The T products listed above are covariant, i.e., they do not involve any c-number Schwinger terms. In general, the T product of any three field operators are not necessarily free of noncovariant terms. We list below two T products that involve such terms:

$$\int d^4x d^4y\, e^{ip\cdot x}e^{-iq\cdot y}\langle T(\partial^\alpha \varphi_a(x)\partial^\beta \varphi_b(y)v^\mu{}_c(0))\rangle = i\epsilon_{abc}\,{}_\pi\Delta(q)\,{}_\pi\Delta(p)\,{}_\rho\Delta^{\mu\nu}(k)g_\rho{}^{-1}p^\alpha q^\beta \times[-m_\rho{}^2(q_\lambda+p_\lambda)-\tfrac{1}{2}\lambda_A(q\cdot kp_\lambda-k\cdot pq_\lambda)]-i\epsilon_{abc}g_\rho{}^{-1}[\delta^\alpha{}_0\delta^\mu{}_0q^\beta\,{}_\pi\Delta(q)+\delta^\beta{}_0\delta^\mu{}_0p^\alpha\,{}_\pi\Delta(p)], \tag{A4}$$

$$\int d^4x d^4y\, e^{ip\cdot x}e^{-iq\cdot y}\langle T(\varphi_a(x)a^\beta{}_b(y)v^\mu{}_c(0))\rangle = i\epsilon_{abc}\,{}_\pi\Delta(p)\,{}_\rho\Delta^{\mu\nu}(k)\,{}_{A_1}\Delta^{\beta\lambda}(q)(1/2F_\pi) \times[-2m_\rho{}^2g_{\nu\lambda}+(q_\nu p_\lambda-q\cdot pg_{\nu\lambda})+(\lambda_A-2)(p\cdot kg_{\nu\lambda}-p_\nu k_\lambda)]-i\epsilon_{abc}\delta^\beta{}_0\delta^\mu{}_0(F_\pi m_A{}^2)^{-1}\,{}_\pi\Delta(p). \tag{A5}$$

On the other hand, it is easily verified that no c-number Schwinger terms appear in the T products of three current operators.

PHYSICAL REVIEW VOLUME 174, NUMBER 5 25 OCTOBER 1968

Application of Hard-Pion Three-Point Functions*

R. ARNOWITT, M. H. FRIEDMAN, AND P. NATH
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 10 June 1968)

Some comparisons with experiment of the hard-pion three-point functions obtained from $SU(2)\times SU(2)$ current algebra are given. Available experimental data involving π, ρ, A_1, and σ mesons are examined. The hard-pion current-algebra method is used to calculate the decays $\rho\to\pi+\pi$, $A_1\to\pi+\rho$, and $\sigma\to\pi+\pi$, and the electromagnetic form factor of the pion. Peripheral processes such as $\pi+N\to\rho+N$ are also examined as a test of meson-vertex functions for spacelike momentum transfers. Here, to reproduce correctly the momentum-transfer dependence at the nucleon vertex, a new extrapolation for the pionic nucleon form factor is introduced, using the Goldberger-Treiman relation. The results of the above calculations are found to be consistent with the present experimental situation. Current-algebra predictions for the $\gamma+N\to A_1+N$ cross section and the decays $A_1\to\pi+\gamma$ and $A_1\to\pi+\sigma$ are given. A cross section of about 0.1 μb is obtained for the A_1 photoproduction, which is on the verge of being detectable.

I. INTRODUCTION

RECENTLY, it has become apparent that soft-pion methods[1] may lead to erroneous results when applied to processes involving energetic pions. Thus the soft-pion method yields a width of approximately 800 MeV for the $A_1\to\pi+\rho$ decay,[2] in contrast to the experimental width of $\approx$100 MeV. These considerations, coupled with the desire to exploit more fully the content of the current algebras, have motivated interest in extending the analysis beyond the domain of the soft-pion method. An important step in this direction was first taken by Weinberg, who used Ward identities, $SU(2)\times SU(2)$ algebra of currents, and the hypothesis of meson dominance of vector and axail-vector currents to obtain the sum rule[3]

$$g_\rho{}^2/m_\rho{}^2=(g_A{}^2/m_A{}^2)+F_\pi{}^2. \tag{1.1}$$

Here, g_ρ and g_A are the coupling strengths of the vector current to the ρ meson and the axial-vector current to the A_1 meson, and F_π is the usual pion-decay amplitude.[4] This result when supplemented by the second Weinberg sum rule[3]

$$g_A=g_\rho \tag{1.2}$$

and the KSRF relation[5]

$$g_\rho\simeq\sqrt{2}F_\pi m_\rho \tag{1.3}$$

yields the well-known result $m_A=\sqrt{2}m_\rho$ which is borne out experimentally.[3]

In the preceding paper,[6] new techniques were described to obtain current-algebra solutions to vertex

* Research supported in part by the National Science Foundation.

[1] For a review of the soft-pion calculations, see R. F. Dashen, in *Proceedings of the Thirteenth International Conference in High-Energy Physics* (University of California Press, Berkeley, 1967).

[2] The soft-pion analysis of the π-ρ-A_1 vertex has been carried out by several authors; see e.g., B. Renner, Phys. Letters **21**, 453 (1966); D. Geffen, Phys. Rev. Letters **19**, 770 (1967).

[3] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). Extension of these results to other vector and axial-vector currents are given by T. Das, V. S. Mathur, and S. Okubo, *ibid.* **18**, 761 (1967); S. L. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967).

[4] We define g_ρ by the relation

$$\langle 0|V_{\mu a}(0)|\rho;k,b,\sigma\rangle\equiv g_\rho\delta_{ab}N_\rho\epsilon_\mu{}^\sigma(k),$$

where a, b=1, 2, 3 are $SU(2)$ isotopic indices, $\epsilon_\mu{}^\sigma(k)$ is the ρ polarization vector normalized by $\epsilon_\mu{}^{\sigma*}\epsilon^{\mu\sigma'}=\delta^{\sigma\sigma'}$, and

$$N_\rho\equiv[(2\pi)^3 2\omega_k]^{-1/2}.$$

Similarly, g_A is defined from the A_1 matrix element $\langle 0|A_{\mu a}(0)|A_1;k,b,\sigma\rangle$ and F_π by $\langle 0|A_{\mu a}(0)|\pi,kb\rangle=iF_\pi\delta_{ab}k_\mu N_\pi$. Our currents are normalized such that the experimental value of F_π is 94 MeV.

