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Hard-Pion Current-Algebra Calculation of Meson Processes—Three-Point Functions*

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Hard-pion techniques are presented for calculating vertex functions involving π, ρ, A_1, and σ mesons. The development of the method involves the following assumptions: (a) saturation of intermediate sums by single mesons, (b) chiral $SU(2)\times SU(2)$ algebra commutation relations, (c) conservation of vector current, and (d) partial conservation of axial-vector current.

I. INTRODUCTION

Several years ago, Gell-Mann[1] suggested that "quark-type" equal-time commutation relations for the vector and axial-vector currents of weak-interaction theory serve as a basis for calculations involving strongly interacting systems. Combined with the assumptions of a conserved vector current (CVC) and a partially conserved axial-vector current (PCAC), this idea has yielded numerous successful predictions in the soft-pion approximation.[2] However, it has become increasingly clear that to treat processes such as $\rho\to 2\pi$, $A_1\to 3\pi$, and $K^*\to K+\pi$, the pion must be kept on its mass shell, since its kinetic energy is not small. In addition to this difficulty, one is necessarily uncomfortable with the soft-pion assumption, since its basis is not clear. There is also a serious lack of uniqueness in the process of continuing the pion momentum off its mass shell, since the mass shell is not a domain but a point.[3]

In this paper, we shall investigate a method for treating T products of three current operators that is not subject to the restrictions imposed by the assumption of soft pions.[4,5] It is based upon the ideas Weinberg[6] introduced in his treatment of the vacuum expectation value of the T product of two currents (two-point functions). In summary, the procedure to be employed consists of the following devices. One assumes: (1) that the T products to be evaluated may be expressed in terms of matrix elements of the currents between single-particle states, i.e., that the sum over intermediate states is saturated by single π, ρ, and A_1 particles; (2) that the resulting particle vertex functions may be approximated by a polynomial in the momenta of the single particles involved; (3) that the currents appearing in the T products satisfy the equal-time commutation relations of Gell-Mann,[1] PCAC, and CVC. In this paper, we shall limit our considerations to the chiral $SU(2)\times SU(2)$ algebra.

In order to carry out the above program it is found convenient to introduce an effective Lagrangian as a calculational device. This Lagrangian is displayed and the rules for its use given. All pertinent three-point functions are then evaluated and listed in the Appendix. In the following paper[7] we use the techniques developed here to calculate a number of physically interesting processes.

II. DYNAMICAL ASSUMPTIONS AND BASIC METHOD

We start by considering the following example of a three-point function:

$$F^{\alpha\mu\beta}(x,y,z)\equiv\langle 0|T(A^{\alpha}{}_{a}(x)V^{\mu}{}_{c}(z)A^{\beta}{}_{b}(y))|0\rangle. \quad (2.1)$$

Here, $A^{\mu}{}_{a}$ and $V^{\mu}{}_{a}$ are the axial-vector and vector currents. The indices a, b, and c are $SU(2)$ isotopic indices. $F^{\alpha\mu\beta}$ may be expanded into its six time orderings. For the moment we restrict our attention to the one corresponding to $x^0>z^0>y^0$. Thus, upon using closure, we obtain

$$F^{\alpha\mu\beta}(x,y,z)=\sum_{n,m}\langle 0|A^{\alpha}{}_{a}(x)|n\rangle \times\langle n|V^{\mu}{}_{c}(z)|m\rangle\langle m|A^{\beta}{}_{b}(y)|0\rangle. \quad (2.2)$$

We now assume that the sum over intermediate states is saturated by single π-, ρ-, and A_1-particle states. The validity of this approximation will, of course, depend upon the eventual comparison of our results with experiment.[7] However, we note at this time that this single-meson-saturation assumption is basically a generalization of the ρ-dominance hypothesis for the vector current. It also produces results in agreement

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[1] M. Gell-Mann, Physics **1**, 63 (1964).

[2] A summary of soft-pion current-algebra calculations is given in the talk by R. F. Dashen, in *Proceedings of the VIII International Conference on High-Energy Nuclear Physics* (University of California Press, Berkeley, 1967).

[3] L. S. Kisslinger, Phys. Rev. Letters **18**, 861 (1967).

[4] A summary of this work was given in R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967).

[5] Results equivalent to those of Ref. 4 but using different techniques have also been obtained by H. Schnitzer and S. Weinberg [Phys. Rev. **164**, 1828 (1967)] and by S. G. Brown and G. W. West [Phys. Rev. Letters **19**, 812 (1967); and Phys. Rev. **168**, 1605 (1968)]. Similar results, but using different physical assumptions, have been obtained by J. Schwinger [Phys. Letters **24B**, 473 (1967)], J. Wess and B. Zumino [Phys. Rev. **163**, 1727 (1967)] and B. Lee and H. T. Nieh [*ibid.* **166**, 1507 (1968)] using a "phenomenological" Lagrangian approach; and by T. Das, V. S. Mathur, and S. Okubo [Phys. Rev. Letters **19**, 859 (1967)] and D. A. Geffen [*ibid.* **19**, 770 (1967)] using dispersion-relation techniques.

