure for continuing the representation, as presented above, may be valuable.

Following the procedure outlined above, we see that it is always possible to continue analytically the partial-wave expansion of any power of $(2i\delta_l)$ to all z, if the phase shift is given explicitly by the modified Cheng representation (1). Sums involving higher products of Q functions may always be reduced with (25) to a form (26a). Other sums are reduced to (26b) and its derivatives. The weight function $\sigma(\mu')$ must, however, be appropriately chosen so that the series and improper integrals noted above converge uniformly.

Finally, from Eqs. (32)–(34), we see that the sums $N=1$ have a cut in z starting at

$$z_0=\cosh\xi \quad \text{or} \quad t=t_0=\mu^2, \tag{52}$$

whereas the sums $N=2$ have cuts starting at

$$z_1=\cosh(2\xi) \quad \text{or} \quad t=t_1=4\mu^2+\mu^4/\nu. \tag{53}$$

Similarly, sums with larger values of N have cuts starting correspondingly farther out in the z plane, and therefore if the phase shift δ_l were exact, the discontinuity across the t axis would be exact in the respective regions

$$\begin{aligned} t_0<t<t_1, &\quad N=1,\\ t_0<t<t_2, &\quad N=2, \end{aligned} \tag{54}$$

etc., as noted above.

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APPENDIX A

We wish to verify the formula

$$U(c,z_0,z)\equiv\sum_{l=0}^{\infty}(2l+1)Q_{l-c}(z_0)P_l(z) = \frac{e^{c\xi}}{z_0-z}-\frac{c}{z_0-z}\int_\xi^\infty e^{cy}\left[\left(\frac{\cosh y-z_0}{\cosh y-z}\right)^{1/2}-1\right]dy, \tag{A1}$$

where $z_0=\cosh\xi$. The procedure is essentially the same that one follows in deriving the usual pole $(z_0-z)^{-1}$ when $c=0$. We use the recurrence formula, valid for both $P_\nu(z)$ and $Q_\nu(z)$[16]

$$(2\nu+1)zP_\nu(z)=(\nu+1)P_{\nu+1}(z)+\nu P_{\nu-1}(z). \tag{A2}$$

For $\nu=l$, we multiply (A2) by $Q_{l-c}(z_0)$ and obtain

$$(2l+1)zQ_{l-c}(z_0)P_l(z)=(l+1)Q_{l-c}(z_0)P_{l+1}(z)+lQ_{l-c}(z_0)P_{l-1}(z). \tag{A3}$$

For $\nu=l-c$, $P_\nu(z)$ replaced by $Q_{l-c}(z_0)$ in (A2), we then multiply by $P_l(z)$ and obtain

$$[2(l-c)+1]z_0Q_{l-c}(z_0)P_l(z)=(l-c+1)Q_{l-c+1}(z_0)P_l(z)+(l-c)Q_{l-c-1}(z_0)P_l(z). \tag{A4}$$

Subtracting respectively (A4) from (A3) and summing on l, we have after some algebra

$$U(c,z_0,z)=-\frac{c}{z_0-z}\sum_{l=0}^{\infty}[Q_{l-c+1}(z_0)-2z_0Q_{l-c}(z_0)+Q_{l-c-1}(z_0)]P_l(z). \tag{A5}$$

Using the integral representation for the Q function (24) and the generating function for the P function (23), (A5) may be written after a little more algebra as

$$U(c,z_0,z)=\frac{e^{c\xi}}{z_0-z}-\frac{c}{z_0-z}\times\int_\xi^\infty e^{cy}\left[\left(\frac{\cosh y-z_0}{\cosh y-z}\right)^{1/2}-1\right]dy \tag{A6}$$

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[15] R. Creighton Buck, in *Advanced Calculus* (McGraw-Hill Book Co., New York, 1956), p. 150, Theorem 27.

[16] See Ref. 10, p. 160, Eq. (2).

with the integral in (A6) converging for $c<1$. When $c=0$, (A6) reduces to the usual result

$$U(0,z_0,z)=\sum_{l=0}^{\infty}(2l+1)Q_l(z_0)P_l(z)=\frac{1}{z_0-z}. \quad \text{(A7)}$$

However, for nonzero values of c, (A6) defines it as an analytic function of z for $c<1$.

APPENDIX B

In Ref. 17, we derived the formula

$$R(\alpha,\xi,z)\equiv\sum_{l=0}^{\infty}(2l+1)\frac{e^{-(l-\alpha)\xi}}{l-\alpha}P_l(z)$$
$$=\frac{\sqrt{2}e^{(\alpha+1/2)\xi}}{(\cosh\xi-z)^{1/2}}+(2\alpha+1)$$
$$\times\left[Q_{-\alpha-1}+\frac{1}{\sqrt{2}}\int_{\xi}^{\eta(z)}\frac{e^{(\alpha+1/2)x}}{(\cosh x-z)^{1/2}}dx\right], \quad \text{(B1)}$$

where $\cosh\eta(z)=z$. Equation (B1) is useful for $\mathrm{Re}(z)>\cosh\xi$ and arbitrary α. However, satisfaction of the crossing relations will require $R(\alpha,\xi,z)$ for $\mathrm{Re}(z)<-\cosh\xi$. A formula useful there as well may be obtained by using the integral representation for the Q function (24). Abbreviating the expression in square brackets in (B1) by $T(\alpha,\xi,z)$, we have

$$T(\alpha,\xi,z)=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{e^{[\alpha-(1/2)]x}}{(\cosh x-z)^{1/2}}dx. \quad \text{(B2)}$$

While (B2) is now useful for all z, it is only valid for $\mathrm{Re}(\alpha)<1$. However, from the generating function for Legendre functions, (23), we may write

$$\Delta_m(x,z)\equiv\frac{1}{(\cosh x-z)^{1/2}}-\sqrt{2}e^{-x/2}\sum_{n=0}^{m}e^{-nx}P_n(z)$$
$$=\sqrt{2}e^{-x/2}\sum_{n=m+1}^{\infty}e^{-nx}P_n(z). \quad \text{(B3)}$$

Equation (B2) may therefore be written

$$T(\alpha,\xi,z)=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\Delta_m(x,z)e^{[\alpha-(1/2)]x}dx$$
$$-\sum_{n=0}^{m}\frac{e^{(\alpha-n-1)\xi}}{\alpha-n-1}P_n(z), \quad \text{(B4)}$$

where now the integral in (B4) converges for $\mathrm{Re}(\alpha)<m+2$, $m=0, 1, 2, \cdots$, and in addition is also valid for all z.

Therefore, we have the desired result, $R(\alpha,\xi,z)$ in (B1) is defined for all z:

$$R(\alpha,\xi,z)=\frac{\sqrt{2}e^{(\alpha+1/2)\xi}}{(\cosh\xi-z)^{1/2}}+(2\alpha+1)T(\alpha+1,\,\xi,\,z), \quad \text{(B5)}$$

where $T(\alpha,\xi,z)$ is given in (B4).

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[17] W. J. Abbe and Y. N. Srivastava, Nuovo Cimento **52A**, 551 (1967). Equation (16) of this reference is incorrect and should be replaced by our equation (B1).