[5] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[6] R. Arnowitt, M. H. Friedman, and P. Nath, preceding paper, Phys. Rev. **174**, 1999 (1968) (hereafter referred to as I). A brief account of these results were given in R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967).

functions involving the π, ρ, A_1 and σ mesons, using also the assumption of single-meson saturation of intermediate sums.[7] In this paper, we describe a comparison of the previously obtained hard-pion three-point functions with experiment. The results of the calculation are grouped in two separate sections. Section III is devoted to the electromagnetic form factor of the pion and a number of decay processes: $\rho\to 2\pi$, $A_1\to\pi+\rho$, $A_1\to\pi+\gamma$, $\sigma\to 2\pi$, and $A_1\to\pi+\sigma$. In Sec. IV, we consider peripheral processes, such as $\pi+N\to\rho+N$ and $\gamma+N\to A_1+N$, which involve a hard-pion vertex. We begin, in Sec. II, with a review of the formalism of Paper I needed for the discussion in Secs. III and IV.

II. CURRENT ALGEBRA AND MESON DOMINANCE

It was demonstrated in I that the assumption of meson dominance can be expressed concisely for three-point functions in terms of an effective Lagrangian involving cubic interactions, which is to be subjected to the requirements of current algebra. For the (σ,π,ρ,A_1) system, the Lagrangian was shown to take the form

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_I\,, \tag{2.1a}$$

where

$$\begin{aligned}\mathfrak{L}_0=&-\varphi^\mu{}_a\partial_\mu\varphi_a+\tfrac{1}{2}(\varphi^\mu{}_a\varphi_{\mu a}-m_\pi{}^2\varphi_a{}^2)-\sigma^\mu\partial_\mu\sigma\\&+\tfrac{1}{2}(\sigma^\mu\sigma_\mu-m_\sigma{}^2\sigma^2)-\tfrac{1}{2}(\partial_\mu v_\nu-\partial_\nu v_\mu)_aG^{\mu\nu}{}_a\\&+\tfrac{1}{4}G^{\mu\nu}{}_aG_{\mu\nu a}-\tfrac{1}{2}m_\rho{}^2v_{\mu a}v^\mu{}_a-\tfrac{1}{2}(\partial_\mu a_\nu-\partial_\nu a_\mu)_aH^{\mu\nu}{}_a\\&+\tfrac{1}{4}H^{\mu\nu}{}_aH_{\mu\nu a}-\tfrac{1}{2}m_A{}^2a_{\mu a}a^\mu{}_a\end{aligned} \tag{2.1b}$$

and $\mathfrak{L}_I=\mathfrak{L}_{3(\pi\rho A)}+\mathfrak{L}_{3(\sigma)}$, where $\mathfrak{L}_{I(\pi\rho A)}$ is given by

$$\begin{aligned}\mathfrak{L}_{3(\pi\rho A)}=\tfrac{1}{2}\epsilon_{abc}[&2g_{\pi\pi\rho}\varphi^\mu{}_b\varphi_cv_{\mu a}+\lambda_{\pi\pi\rho}\varphi_{\mu a}\varphi_{\nu b}G^{\nu\mu}{}_c\\&+2g_{\pi\rho A}v_{\mu a}\varphi_ba^\mu{}_c+2\mu_{\pi\rho A}\varphi_aG^{\mu\nu}{}_bH_{\mu\nu c}\\&+2\lambda_{\pi\rho A}v_{\mu a}\varphi_{\nu b}H^{\mu\nu}{}_c+2\tilde{\lambda}_{\pi\rho A}a_{\mu a}\varphi_{\nu b}G^{\mu\nu}{}_c\\&+g_{\rho\rho\rho}v_{\mu a}v_{\nu b}G^{\nu\mu}{}_c+2g_{\rho AA}v_{\mu a}a_{\nu b}H^{\nu\mu}{}_c\\&+\lambda_{\rho AA}a_{\mu a}a_{\nu b}G^{\nu\mu}{}_c+\mu_{\rho\rho\rho}G_{\mu\nu a}G^{\nu\lambda}{}_bG_\lambda{}^\mu{}_c\\&+\mu_{\rho AA}G_{\mu\nu a}H^{\nu\lambda}{}_bH_\lambda{}^\mu{}_c]\,,\end{aligned} \tag{2.1c}$$

$\mathfrak{L}_{3(\sigma)}$ being given by

$$\begin{aligned}\mathfrak{L}_{3(\sigma)}=&\tfrac{1}{2}g_{\sigma\pi\pi}\varphi_a\varphi_a\sigma+\tfrac{1}{2}\lambda_{\sigma\pi\pi}\varphi^\mu{}_a\varphi_{\mu a}\sigma+\tfrac{1}{2}g_{\sigma\rho\rho}v^\mu{}_av_{\mu a}\sigma\\&+\tfrac{1}{4}\lambda_{\sigma\rho\rho}G^{\mu\nu}{}_aG_{\mu\nu a}\sigma+\tfrac{1}{2}g_{\sigma AA}a^\mu{}_aa_{\mu a}\sigma\\&+\tfrac{1}{4}\lambda_{\sigma AA}H^{\mu\nu}{}_aH_{\mu\nu a}\sigma+\lambda_{\sigma\pi A}\varphi_aa^\mu{}_a\sigma_\mu\\&+\tilde{\lambda}_{\sigma\pi A}a^\mu{}_a\varphi_{\mu a}\sigma+\mu_{\sigma\rho\rho}v_{\mu a}G^{\mu\nu}{}_a\sigma_\nu\\&+\mu_{\sigma AA}a_{\mu a}H^{\mu\nu}{}_a\sigma_\nu+\mu_{\sigma\pi A}\varphi_{a\mu}H^{\mu\nu}{}_a\sigma_\nu\\&+\mu_{\sigma\pi\pi}\varphi_a\varphi^\nu{}_a\sigma_\nu+g_{\sigma\sigma\sigma}\sigma^3+\lambda_{\sigma\sigma\sigma}\sigma\sigma_\mu\sigma^\mu\,.\end{aligned} \tag{2.1d}$$

Here, σ, φ_a, $v^\mu{}_a$, $a^\mu{}_a$ are the Heisenberg interpolating fields for the σ, π, ρ, A_1 mesons, respectively, and m_σ, m_π, m_ρ, m_{A_1} are their corresponding physical masses. Using Eqs. (2.1), the $SU(2)$ multiplets of vector and axial-vector currents are generated by the relations

$$\begin{aligned}V^\mu{}_a&=g_\rho v^\mu{}_a\,,\\A^\mu{}_a&=g_Aa^\mu{}_a+F_\pi\partial^\mu\varphi_a\,.\end{aligned} \tag{2.2}$$

The currents obtained from Eq. (2.2) are then subject to the following conditions: the conservation of the vector current,

$$\partial_\mu V^\mu{}_a(x)=0\,, \tag{2.3}$$

the partial conservation of the axial current,

$$\partial_\mu A^\mu{}_a(x)=m_\pi{}^2F_\pi\varphi_a(x)\,, \tag{2.4}$$

and the current commutation relations

$$\begin{aligned}\delta(x^0-y^0)[V^0{}_a(x),V^\mu{}_b(y)]&=i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}\,,\\\delta(x^0-y^0)[V^0{}_a(x),A^\mu{}_b(y)]&=i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}\,,\\\delta(x^0-y^0)[A^0{}_a(x),V^\mu{}_b(y)]&=i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}\,,\\\delta(x^0-y^0)[A^0{}_a(x),A^\mu{}_b(y)]&=i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}\end{aligned} \tag{2.5}$$

The last terms in Eqs. (2.5) stand for c-number Schwinger terms.