[6] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[7] R. Arnowitt, M. H. Friedman, and P. Nath, following paper, Phys. Rev. **174**, 2008 (1968).

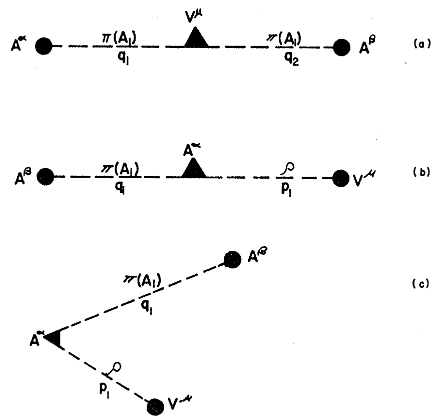

FIG. 1. (a) Diagram representing the time ordering $x^0>z^0>y^0$ in the three-point function $\langle 0|T(A^\alpha{}_a(x)V^\mu{}_c(z)A^\beta{}_b(y))|0\rangle$ for the case of π or A_1 intermediate states. The circles (●) represent the vacuum-to-one-particle matrix elements of the current, while the solid triangles (▲) represent the one-particle matrix elements. (b) Diagram for the time ordering $x^0>y^0>z^0$ with a π or A_1 for the first intermediate state and a ρ for the second. (c) Diagram with time ordering of (a) for a two-body intermediate state where π (or A_1) is a "spectator." This diagram is the crossed diagram of (b).

with the soft-pion calculations when the latter are valid, and thus provides a basis for the justification of the "gentleness" hypothesis. Returning to Eq. (2.2), the only single-particle states that can enter into the intermediate sums are the π and A_1 states. The contribution from the π states alone yields

$$F^{\alpha\mu\beta}(x,y,z)=\sum_{a_1,a_2}\int d^3q_1d^3q_2\langle 0|A^\alpha{}_a|\pi q_1a_1\rangle \times\langle\pi,q_1a_1|V^\mu{}_c|\pi q_2a_2\rangle\langle\pi q_2a_2|A^\beta{}_b|0\rangle \quad (2.3)$$

with additional terms obtained when $|n\rangle$ or $|m\rangle$, or both, refer to A_1 states.

We next consider Eq. (2.1) when $y^0>x^0>z^0$. This gives

$$F^{\alpha\mu\beta}=\sum_{n,m}\langle 0|A^\beta{}_b|n\rangle\langle n|A^\alpha{}_a|m\rangle\langle m|V^\mu{}_c|0\rangle. \quad (2.4)$$

Again, assuming single-particle dominance, the state $|n\rangle$ may be either a π or an A_1 state, while $|m\rangle$ must be a ρ meson. This may be represented diagrammatically as in Figs. 1(a) and 1(b).

Figure 1(b) represents a time-ordered Heitler diagram. Figure 1(c) is another Heitler diagram which is a part of the same Feynman diagram as is Fig. 1(b). It is clear that if we are to maintain Lorentz covariance (and hence, crossing) it is necessary that it also be included. This is accomplished by returning to Eq. (2.2) and including all two-particle intermediate states where, however, one of the particles is a "spectator," i.e., there is no sum over its momentum. In order to see this, we examine the following additional contribution from Eq. (2.2):

$$F^{\alpha\mu\beta}=\sum_{a_1,a_2,a_3}\int d^3q_1d^3q_2d^3q_3\langle 0|A^\alpha{}_a|\pi q_1a_1,\rho p_1a_3\rangle \times\langle\pi q_1a_1,\rho p_1a_3|V^\mu{}_c|\pi q_2a_2\rangle\langle\pi q_2a_2|A^\beta{}_b|0\rangle. \quad (2.5)$$

Upon using the usual Lehman-Symanzik-Zimmermann (L. S. Z.) reduction, the matrix element of the vector current becomes

$$\begin{aligned}\langle\pi q_1a_1,\rho p_1a_3|V^\mu{}_c(z)|\pi q_2a_2\rangle &=\delta^3(\mathbf{q}_1-\mathbf{q}_2)\delta a_1a_2\langle\rho p_1a_3|V^\mu{}_c|0\rangle\\ &-N_\pi(q_1)N_\pi(q_2)\int d^4xd^4y\, e^{iq_2x-iq_1y}K_xK_y\\ &\times\langle\rho p_1a_3|T(\varphi_{a_2}(x)\varphi_{a_1}(y)V^\mu{}_c(z))|0\rangle.\end{aligned} \quad (2.6a)$$

Here, $N_\pi(q)$ are the Bose normalization factors,[8] while $K_x=-\Box_x{}^2+m_\pi{}^2$. The second term on the right-hand side of Eq. (2.6) contains all the dynamical effects of the two-particle intermediate state, whereas the first term has the same content as the one-particle states. We shall therefore retain only the latter, in keeping with the assumption of single-particle dominance, and make the approximation

$$\langle\pi q_1a_1,\rho p_1a_3|V^\mu{}_c(z)|\pi q_2a_2\rangle \cong\delta^3(\mathbf{q}_1-\mathbf{q}_2)\delta_{a_1a_2}\langle\rho p_1a_3|V^\mu{}_c(z)|0\rangle. \quad (2.6b)$$

When this replacement is made in Eq. (2.5), we obtain the Heitler diagram corresponding to Fig. 1(c).

The various one-particle-to-vacuum matrix elements of the currents that are encountered in the above expansions serve to define the coupling strengths of these currents to the particles. Thus F_π, g_A, and g_ρ are defined by the equations

$$\langle 0|A^\mu{}_a(0)|\pi qb\rangle=iq^\mu N_\pi\delta_{ab}F_\pi, \quad (2.7a)$$

$$\langle 0|A^\mu{}_a(0)|A_1qb\sigma\rangle=\delta_{ab}N_Ag_A\,{}_A\epsilon^{\mu\sigma}(q), \quad (2.7b)$$

$$\langle 0|V^\mu{}_a(0)|\rho qb\sigma\rangle=\delta_{ab}N_\rho g_\rho\,{}_\rho\epsilon^{\mu\sigma}(q). \quad (2.7c)$$

In Eqs. (2.7), ${}_A\epsilon^{\mu\sigma}$ and ${}_\rho\epsilon^{\mu\sigma}$ are the polarization vectors of helicity σ normalized by $\epsilon^{\mu\sigma*}\epsilon_\mu{}^{\sigma'}=\delta^{\sigma\sigma'}$.

The other types of matrix elements arising in the above expansion of the T product are the one-particle-to-one-particle elements of $V^\mu{}_a$ and $A^\mu{}_a$, as well as the vacuum-to-two-particle matrix elements. This latter is determined by the former through the demands of crossing symmetry. Single-particle dominance suggests that the vector current links to the particles through the ρ meson while the axial-vector current links through the π and A_1 mesons. In fact, without any loss of generality, we may write

$$\langle Bq_1a|V^\mu{}_c(0)|Cq_2b\rangle =i\epsilon_{abc}N_BN_C\,{}_\rho\Delta^\mu{}_\lambda(k)\Gamma_{B\rho C}(q_1,q_2) \quad (2.8a)$$

[8] We normalize states so that $N_\pi(q)=[2\omega_q(2\pi)^3]^{-1/2}$, where $\omega_q=(q^2+m_\pi{}^2)^{1/2}$.

and

$$\langle Bq_1a|A^\mu{}_c(0)|Cq_2b\rangle = i\epsilon_{abc}N_BN_C[{}_A\Delta^\mu{}_\lambda(k)\Gamma_{BAC}{}^\lambda(q_1,q_2) + {}_\pi\Delta(k)\Gamma_{B\pi C}{}^\mu(q_1,q_2)], \quad (2.8b)$$

where ${}_\rho\Delta^\mu{}_\lambda$, ${}_A\Delta^\mu{}_\lambda$, and ${}_\pi\Delta$ are the ρ, A_1, and π propagators

$${}_\rho\Delta^\mu{}_\lambda(k) = (k^2+m_\rho{}^2)^{-1}(\delta^\mu{}_\lambda + k^\mu k_\lambda m_\rho{}^{-2}), \quad (2.9a)$$

$${}_A\Delta^\mu{}_\lambda(k) = (k^2+m_A{}^2)^{-1}(\delta^\mu{}_\lambda + k^\mu k_\lambda m_A{}^{-2}), \quad (2.9b)$$

$${}_\pi\Delta(k) = (k^2+m_\pi{}^2)^{-1}. \quad (2.9c)$$

In these propagators, the masses are those of the physical particles. In Eqs. (2.8) the labels B and C are particle labels (π, ρ, or A_1) and take on any values allowed by G parity, while the momentum transfer k^μ is $q_1{}^\mu - q_2{}^\mu$. Expressions (2.8) involve no loss of generality, since ${}_\rho\Delta^\mu{}_\lambda$, ${}_A\Delta^\mu{}_\lambda$, and $\delta^\mu{}_\lambda\,{}_\pi\Delta$ are nonsingular matrices and thus these equations may be viewed as defining relations for the particle vertex functions $\Gamma_{B\rho C}{}^\lambda$, etc.

We now assume that the particle vertices can be approximated by a polynomial in the momentum transfer, k^2. Thus, since we have already extracted the particle poles, it is reasonable to assume that the factors $\Gamma_{B\rho C}{}^\lambda$, etc., are relatively smooth functions for k^2 not too large. [In applications,[7] the approximation appears successful for k^2 up to the order of (1 GeV)^2.] Thus, for the π-ρ-π vertex we write

$$\Gamma_{\pi\rho\pi}{}^\mu(q_1,q_2) = (q_1{}^\mu + q_2{}^\mu)(\alpha_1 + \alpha_2 k^2 + \cdots), \quad (2.10)$$

with similar expressions for the other vertex functions. Equations (2.9) and (2.10) may now be inserted into Eqs. (2.8), and the resultant forms for the matrix elements of Eqs. (2.7) and (2.8) inserted back into the expansion of the T products. Thus one obtains expressions for the latter which are covariant and crossing-symmetric. They are given in terms of unknown parameters α_1, α_2, etc.