Consistent with our requirement that the effective Lagrangian Eq. (2.1) is to be used for calculating three-point processes, the condition of single-meson saturation requires that Eqs. (2.3)–(2.5) be satisfied only up to first order in the coupling constants. For the σ-independent coupling constants, the following relations were derived in I:

$$\begin{aligned}g_{\rho\rho\rho}=g_{\rho AA}=g_{\pi\pi\rho}&=m_\rho{}^2g_\rho{}^{-1}\,,\\g_{\pi\rho A}=-m_A{}^2\lambda_{\pi\rho A}&=m_\rho{}^2(F_\pi x^2yz^2)^{-1}\,,\\g_\rho\lambda_{\pi\pi\rho}&=x^4y^2z^2\tfrac{1}{2}\lambda_A+2(1-z^2)\,,\\F_\pi\tilde{\lambda}_{\pi\rho A}=-y(1-x^2\tfrac{1}{2}\lambda_A)\,,&\quad 2F_\pi\mu_{\pi\rho A}=y-y^{-1}\,,\end{aligned} \tag{2.6}$$

where λ_A (which has the physical interpretation of being the anomalous magnetic moment of the A_1 meson) is given by

$$\lambda_A\equiv g_\rho m_\rho{}^{-2}\lambda_{\rho AA}\,, \tag{2.7}$$

and the three parameters x, y, z are

$$x\equiv\sqrt{2}m_\rho/m_A\,,\quad y\equiv g_A/g_\rho\,,\quad z\equiv g_\rho/(\sqrt{2}m_\rho F_\pi)\,. \tag{2.8}$$

The condition that the current commutation relations Eqs. (2.5) have no q-number Schwinger terms implies that x, y, and z obey the first Weinberg sum rule [Eq. (1.1)]:

$$x^2y^2z^2-2z^2+1=0\,. \tag{2.9}$$

Thus, only two of the parameters x, y, z are independent.

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[7] Results equivalent to those of Ref. 6 (for the π-ρ-A_1 system) but using different techniques have also been obtained by H. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967); by S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967); Phys. Rev. **174**, 1777 (1968). Results similar to those of Ref. 6 based on chiral-invariant Lagrangians have been obtained by J. Schwinger, Phys. Letters **24B**, 473 (1967); Phys. Rev. **165**, 1714 (1968); lectures at Brandeis Summer Institute, 1967 (unpublished); J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968).

TABLE I. Γ_ρ ($\rho\to\pi+\pi$ decay width), $\Gamma_{A\rho\pi}$ ($A_1\to\pi+\rho$ decay width), $\Gamma_{A_1\pi\gamma}$ ($A_1\to\pi+\gamma$ decay width), and r_e (the electromagnetic charge radius of the pion), calculated for various values of λ_A (the anomalous moment of the A_1 meson).

λ_A	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Γ_ρ (MeV)	96	102	108	114	121	127	134
$\Gamma_{A\rho\pi}$ (MeV)	129	116	104	93	83	73	63
$A_1\pi\gamma$ (MeV)	0.028	0.050	0.078	0.11	0.15	0.20	0.25
γ_e (F)	0.57	0.58	0.59	0.60	0.61	0.62	0.62

For the σ-dependent coupling constants, the following relations were obtained:

$$\begin{aligned} g_{\sigma\rho\rho}&=0=\mu_{\sigma\rho\rho}\,,\\ F_\pi g_{\rho\pi\pi}&=(m_\sigma^2\lambda_3-m_\pi^2\lambda_1)\,,\\ F_\pi\lambda_{\sigma\pi\pi}&=-(\lambda_1+\lambda_2)\,,\\ F_\pi g_{\sigma AA}&=(x^2yz)^{-2}2m_\rho^2(\lambda_1-\lambda_2)\,,\\ \sqrt{2}m_\rho\mu_{\sigma\pi A}&=-x^2yz\mu_{\sigma AA}\,,\end{aligned}\tag{2.10}$$

where

$$\lambda_1\equiv(g_Am_A^{-2})\lambda_{\sigma\pi A}\,,\quad \lambda_2\equiv(g_Am_A^{-2})\tilde{\lambda}_{\sigma\pi A}\,,\quad \lambda_3\equiv\lambda_1+F_\pi\mu_{\sigma\pi\pi}\,.\tag{2.11}$$

It is possible to extend the results of I to N-point functions.[8] Equations (2.6)–(2.11) remain valid and one further constraint arises:

$$\lambda_1\lambda_3=1\,.\tag{2.12}$$

III. FORM FACTORS AND DECAYS

We now proceed to compute the $\rho\to\pi+\pi$ and $A_1\to\pi+\rho$ decay widths. Using the effective Lagrangian of Sec. II to first order (as required by the single-particle saturation assumption), one finds in a straightforward fashion

$$\Gamma_\rho=m_\rho^3(96\pi F_\pi^2)^{-1}[(1-\tfrac{1}{4}\lambda_A)^2+2\delta(1-\tfrac{1}{4}\lambda_A)\times(1-\tfrac{3}{4}\lambda_A)](1-4m_\pi^2m_\rho^{-2})^{3/2}\tag{3.1}$$

and

$$\begin{aligned}\Gamma_{A\rho\pi}=2^{-1/2}m_\rho^3(1536\pi F_\pi^2)^{-1}\{&(8+12\lambda_A+5\lambda_A^2)-6\lambda_A^2m_\pi^2m_\rho^{-2}+(8-12\lambda_A+5\lambda_A^2)8m_\pi^4m_\rho^{-4}\}\\ &\times(1-6m_\pi^2m_\rho^{-2}+m_\pi^4m_\rho^{-4})^{1/2}(1+2\delta)\,,\end{aligned}\tag{3.2}$$

where we have used the experimental result $x=1$, eliminated $y(\equiv g_A/g_\rho)$ by the Weinberg sum rule [Eq. (2.9)], and allowed for a correction to the KSRF condition Eq. (3) by setting

$$z=1+\delta\,.\tag{3.3}$$

Terms of $O(\delta^2)$ have been neglected. From Eqs. (3.1) and (3.2) one sees that changes in the widths due to deviations from the KSRF condition are proportional to 2δ, and thus a 10% correction to this condition can lead to $\approx$10–20% correction to Γ_ρ (depending on the value of λ_A) and $\approx$20% change in $\Gamma_{A\rho\pi}$. It is thus necessary to first investigate the validity of Eq. (1.3) (i.e., $z=1$). As has recently been observed by several authors, Eq. (1.3) is not a current-algebra result.[6,7,9,10] On the other hand, Eq. (1.3) can be tested experimentally from the $\rho^0\to l^++l^-$ lepton decay since the decay amplitude is directly proportional to g_ρ.[11] Thus z can be expressed directly in terms of the lepton-pair branching ratio B_ρ by

$$z^2=B_\rho\Gamma_\rho\left(\frac{3}{8\pi\alpha^2}\frac{m_\rho}{F_\pi^2}\right)\left(1+2\frac{m_l^2}{m_\rho^2}\right)^{-1}\left(1-4\frac{m_l^2}{m_\rho^2}\right)^{-1/2}.\tag{3.4}$$