We next subject the T products to the restrictions imposed by the current algebra. We will see that this in part determines the parameters α_1, α_2, etc. The commutation relations to be satisfied by the current densities are those of chiral $SU(2)\times SU(2)$:

$$\delta(x^0-y^0)[V^0{}_a(x),V^\mu{}_b(y)] = i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y) + c\text{-No. S.T.}, \quad (2.11a)$$

$$\delta(x^0-y^0)[V^0{}_a(x),A^\mu{}_b(y)] = i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y) + c\text{-No. S.T.}, \quad (2.11b)$$

$$\delta(x^0-y^0)[A^0{}_a(x),V^\mu{}_b(y)] = i\epsilon_{abc}A^\mu{}_c(x)\delta^4(x-y) + c\text{-No. S.T.}, \quad (2.11c)$$

$$\delta(x^0-y^0)[A^0{}_a(x),A^\mu{}_b(y)] = i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y) + c\text{-No. S.T.}, \quad (2.11d)$$

where "c-No. S.T." stands for c-number Schwinger terms. We consider the vacuum–one-particle matrix elements of Eqs. (2.11). For example, from Eq. (2.11a) one has

$$\delta(x^0-y^0)\langle 0|[V^0{}_a(x),V^\mu{}_b(y)]|\rho,k e\rangle = i\epsilon_{abd}\delta^4(x-y)\langle 0|V^\mu{}_d(x)|\rho,k e\rangle. \quad (2.12)$$

Since the $V^\mu{}_a$ may be used as an interpolating field for the ρ meson, contracting down that particle shows that the left-hand side is proportional to three-point functions. (Similarly, $A^\mu{}_a$ may be used as an interpolating field for the π and A_1 mesons, and so the same is true for all the vacuum-one-particle matrix elements.) We therefore invoke single-meson saturation in the left-hand side of Eq. (2.12) to obtain

$$\delta(x^0-y^0)\sum_n [\langle 0|A^0{}_a(x)|n\rangle\langle n|A^\beta{}_b(y)|\rho c p\rangle - \langle 0|A^\beta{}_b(y)|n\rangle\langle n|A^0{}_a(x)|\rho c p\rangle] = i\epsilon_{abd}\delta^4(x-y)\langle 0|V^\beta{}_d(x)|\rho c p\rangle, \quad (2.13)$$

where the states $|n\rangle$ are single-particle states of π or A_1 mesons. Upon inserting our previous evaluation for the matrix elements appearing in Eq. (2.13), we obtain an algebraic constraint upon the unknown parameters α_1, α_2, etc., appearing in our expansion of the vertex functions [e.g., Eq. (2.10)]. Equations analogous to Eq. (2.13) may be obtained from the remaining commutation relations of Eq. (2.11) with the resulting algebraic system of equations relating the parameters.

The last requirements are those of PCAC and CVC:

$$\partial_\mu A^\mu{}_a(x) = F_\pi m_\pi{}^2\varphi_a(x), \quad (2.14a)$$

$$\partial_\mu V^\mu{}_a(x) = 0, \quad (2.14b)$$

where Eq. (2.14a) may be viewed as the defining equation for $\varphi_a(x)$. One may use $\varphi_a(x)$ as an interpolating field for the pion field. The vacuum-one-particle matrix elements of Eqs. (2.14) are automatically satisfied by Eqs. (2.7). The one-meson–one-meson matrix elements of $A^\mu{}_a$ and $V^\mu{}_a$ are to be used in constructing the T products, and hence must satisfy Eqs. (2.14). Thus from Eqs. (2.14b) and (2.8a) one learns that

$$k_\lambda\Gamma_{B\rho C}{}^\lambda(q_1,q_2) = 0. \quad (2.15)$$

In Eq. (2.8b) two vertex functions have been defined and so we are free to specify one in the most convenient fashion without loss of generality. Let us therefore define $\Gamma_{B\pi C}{}^\mu$ by

$$i\epsilon_{abc}N_BN_C\,{}_\pi\Delta(k)\Gamma_{B\pi C}{}^\mu(q_1,q_2) \equiv F_\pi\langle Bq_1a|\partial^\mu\varphi_c|Cq_2b\rangle, \quad (2.16)$$

where φ_c is given by Eq. (2.14a). The significance of this choice resides in the fact that $\Gamma_{B\pi C}{}^\mu$ then will represent the spin-zero part of the axial-vector current vertex function on the pion mass shell. We may thus write

$$\Gamma_{B\pi C}{}^\mu = -F_\pi k^\mu f_{B\pi C}(q_1,q_2), \quad (2.17)$$

where $f_{B\pi C}$ is the vertex function of the pion field as defined by Eq. (2.14a). From Eqs. (2.8b) and (2.14a), we now obtain

$$f_{B\pi C}(q_1,q_2)=(F_\pi m_A{}^2)^{-1}k_\lambda\Gamma_{BAC}{}^\lambda(q_1,q_2)\,. \tag{2.18}$$

Thus, Eqs. (2.13), (2.15), and (2.18) yield a set of algebraic equations that restrict the allowed values of the unknown coupling constants appearing in the expansion of the vertex functions. In principle, these equations can be solved, thus determining the matrix elements of the vector and axial-vector currents. Finally, one uses the latter for computing the T products with which we started.

While the program described above will allow the calculation of the T products of three vector and axial-vector currents obeying the current-algebra constraints, it is somewhat tedious to actually calculate three-point functions in this fashion, and we proceed next to introduce an "effective Lagrangian" as a convenient calculational tool to simplify the analysis.

III. EFFECTIVE LAGRANGIAN

In order to generate the matrix elements of Eqs. (2.7) and (2.8), it is convenient to introduce a set of in-field operators[9] $\tilde{\varphi}_a(x)$, $\tilde{v}^\mu{}_a(x)$, and $\tilde{a}^\mu{}_a(x)$, which correspond to π, ρ, and A_1 particles, respectively. We now note that Eqs. (2.7) are reproduced if $V^\mu{}_a$ and $A^\mu{}_a$ are replaced by

$$V^\mu{}_a\to g_\rho\tilde{v}^\mu{}_a\,, \tag{3.1a}$$

$$A^\mu{}_a\to g_A\tilde{a}^\mu{}_a+F_\pi\partial^\mu\tilde{\varphi}_a\,. \tag{3.1b}$$

Similarly, Eqs. (2.8) may be reproduced if one replaces $A^\mu{}_a$ and $V^\mu{}_a$ by bilinear operators in the in-fields. Thus if, in Eq. (2.8a), we consider B and C to be π-mesons, then the matrix element is correctly reproduced [with $\Gamma^\lambda{}_{\pi\rho\pi}$ expanded as in Eq. (2.10)] if $V^\mu{}_c$ is replaced by

$$V^\mu{}_c\to\int d^4y\;{}_\rho\Delta^\mu{}_\lambda(x-y)\,\epsilon_{abc}\times[\alpha_1-\alpha_2\Box+\cdots]\tilde{\varphi}_a(y)\partial^\lambda\tilde{\varphi}_b(y)\,. \tag{3.2a}$$

Similarly, in Eq. (2.8b), $B=\pi$ and $C=\rho$, the matrix element is correctly generated by setting $A^\mu{}_c$ to

$$A^\mu{}_c(x)\to\epsilon_{abc}\int d^4y\{{}_A\Delta^\mu{}_\lambda(x-y)[\beta_1\tilde{\varphi}_a(y)\tilde{v}^\lambda{}_b(y)+\cdots]+{}_\pi\Delta(x-y)[\gamma_1\tilde{\varphi}_a(y)\tilde{v}^\mu{}_b(y)+\cdots]\}\,. \tag{3.2b}$$

The omitted terms are those additional bilinear structures necessary to reproduce correctly all the matrix elements corresponding to Eqs. (2.8) (i.e., when B and C range over their allowed values). The form of $V^\mu{}_c$ and $A^\mu{}_c$ in Eqs. (3.1) and (3.2) automatically guarantees crossing symmetry and hence may also be used for evaluating the vacuum-to-two-particle matrix elements.

[9] One may of course, employ out-field operators instead.

If we now add the results of Eqs. (3.1) to those of Eqs. (3.2), we note that we have an expansion of the currents in terms of in-field operators for particles. Thus one has that the phenomenological expressions

$$V^\mu{}_c(x)\simeq g_\rho\tilde{v}^\mu{}_c(x)+\epsilon_{abc}\int d^4y\;{}_\rho\Delta^\mu{}_\lambda(x-y)\times[\alpha_1-\alpha_2\Box+\cdots]\tilde{\varphi}_a(y)\partial^\mu\tilde{\varphi}_b(y)+\cdots \tag{3.3a}$$

and

$$A^\mu{}_c(x)\simeq g_A\tilde{a}^\mu{}_c(x)+F_\pi\partial^\mu\tilde{\varphi}_a(x)+\epsilon_{abc}\int d^4y\{{}_A\Delta^\mu{}_\lambda(x-y)[\beta_1\tilde{\varphi}_a\tilde{v}^\lambda{}_b+\cdots]+{}_\pi\Delta(x-y)[\gamma_1\tilde{\varphi}_a\tilde{v}^\mu{}_b+\cdots]\}+\cdots \tag{3.3b}$$

will correctly reproduce all the matrix elements needed to calculate the three-point functions in the single-meson-saturation approximation. (In relation to the order in which we are working, we need not concern ourselves with the difference between these expansions and a normal-ordered expansion.)