One difficulty associated with the experimental determination of z in this manner is the uncertainty involved in the experimental value of the ρ width. Since the ρ is generally produced in the presence of nucleons, the final-state interactions appear to broaden the resonance. An attempt to correct for this effect has been made by Roos,[12] leading to the result $\Gamma_\rho=128\pm5$ MeV. All such complications, however, are absent in an e^+-e^- colliding-beam production of the ρ. Indeed, recent data[13] yield the apparently "low" value of $\Gamma_\rho=93\pm15$ MeV. We choose, for the present discussion, a midvalue of $\Gamma_\rho=111\pm17$ MeV. The recent experimental determination of B_ρ from Novosibirsk[13,14] gives

$$B_\rho\simeq(4.8\pm0.8)\times10^{-5}.$$

With this choice of Γ_ρ and B_ρ, z is then determined to be

$$z=1.05\pm0.15\tag{3.5}$$

consistent with the KSRF relation $z=1$. For the sake of simplicity, we shall therefore set $z=1$ in all our succeeding discussions [which then implies $y=1$ by Eq. (2.9), since $x=1$ has already been assumed]. Equations (3.1) and (3.2) then yield the result[15]

$$\Gamma_\rho=141(1-\tfrac{1}{4}\lambda_A)^2\ \text{MeV}\,,\tag{3.6}$$

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[8] This analysis is carried out in R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. (to be published) (hereafter referred to as III).

[9] D. A. Geffen, Phys. Rev. Letters 19, 770 (1967).

[10] A reexamination of the earlier derivations in Ref. 4 have shown to lead to identities rather than a derivation of Eq. (3). See R. Arnowitt, M. H. Friedman, and P. Nath, Nucl. Phys. **B5**, 115 (1968).

[11] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966). No assumption of ρ dominance is necessary in this determination of g_ρ.

[12] M. Roos, Nucl. Phys. **B2**, 615 (1967).

[13] V. L. Auslander, G. I. Budker, Ju. N. Pestov, V. A. Sidorov, A. N. Skrinsky, and A. G. Khabakhpashev, Phys. Letters **25B**, 433 (1967).

[14] This value of B_ρ is consistent with the recent analyses of R. G. Parsons and R. Weinstein, Phys. Rev. Letters **20**, 1314 (1968) and M. Davier (SLAC Report) on the photoproduced ρ mesons (unpublished). These authors correct for the interference due to lepton pairs from the ω decay.

[15] Equations (3.6) and (3.7) have also been obtained by H. Schnitzer and S. Weinberg and by S. G. Brown and G. W. West, Ref. 7.

$$\Gamma_{A\rho\pi}=6.85(8+12\lambda_A+5\lambda_A^2)\ \text{MeV}. \qquad (3.7)$$

Γ_ρ and $\Gamma_{A\rho\pi}$ are listed in Table I for a range of values of λ_A. One sees that the experimental value of $\Gamma_\rho=111\pm17$ MeV is obtained with the choice

$$\lambda_A=0.4\pm0.3. \qquad (3.8)$$

The experimental value for $\Gamma_{A\rho\pi}$ is more uncertain. The most recent survey[16] lists $\Gamma_A=80\pm35$ MeV. Equation (3.8) is consistent with this value, if we assume that the major mode of decay of the A_1 is small. A discussion of other possible decay modes of the A_1 is given below.

We next consider the pion electromagnetic form factors obtained by examining the vertex $\langle\pi q|V^\mu{}_a|\pi p\rangle$. This process necessarily involves the coupling of an off-shell ρ to two on-shell pions in contrast to the $\rho\to2\pi$ decay where all the particles involved are on their respective mass shells. It thus tests the soundness of interpolating current-algebra conditions off the particle-mass shells. Replacing $V^\mu{}_a$ by $g_\rho v^\mu{}_a$ and using Eq. (2.1), we obtain for the form factor, the result

$$f(k^2)\equiv m_\rho^2(k^2+m_\rho^2)^{-1}(1+\tfrac{1}{4}\lambda_A k^2 m_\rho^{-2}). \qquad (3.9)$$

The electromagnetic charge radius of the pion γ_e is defined through the expansion

$$f(k^2)=1-r_e^2(\tfrac{1}{6}k^2)+\cdots. \qquad (3.10)$$

This yields for r_e the value $\surd6(1-\frac{1}{4}\lambda_A)^{1/2}/m_\rho$. For $\lambda_A=0.4$, one finds $r_e=0.6$ F to be compared with the experimental value[17] of 0.8 ± 0.1 F. We note that the ρ-dominance model would yield $r_e=(\surd6)/m_\rho$, which in this case appears to be a good approximation because of the accidental smallness of $\frac{1}{4}\lambda_A$.

The decay $A_1{}^\pm\to\pi^\pm+\gamma$ may also be calculated easily. Since the outgoing photon is a pure isovector, it couples only to the ρ, which must be on the photon-mass shell to conserve energy and momentum (Fig. 1). Thus again, the ρ leg of the vertex is off the ρ-mass shell. The decay widths for both the charged states are the same and we find, for the partial width of $A_1\to\pi+\gamma$, the result

$$\Gamma_{A\pi\gamma}=\frac{\alpha}{24}(1-\lambda_A)^2 m_A\left(1-\frac{m_\pi^2}{m_A^2}\right)^3, \qquad (3.11)$$

where α $(=e^2/4\pi)$ is the fine-structure constant.
The partial width $\Gamma_{A\pi\gamma}$ is given in Table I for a range of values of λ_A. While it appears doubtful that a direct measurement of this decay mode is likely in the near future, the process $A_1\to\gamma+\pi$ is of theoretical interest as the value of $\Gamma_{A\pi\gamma}=0.3$ MeV has been obtained by Singh[18] from Compton-scattering sum rules. The results

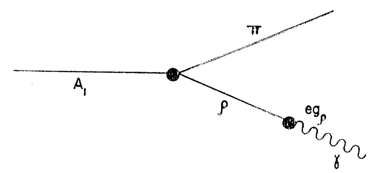

FIG. 1. Feynman diagram for the decay $A_1\to\pi+\gamma$.

of Table I show that the hard-pion current algebra predicts a somewhat smaller decay width than do the superconvergent dispersion relations. Agreement with the calculation of Ref. 18 can be achieved if we set $\lambda_A\simeq0$. On the other hand, the chiral symmetric Yang-Mills limit would yield $\lambda_A=1$ and hence $\Gamma_{A\pi\gamma}=0$. Thus the experimental observation of the $A_1\to\pi+\gamma$ decay mode would determine the deviation from the Yang-Mills limit (i.e., $\lambda_A=1$).