One may next rephrase this result by noting that Eqs. (3.3) would automatically arise upon solving a set of coupled Heisenberg equations of motion, provided we appropriately truncated the solutions. Thus, let $v^\mu{}_c(x)$, $a^\mu{}_c(x)$, and $\varphi_c(x)$ be a set of ρ, A_1, and π *Heisenberg* field operators obeying

$${}_\rho P^\mu{}_\lambda(x)v^\lambda{}_c(x)=g_\rho{}^{-1}\epsilon_{abc}[\alpha_1-\alpha_2\Box+\cdots]\varphi_a(x)\partial^\mu\varphi_b(x)+\cdots\,, \tag{3.4a}$$

$${}_AP^\mu{}_\lambda(x)a^\lambda{}_c(x)=g_A{}^{-1}\epsilon_{abc}[\beta_1\varphi_a(x)v^\mu{}_b(x)+\cdots]+\cdots\,, \tag{3.4b}$$

$${}_\pi K(x)\varphi_c(x)=F_\pi{}^{-1}\epsilon_{abc}[\gamma_1\varphi_{a\mu}(x)v^\mu{}_b(x)+\cdots]+\cdots\,, \tag{3.4c}$$

where ${}_\rho P^\mu{}_\lambda$ and ${}_AP^\mu{}_\lambda$ are the ρ and A_1 Proca operators $[{}_\rho P^\mu{}_\lambda=(-\Box+m_\rho{}^2)\delta^\mu{}_\lambda+\partial^\mu\partial_\lambda]$ and ${}_\pi K=-\Box+m_\pi{}^2$. Then the desired expressions for the currents follow, if we set

$$V^\mu{}_c(x)=g_\rho v^\mu{}_c(x) \tag{3.5a}$$

and

$$A^\mu{}_c(x)=g_Aa^\mu{}_c(x)+F_\pi\partial^\mu\varphi_c(x)\,, \tag{3.5b}$$

provided we will carry out the solutions of Eqs. (3.4) in an in-field expansion to first order only in the coupling constants α_i, β_i, and γ_i.

One now sees that the T products of currents, e.g., Eq. (2.1), may be expressed in terms of T products of Heisenberg particle fields by using Eqs. (3.4). Furthermore, we arrive at the Fundamental result that the single-particle saturation condition is equivalent to evaluating these T products by using the equation of motion (3.3) and evaluating to first order in the coupling constants α_i, β_i, and γ_i, etc.

Our next task is to satisfy the commutation relations Eqs. (2.11), but only to within the demands of single-particle saturation as expressed by Eq. (2.13). We note that this is equivalent to using the expressions given by Eqs. (3.3), but evaluating the commutators only to first order in α_i, β_i, and γ_i, while the right-hand side of Eqs. (2.11) are to be evaluated to zeroth order in these parameters. However, the currents of Eq. (3.3) are not in general even local-field operators whose commutators vanish for spacelike separations. This is due to the presence of the nonlocal propagators ${}_A\Delta^{\mu\nu}(x-y)$, etc. The locality condition on the current will be satisfied if one requires that the Heisenberg fields appearing there ($v^\mu{}_a$, $a^\mu{}_a$, and φ_a), in fact be, *local*-field operators. The simplest way of guaranteeing this (and probably the only way) is to require that the equations of motion, Eqs. (3.4), satisfied by them be obtained from a local-field Lagrangian. Since the source terms that appear or the right-hand side of these equations are bilinear in the fields, the interaction Lagrangian must be cubic. We therefore construct the following *effective Lagrangian* as a device for generating Eqs. (3.4).

$$\mathfrak{L}=\mathfrak{L}_{0\pi}+\mathfrak{L}_{0\rho}+\mathfrak{L}_{0A}+\mathfrak{L}_{(3)\pi\rho A}\,, \tag{3.6a}$$

where the free-particle Lagrangian is given by

$$\begin{aligned}\mathfrak{L}_{0\pi}+\mathfrak{L}_{0\rho}+\mathfrak{L}_{0A}=&-\varphi^\mu{}_a\partial_\mu\varphi_a+\tfrac{1}{2}(\varphi^\mu{}_a\varphi_{\mu a}-m_\pi{}^2\varphi_a{}^2)\\&-\tfrac{1}{2}G^{\mu\nu}{}_a(\partial_\mu v_{\nu a}-\partial_\nu v_{\mu a})+\tfrac{1}{4}G^{\mu\nu}{}_aG_{\mu\nu a}\\&-\tfrac{1}{2}m_\rho{}^2v^\mu{}_av_{\mu a}-\tfrac{1}{2}H^{\mu\nu}{}_a(\partial_\mu a_{\nu a}-\partial_\nu a_{\mu a})\\&+\tfrac{1}{4}H^{\mu\nu}{}_aH_{\nu\mu a}-\tfrac{1}{2}m_A{}^2a^\mu{}_aa_{\mu a}\end{aligned} \tag{3.6b}$$

and the interaction Lagrangian is

$$\begin{aligned}\mathfrak{L}_{(3)\pi\rho A}=\tfrac{1}{2}\epsilon_{abc}[&2g_{\pi\pi\rho}\varphi^\mu{}_b\varphi_cv_{\mu a}+\lambda_{\pi\pi\rho}\varphi_{\mu a}\varphi_{\nu b}G^{\nu\mu}{}_c\\&+2g_{\pi\rho A}v_{\mu a}\varphi_ba^\mu{}_c+2\mu_{\pi\rho A}\varphi_aG^{\mu\nu}{}_bH_{\mu\nu c}+2\lambda_{\pi\rho A}v_{\mu a}\varphi_{\nu b}H^{\mu\nu}{}_c\\&+2\bar{\lambda}_{\pi\rho A}a_{\mu a}\varphi_{\nu b}G^{\mu\nu}{}_c+g_{\rho\rho\rho}v_{\mu a}v_{\nu b}G^{\nu\mu}{}_c+2g_{\rho AA}v_{\mu a}a_{\nu b}H^{\nu\mu}{}_c\\&+\lambda_{\rho AA}a_{\mu a}a_{\nu b}G^{\nu\mu}{}_c+\mu_{\rho\rho\rho}G_{\mu\nu a}G^{\nu\lambda}{}_bG_\lambda{}^\mu{}_c\\&+\mu_{\rho AA}G_{\mu\nu a}H^{\nu\lambda}{}_bH_\lambda{}^\mu{}_c].\end{aligned} \tag{3.6c}$$

We have here chosen to use a "first-order" formalism for later convenience, where (φ_μ,φ), $(G_{\mu\nu},v_\mu)$, and $(H_{\mu\nu},a_\mu)$ are to be varied independently to yield first-order, coupled differential equations. Thus (φ_{0a},φ_a), (G_{0ia},v_{ia}), and (H_{0ia},a_{ia}) are the canonically conjugate pairs of variables for the π, ρ, and A_1 fields, respectively. The coupling constants $g_{\pi\pi\rho}$, $\lambda_{\pi\pi\rho}$, etc., are totally arbitrary at this point of the analysis and can be related to the α_i, β_i, and γ_i that appeared earlier [e.g., in Eqs. (3.3)]. The above $\mathfrak{L}_{(3)\pi\rho A}$ is the most general cubic interaction not containing explicit derivatives of the π, ρ, and A_1 fields in the first-order formalism. This is tantamount to having fixed the number of derivatives that are allowed on the right-hand side of Eqs. (3.4), and hence the amount of momentum transfer allowed in the vertex functions of Eqs. (2.8). Whether the truncation of the expansion of the vertex functions in powers of the momentum transfer to this order is correct or not, is a question that must be answered by comparison with experiment. It should be emphasized that the Lagrangian that has been introduced is a purely calculational device.

One may now employ this Lagrangian to compute a T product of currents such as appears in Eq. (2.1), using the defining relations Eqs. (3.5) for the currents. Furthermore, the condition of single-particle saturation requires that we carry out this computation only to *first order* in the coupling constants. Thus first-order perturbation theory is the total domain of validity of the Lagrangian.

We note that up to this point the only symmetries that have been imposed on $\mathfrak{L}$, and hence on the T products, is the conservation of isotopic spin and G parity. We are now in a position to determine the constraints imposed on the values of the coupling constants by the current algebra, i.e., current commutation relations (CCR), CVC, and PCAC. It is these physical conditions *alone* that will govern the amount of chiral $SU(2)\times SU(2)$ symmetry remaining in the T products of the currents. We will impose no chiral $SU(2)\times SU(2)$ symmetry requirements on the Lagrangian itself.

IV. CURRENT-ALGEBRA CONSTRAINTS ON THE COUPLING CONSTANTS

We start by examining the requirements for satisfying the conditions imposed by the commutation relations Eqs. (2.11), but only to the order required by single-particle saturation as given in Eq. (2.13). The perturbation expansion of $V^\mu{}_a$ and $A^\mu{}_a$ in in-field operators has been given in Eqs. (3.3). We note that the terms linear in these operators are of zeroth order in the coupling constants, while the quadratic terms are of first order. Hence, the left-hand side of the commutators in Eqs. (2.11) need only be carried out to first order in the coupling constants, while the right-hand side is evaluated to zeroth order. This is completely equivalent to Eq. (2.13). Rather than working directly with the in-field expansions, it is more convenient to express the currents in terms of the Heisenberg canonical variables by using the field equations. One then evaluates the commutators to first order in the coupling constants.