Next, we proceed to consider the decays involving the σ meson. There now appears to be growing evidence for such an $I=0=J$ resonance. The analysis of $\pi\pi$ phase shifts from the pion-production data in the reaction $\pi+N\to2\pi+N$ by Schlein and Malamud[19] and by Walker *et al.*[20] indicate an $I=0$, S-wave resonance in the region between 700 MeV and 1 GeV. The $I=0$ σ decay proceeds through two modes $\sigma\to\pi^0\pi^0$ and $\sigma\to\pi^+\pi^-$, the charged mode being suppressed by a factor of $\frac{1}{2}$ over the neutral mode because of Bose statistics. Thus, the total decay width Γ_σ, on using Eq. (2.1), is given by

$$\Gamma_\sigma=\frac{3m_\sigma^3}{\pi(128)F_\pi^2}(1-4\epsilon_\sigma)^{1/2}[(\lambda_1-\lambda_2)+2\epsilon_\sigma\lambda_2]^2, \qquad (3.12)$$

where $\epsilon_\sigma=m_\pi^2/m_\sigma^2$. The decay width Γ_σ is thus determined by the coupling parameter $\lambda=(\lambda_1-\lambda_2)+2\epsilon_\sigma\lambda_2$. From the $\pi\pi$ phase-shift analysis of Schlein and Malamud,[19] one has $m_\sigma=730$ MeV, and for $\lambda^2=1$ we obtain $\Gamma_\sigma\simeq350$ MeV consistent with the data. From the analysis of Walker *et al.*[20] $m_\sigma=930$ MeV, and for $\lambda^2=0.9$ we have $\Gamma_\sigma\simeq650$ MeV, also consistent with the data.

The 3π Dalitz plot for the A_1 decay does not allow one to determine clearly what fraction of the A_1 width is due to the $\rho+\pi$ decay mode.[16] It is of interest, therefore, to consider the partial widths due to other possible decay modes. Using the Lagrangian of Eq. (2.1), it is straightforward to calculate the $A_1\to\pi+\sigma$ decay width. We find

$$\begin{aligned}\Gamma_{A_1\pi\sigma}=&(24\pi F_\pi^2)^{-1}[(m_A^2-m_\sigma^2-m_\pi^2)^2\\ &\times(4m_A^2)^{-1}-m_\pi^2m_\sigma^2m_A^{-2}]^{3/2}\\ &\times(\lambda_1-\lambda_2-g_A\mu_{\sigma\pi A})^2. \qquad (3.13)\end{aligned}$$

Since $\epsilon_\sigma\ll1$, one may reasonably approximate $\lambda_1-\lambda_2$

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[19] P. E. Schlein, Phys. Rev. Letters **19**, 1052 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967).
[20] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967).

TABLE II. Comparison of the Amaldi-Selleri form factor $F(q^2)$ with $K(q^2)$ obtained from the Goldberger-Treiman relation, for a range of values of the momentum transfer.

q^2/m_π^2	-1	0	1	2	3	4	5	10
$F(q^2)$	1.00	0.87	0.79	0.72	0.67	0.63	0.60	0.50
$K(q^2)$	1.00	0.87	0.77	0.69	0.63	0.57	0.53	0.38
$[F(q^2)-K(q^2)]/F(q^2)$	0	0	2.5%	4.2%	6.0%	9.5%	12%	24%

by λ. The coupling constant $\mu_{\sigma\pi A}$ is undetermined by the current algebra. The simplest assumption, $\mu_{\sigma\pi A}=0$, yields, for the Schlein and Malamud[19] σ meson,

$$\Gamma_{A_1\pi\sigma}\simeq 225 \text{ MeV} \tag{3.14}$$

for the values $m_\sigma=730$ MeV and $\lambda^2=1$. For the σ meson of Walker *et al.*,[20] one obtains

$$\Gamma_{A_1\pi\sigma}\simeq 60 \text{ MeV} \tag{3.15}$$

for the values $m_\sigma=930$ MeV and $\lambda^2=0.9$. Since the $\pi\rho$ decay mode of the A_1 already accounts for most of the experimental value of the A_1 width, the $\pi\sigma$ mode contribution for either Eq. (3.14) or (3.15) appears to be too large. One may of course adjust $g_A\mu_{\sigma\pi A}$ to suppress this mode. However, there exists, in addition, a nonresonant $A_1\to 3\pi$ direct decay mode whose amplitude could conceivably destructively interfere with the $\pi\sigma$ decay amplitude due to the broadness of the σ resonance. To investigate this matter further, one must use four-point functions and this requires the techniques developed in III (Ref. 8). A detailed analysis of the A_1 decay will be discussed elsewhere.

IV. PIONIC FORM FACTOR OF THE NUCLEON AND PERIPHERAL PROCESSES

An application of the hard-pion vertex functions arises naturally in peripheral processes and we consider here two such processes whose meson vertices involve the π, ρ, and A_1. The reactions we consider are $\pi+N\to\rho+N$ and $\gamma+N\to A_1+N$. We consider first the ρ-production process. The peripheral diagram governing this reaction is given in Fig. 2. The π-π-ρ vertex appearing there can be directly determined from the Lagrangian of Eq. (2.1). Here, one of the pions is off the mass shell. In addition, one needs to know the

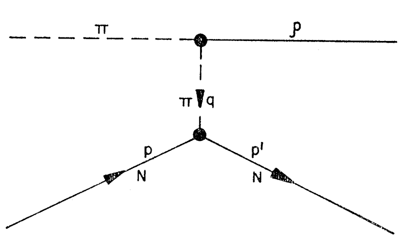

FIG. 2. The Feynman diagram representing the peripheral process $\pi+N\to\rho+N$.

pion-nucleon vertex[21] $K(q^2)$. To estimate $K(q^2)$ we note that $K(0)$ appears in the partially conserved axial-vector current (PCAC) constant[22]

$$F_\pi=m(G_A/G_V)[gK(0)]^{-1}, \tag{4.1}$$

where m is the nucleon mass, and G_A/G_V $(=1.18)$ is the ratio of the renormalized axial-vector and vector coupling constants of β decay. Using the experimental value of F_π in Eq. (4.1), we find

$$K(0)=0.87. \tag{4.2}$$

On the other hand, $K(q^2)$ is normalized such that $K(-m_\pi^2)=1$, and hence a simple pole extrapolation yields[23]

$$K(q^2)=\frac{0.87}{1+0.13(q^2/m_\pi^2)}. \tag{4.3}$$

A phenomenological π-N form factor $F(q^2)$ has previously been suggested by Amaldi and Selleri.[24,25] For $q^2\ll 10m_\pi^2$ one may approximate $F(q^2)$ by

$$F(q^2)=0.28+\frac{0.72}{1+(1/4.73)[(q^2+m_\pi^2)/m_\pi^2]}. \tag{4.4}$$

In Table II, we compare the numerical values of the form factors Eq. (4.3) and Eq. (4.4) over a range of momentum transfers up to $10m_\pi^2$. The two form factors are in agreement within 10% for $q^2\lesssim 5m_\pi^2$. Returning to the peripheral process $\pi+N\to\rho+N$, the differential cross section in the universal form is given by

$$\left(k_L^2\frac{d\sigma}{dt}\right)_{\pi N\to\rho N}=\frac{\pi}{16}\frac{g^2}{4\pi}\left(\frac{m_\rho^2}{g_\rho}\right)^2\frac{(1-\frac{1}{4}\lambda_A)^2}{m^2m_\rho^2}\times[q^2+(m_\rho-m_\pi)^2][q^2+(m_\rho+m_\pi)^2]\times q^2K^2(q^2)\Delta_\pi^2(q^2), \tag{4.5}$$

[21] The pion-nucleon vertex $K(q^2)$ is defined by the nucleon matrix element of the pion field:

$$\langle p|\varphi_a(0)|p'\rangle=N_pN_{p'}\bar{u}(p)\gamma^5\tau_a u(p')(q^2+\mu^2)^{-1}gK(q^2),$$

where $g^2/4\pi\cong 14.7$ is the pion-nucleon coupling constant, $u(p)$ the nucleon spinors and $q^2=(p'-p)^2$.

[22] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958); L. Wolfenstein, Nuovo Cimento **8**, 882 (1958).

[23] One can also use a different extrapolation $K(q^2)=0.87-0.13(q^2/m_\pi^2)$. The pole extrapolation of Eq. (4.3) and the linear extrapolation above yield essentially the same result for small momentum transfers.

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[25] J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965).

TABLE III. Differential cross sections for $\gamma+N\rightarrow A_1+N$. The second row is the universal form (independent of c.m. energy) while in the third row $d\sigma/dt$ is calculated for the c.m. energy of 4 GeV.

t (in m_π^2)	0.5	7.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0
$k_L^2 d\sigma/dt$ (in μb)	60.22	63.0	61.1	53.4	45.2	32.4	23.8	18.1	14.2	11.3
$d\sigma/dt$ (in μb/GeV^2)	0.93	0.97	0.94	0.82	0.699	0.5	0.368	0.28	0.22	0.175

where k_L is the pion momentum in the laboratory frame, $q^2=t=(p-p')^2$ (see Fig. 2), and $\Delta_\pi(q^2)=(q^2+m_\pi^2)^{-1}$. From the ρ-decay width, we obtain

$$\frac{1}{4\pi}\left(\frac{m_\rho^2}{g_\rho}\right)^2(1-\tfrac{1}{4}\lambda_A)^2=2.16. \qquad (4.6)$$

Using Eqs. (4.3) and (4.6) in Eq. (4.5), one obtains good agreement between theory and experiment over a wide range of energies.[25] Here, current algebra gives the same result as ρ dominance, even though one of the pions if off the mass shell.

The success of the peripheral analysis, in treating ρ production, leads one to attempt a similar analysis of the photoproduction of A_1. The A_1^0 peripheral photoproduction is forbidden by isospin invariance and the allowed processes are

$$\begin{aligned}\gamma+p&\rightarrow A_1^{+}+n,\\ \gamma+n&\rightarrow A_1^{-}+p.\end{aligned} \qquad (4.7)$$

The peripheral diagrams for these processes are given in Fig. 3. We note that these diagrams are automatically gauge-invariant and the G-parity invariance implies that only the isovector part of the photon contributes. A brief discussion of the calculations of the photoproduction process is given in the Appendix. We quote the result here in the universal form

$$\left(k_L^2\frac{d\sigma}{dt}\right)_{\gamma N\rightarrow A_1N}=\tfrac{1}{16}\pi\alpha\frac{g^2}{4\pi}\frac{(1-\lambda_A)^2}{m^2m_\rho^2}\times(p^2+m_A^2)^2p^2K^2(p^2)\Delta_\pi^2(p^2). \qquad (4.8)$$

Here, k_L is the photon momentum in the laboratory frame and $t=p^2=(p_1-p_2)^2$. We compare the result of Eq. (4.8) with the corresponding expression for the process $\pi+N\rightarrow\rho+N$ given by Eq. (4.5). For $t\ll m_\rho^2$, we get

$$\frac{(k_L^2d\sigma/dt)_{\gamma N\rightarrow A_1N}}{(k_L^2d\sigma/dt)_{\pi N\rightarrow\rho N}}=4\alpha\left(\frac{m_\rho^2}{g_\rho}\right)^{-2}\frac{(1-\lambda_A)^2}{(1-\frac{1}{4}\lambda_A)^2}. \qquad (4.9)$$

For $\lambda_A\simeq\frac{1}{2}$, Eq. (4.9) gives for the right-hand side $\simeq0.4\times10^{-3}$. In Table III, we give $(d\sigma/dt)_{\gamma N\rightarrow A_1N}$ at 4-GeV c.m. energy for momentum transfers up to $7m_\pi^2$. The universal form $(k_L^2d\sigma/dt)_{\gamma N\rightarrow A_1N}$ is also given. A rough estimate of the total cross section $\sigma_{\gamma N\rightarrow A_1N}$ at 4-GeV c.m. energy yields ≈0.1 μb, which is just the margin of the statistical errors in the present photoproduction experiments. However, it is possible that in more detailed analyses of the data for the reaction $\gamma+p\rightarrow n\pi^+\pi^-\pi^+$ such a process may be observed.[26] Equation (4.9) would then represent an interesting test of the hard-pion current-algebra analysis.

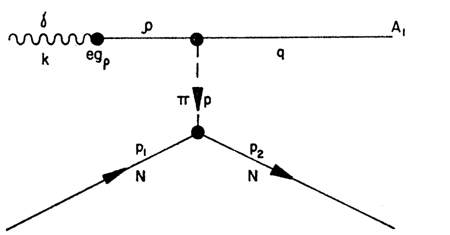

FIG. 3. The Feynman diagram representing the peripheral process for the photoproduction of A_1 from nucleons; $\gamma+N\rightarrow A_1+N$.

V. CONCLUSIONS

We should like to discuss briefly some of the achievements and limitations of the hard-pion current-algebra method for the determination of vertex functions involving π, ρ, A_1, and σ mesons. As was discussed in I,[6] the equal-time $SU(2)\times SU(2)$ current-algebra commutation relations, conserved vector current (CVC), PCAC, and the hypothesis of single-particle saturation of intermediate sums in T products put a number of constraints on the meson coupling constants. However, not all the constants are determined. Thus, of the eleven interactions involving the π, ρ, and A_1 mesons, two of the constants ($\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$) are totally unconstrained. These constants, however, do not enter into any of the calculations of this paper. Of the remaining nine, six are determined in terms of two of the three parameters x, y, z [Eq. (2.8)] and two more involve also the ninth constant λ_A (the existence of many undetermined parameters that are found in chiral-symmetric Lagrangians[7] is due in part to the fact that while the current commutation relations are chirally symmetric they are not sufficient to impose chiral invariance on the Lagrangian). If one uses the experimental values, $x=1$, $y\simeq z\simeq1$, there then remains only one undetermined parameter λ_A appearing in all the π, ρ, A_1 vertex functions considered here. The value of λ_A is perhaps best determined from the experimental value of the ρ width and one finds $\lambda_A=0.4\pm0.3$. The largeness of the error quoted is due in part to the present uncertainty

[26] We would like to thank Dr. Marvin Gettner for discussions concerning experimental aspects of this problem.

TABLE IV. The range over which the hard-pion vertices have been examined in this paper. Various criteria which test the "hardness" of a given process are listed.

Process	Hardness of the pion measured in pion energy	Particles off the mass shell	Two largest (mom. trans.)^2 across vertex	Amount of separation off the mass shell
$\rho\to 2\pi$	$E_\pi=\frac{1}{2}m_\rho$		$-m_\pi^2,\ -m_\rho^2$	
$A_1\to\pi\rho$	$E_\pi\simeq\frac{1}{2}m_\rho$		$-m_A^2,\ -m_\rho^2$	
$A_1\to\pi\gamma$	$E_\pi\approx\frac{1}{2}m_A$	ρ	$-m_A^2,\ -m_\pi^2$	$\Delta_\rho^{-1}=m_\rho^2$
Electromagnetic form factor of π		ρ	$-m_\pi^2,\ \lesssim m_\rho^2$	$\Delta_\rho^{-1}\geq m_\rho^2$
$\sigma\to 2\pi$	$E_\pi=\frac{1}{2}m_\sigma$		$-m_\sigma^2,\ -m_\pi^2$	
$A_1\to\pi\sigma$	$E_\pi\simeq\frac{1}{2}m_\rho$		$-m_A^2,\ -m_\sigma^2$	
$\pi+N\to\rho+N$	$E_\pi\gg m_\rho$	π	$-m_\rho^2,\ \lesssim 10m_\pi^2$	$\Delta_\pi^{-1}\lesssim 10m_\pi^2$
$\gamma+N\to A_1+N$		$\pi,\ \rho$	$-m_A^2,\ \lesssim 10m_\pi^2$	$\Delta_\pi^{-1}\lesssim 10m_\pi^2,\ \Delta_\rho^{-1}=m_\rho^2$

in the ρ width, and in part to the fact that λ_A does not enter too sensitively in the meson vertices. Clearly, λ_A will become better determined as the colliding-beam determination of the ρ width improves. The above value of λ_A appears to be consistent with the present data concerning the $A_1\to\pi+\rho$ partial width, the charge radius of the pion, and the peripheral ρ production in $\pi+N\to\rho+N$. In addition, predictions of the photodecay of A_1 and the peripheral production $\gamma+N\to A_1+N$ were obtained. The latter appears to be on the verge of being observed, and, while the former is too small to be seen, it is interesting to note that the hard-pion techniques predict a partial width for the decay $A_1\to\pi+\gamma$ somewhat smaller than that obtained from the superconvergent dispersion relations.[18] In general, we might note that at present λ_A is constrained by the fact that too small a value will give too large a ρ width (if one believes in the colliding-beam values[13] for Γ_ρ) and too large a value would give too large an $A_1\to\pi+\rho$ partial width (see Table I). At present, no experimental determination of $\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$ exists. One can set them equal to zero without disturbing the above discussion. An alternative would be to try to choose them so that these interactions cancel the logarithmic infinities in the electromagnetic mass differences of the ρ and the A_1 mesons. A discussion of this possibility will be given elsewhere.

For the fourteen coupling constants involving the σ meson, four constants ($\lambda_{\sigma\rho\rho}$, $\lambda_{\sigma AA}$, $g_{\sigma\sigma\sigma}$, and $\lambda_{\sigma\sigma\sigma}$) are totally unconstrained. Again, these constants, which are "orthogonal" to the current-algebra conditions, do not enter into any of the considerations of this paper. Seven of the remaining ten constants can be determined in terms of the three constants λ_1, λ_2, and $\mu_{\sigma\pi A}$ [Eqs. (2.11) and (2.12)]. The decay $\sigma\to\pi+\pi$ was found to depend on the combination $\lambda=\lambda_1-\lambda_2+2m_\pi^2m_\sigma^{-2}\lambda_2$, and the experimental data[19,20] indicates $\lambda^2\approx 1$. The constant $\mu_{\sigma\pi A}$ enters directly into the decay $A_1\to\pi+\sigma$, and the possibility that this mode may be suppressed would be a strong constraint fixing this constant.

An important test of the vertex functions resides in their validity for a wide range of momentum transfers at meson vertices. Several different criteria can be used for this purpose. In Table IV we examine all the processes considered in this paper from the point of view of the different aspects of the hard-pion calculation they test. These include the hardness of the process measured in pion kinetic energy, amount of separation from the mass shell, particles off the mass shell, nature of momentum transfers involved (spacelike or timelike), and the magnitudes of such momentum transfers.

In summary, we find that the computed vertex functions are consistent with the present experimental situation, and are good for momentum transfers up to 1 GeV. Further refined experimental data are needed to test the hard-pion current-algebra predictions more accurately.

APPENDIX

We give here a brief description of the peripheral calculation of the process

$$\gamma+N\to A_1+N. \tag{A1}$$

A convenient way of proceeding is to first contract the photon. In the notation of Fig. 3, one has

$$S\equiv\langle p_2\tau_2\sigma_2;\,qa\sigma'\ \text{out}\,|\,p_1\tau_1\sigma_1;\,k\sigma\ \text{in}\rangle$$
$$=ieN_k\int d^4x\ e^{ikx}\langle p_2\tau_2\sigma_2;\,qa\sigma'\,|\,j^\mu(x)\,|\,p_1\tau_1\sigma_1\rangle \times\epsilon_\mu{}^\sigma(k)\,, \tag{A2}$$

where $ej^\mu(x)$ is the electromagnetic current and $\epsilon_\mu{}^\sigma(k)$ is the photon's polarization vector of helicity σ, defined by

$$k^\mu\epsilon_\mu{}^\sigma=0\,,\quad \epsilon_\mu{}^{\sigma*}\epsilon^{\mu\sigma'}=\delta^{\sigma\sigma'}. \tag{A3}$$

To lowest order in the electromagnetic coupling, one may evaluate the matrix element of Eq. (A2) in the limit $e\to 0$. Thus only the isovector part of the electromagnetic current contributes (by G-parity invariance) and one may replace j^μ by $V^\mu{}_3=g_\rho v^\mu{}_3$. Thus,

$$S=iN_k(2\pi)^4\delta^4(p_1+k-q-p_2)eg_\rho \times\langle p_2\tau_2\sigma_2;\,qa\sigma'\,|\,v^\mu{}_3(0)\,|\,p_1\tau_1\sigma_1\rangle\epsilon_\mu{}^\sigma(k)\,. \tag{A4}$$

The field equations for the ρ field obtained from Eqs. (2.1) read

$$ {}_{\rho}K^{\mu}{}_{\nu}\, v^{\nu}{}_{a} = S^{\mu}{}_{a}\,, \tag{A5} $$

where ${}_{\rho}K^{\mu}{}_{\nu}$ is the ρ Proca operator and the relevant part of $S^{\mu}{}_{a}$ is given by

$$ S^{\mu}{}_{a} = \epsilon_{abc}[g_{\pi\rho A}\varphi_b a^{\mu}{}_{c} + \lambda_{\pi\rho A}\varphi_{\lambda b}H^{\mu\lambda}{}_{c}] - 2\epsilon_{abc}\partial_{\nu} \times[\mu_{\pi\rho A}\varphi_b H^{\mu\nu}{}_{c} - \tfrac{1}{2}\bar{\lambda}_{\pi\rho A}(a^{\mu}{}_{b}\varphi^{\nu}{}_{c} - a^{\nu}{}_{b}\varphi^{\mu}{}_{c})]. \tag{A6} $$

The matrix element of Eq. (A4) thus becomes

$$ \langle p_2,q|v^{\mu}{}_{3}(0)|p_1\rangle = {}_{\rho}\Delta^{\mu}{}_{\alpha}(k)\langle p_2;q|S^{\alpha}{}_{a}(0)|p_1\rangle\,, \tag{A7} $$

where ${}_{\rho}\Delta^{\mu}{}_{\alpha}(k)$ is the ρ propagator,

$$ {}_{\rho}\Delta^{\mu}{}_{\alpha} \equiv (k^2+m_\rho^2)^{-1}(\delta^{\mu}{}_{a} + k^{\mu}k_{\alpha}m_\rho^{-2})\,. \tag{A8} $$

We now make the peripheral approximation by replacing the A_1 fields $a^{\mu}{}_{a}$ and $H^{\mu\nu}{}_{a}$ by their free out-fields. They then annihilate the A_1 in the out state yielding

$$ S = i(2\pi)^4\delta^4(p_1+k-q-p_2)N_kN_q e g_\rho m_\rho^{-2} \times(\lambda_{\pi\rho A} - 2\mu_{\pi\rho A} - \bar{\lambda}_{\pi\rho A})\epsilon_{\nu}{}^{\sigma'*}(q)\epsilon_{\mu}{}^{\sigma}(k) \times(q^{\mu}k^{\nu} - kqg^{\mu\nu})\epsilon_{ab3}\langle p_2\sigma_2\tau_2|\varphi_b(0)|p_1\sigma_1\tau_1\rangle\,. \tag{A9} $$

We note that Eq. (A9) is gauge-invariant as S vanishes when $\epsilon_{\mu}{}^{\sigma}(k)$ is replaced by k_μ. (This is in contrast to the photoproduction of the pion where the peripheral diagram itself is *not* gauge-invariant.) The pion matrix element of Eq. (A9) is defined in Ref. 21. The remainder of the calculation of the cross section of Eq. (4.8) is now straightforward.

PHYSICAL REVIEW VOLUME 174, NUMBER 5 25 OCTOBER 1968

Analytic Continuation of an Amplitude*

W. J. Abbe and G. A. Gary
Physics Department, University of Georgia, Athens, Georgia 30601
(Received 1 May 1968; revised manuscript received 5 June 1968)

A general procedure is given for successively continuing the invariant amplitude to arbitrary regions of the z plane when the partial-wave amplitude is given explicitly by the most general modified Cheng representation. The partial-wave expansion is summed in terms of elementary functions or integrals thereof, and no dispersion or background integrals are required.

I. INTRODUCTION

Attempts to calculate strongly-interacting particle cross-sections self-consistently, require knowledge of the invariant amplitude in unphysical regions of the dynamical variables of energy squared (s) and momentum transfer squared (t), since only in these regions are the crossing relations nonempty.[1] However, if the invariant amplitude is defined in terms of a partial-wave expansion over various orbital angular-momentum states, as it usually is, such an expansion will in general only converge in a finite region of the s-t plane, and the problem therefore revolves around how one can make analytic continuations of such an expansion.

In several previous reports, a representation for the two-body, single-channel, partial-wave S-matrix element was studied and found to withstand, quite well, a number of tests to which it was subjected.[2–6] It has also been written for multichannel reactions, although no numerical comparisons have yet been made in this case.[7] It should be noted that while a comparison of a given conjectured partial-wave S-matrix with exact potential-theory results is negative, in the sense that a favorable comparison would clearly not necessarily imply the representation to be a valid relativistic one, an unfavorable comparison can at least be used to exclude many representations, since intuitively we expect any conjectured relativistic S-matrix element to also be valid in a "correspondence principle" or nonrelativistic limit.

Although the Regge method,[8] of rewriting the partial-wave expansion as a background integral plus pole terms in the angular-momentum plane, provides an analytic continuation of the invariant amplitude to all t in principle, in practice if one wishes to satisfy the crossing relations in threshold and intermediate regions of the s-t plane, it then becomes necessary to evaluate the background integral explicitly, and this is a formidable task because of the poor convergence properties of this integral.

The purpose of this report is to show that it is possible, however, to avoid the above difficulties with the background integral when one uses explicitly the modi-

* This work was partially supported by a grant from Research Corporation.
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