In order to satisfy the conditions of PCAC and CVC, it is again more convenient to work with the Heisenberg fields directly, rather than the matrix elements. From Eqs. (2.14b), (2.15), and (3.5a), we find that CVC means that we are to require that

$$\partial_\mu V^\mu{}_c(x)=g_\rho\partial_\mu v^\mu{}_c(x)=0 \tag{4.1}$$

be satisfied only to first order in the coupling constants. Similarly, from Eqs. (2.14a), (2.18), and (3.5b) we require that

$$\partial_\mu A^\mu{}_c(x)=g_A\partial_\mu a^\mu{}_c(x)+F_\pi\Box\varphi_c(x)=F_\pi m_\pi{}^2\varphi_c(x) \tag{4.2}$$

or

$$g_A\partial_\mu a^\mu{}_c(x)-F_\pi(-\Box+m_\pi{}^2)\varphi_c(x)=0 \tag{4.3}$$

also be satisfied only to first order in the coupling constants. We note that Eq. (4.3) is totally equivalent to Eq. (2.18).

The Lagrangian, Eqs. (3.6), yields the following set of field equations for the pion fields:

$$\varphi^\mu{}_a=\partial^\mu\varphi_a+\epsilon_{abc}(g_{\pi\pi\rho}v^\mu{}_b\varphi_c-\lambda_{\pi\pi\rho}\varphi_{\lambda a}G^{\lambda\mu}{}_c+\lambda_{\pi\rho A}v_{\lambda b}H^{\lambda\mu}{}_c+\bar{\lambda}_{\pi\rho A}a_{\lambda b}G^{\lambda\mu}{}_c),\quad(4.4\text{a})$$

$$-\partial_\mu\varphi^\mu{}_a+m_\pi{}^2\varphi_a=\epsilon_{abc}(g_{\pi\pi\rho}v_{\mu b}\varphi^\mu{}_c-g_{\pi\rho A}v_{\mu b}a^\mu{}_c+\mu_{\pi\rho A}G^{\mu\nu}{}_bH_{\mu\nu c});\quad(4.4\text{b})$$

for the ρ fields:

$$G^{\mu\nu}{}_a=(\partial^\mu v^\nu{}_a-\partial^\nu v^\mu{}_a)+\epsilon_{abc}[g_{\rho\rho\rho}v^\mu{}_bv^\nu{}_c+\lambda_{\rho AA}a^\mu{}_ba^\nu{}_c+\lambda_{\pi\pi\rho}\varphi^\mu{}_b\varphi^\nu{}_c+2\mu_{\pi\rho A}\varphi_bH^{\mu\nu}{}_c-\bar{\lambda}_{\pi\rho A}(a^\mu{}_b\varphi^\nu{}_c-a^\nu{}_b\varphi^\mu{}_c)-3\mu_{\rho\rho\rho}G^{\nu\lambda}{}_bG_\lambda{}^\mu{}_c-\mu_{\rho AA}H^{\nu\lambda}{}_bH_\lambda{}^\mu{}_c],\quad(4.5\text{a})$$

$$\partial_\nu G^{\mu\nu}{}_a+m_\rho{}^2v^\mu{}_a=\epsilon_{abc}(g_{\rho\rho\rho}v_{\lambda b}G^{\lambda\mu}{}_c+g_{\rho AA}a_{\lambda b}H^{\lambda\mu}{}_c-g_{\pi\pi\rho}\varphi_b\varphi^\mu{}_c+g_{\pi\rho A}\varphi_ba^\mu{}_c+\lambda_{\pi\rho A}\varphi_{\lambda b}H^{\mu\lambda}{}_c);\quad(4.5\text{b})$$

and for the A_1 fields:

$$H^{\mu\nu}{}_a=(\partial^\mu a^\nu{}_a-\partial^\nu a^\mu{}_a)+\epsilon_{abc}[g_{\rho AA}(v^\mu{}_ba^\nu{}_c-v^\nu{}_ba^\mu{}_c)-2\mu_{\pi\rho A}\varphi_bG^{\mu\nu}{}_c-\lambda_{\pi\rho A}(v^\mu{}_b\varphi^\nu{}_c-v^\nu{}_b\varphi^\mu{}_c)+2\mu_{\rho AA}H^{\mu\lambda}{}_bG_\lambda{}^\nu{}_c],\quad(4.6\text{a})$$

$$\partial_\nu H^{\mu\nu}{}_a+m_A{}^2a^\mu{}_a=\epsilon_{abc}(g_{\rho AA}v_{\lambda b}H^{\lambda\mu}{}_c+\lambda_{\rho AA}a_{\lambda b}G^{\lambda\mu}{}_c+g_{\pi\rho A}v^\mu{}_b\varphi_c+\bar{\lambda}_{\pi\rho A}\varphi_{\lambda b}G^{\mu\lambda}{}_c).\quad(4.6\text{b})$$

The canonically conjugate pairs of variables are (φ_{0a},φ_a), (G_{0ia},v_{ia}), and (H_{0ia},a_{ia}).

In order to satisfy Eq. (4.1) (CVC), we take the divergence of both sides of Eq. (4.5b). This yields

$$m_\rho{}^2\partial_\mu v^\mu{}_a=\epsilon_{abc}[g_{\pi\rho A}(\partial_\mu\varphi_b)a^\mu{}_c+g_{\pi\rho A}\varphi_b(\partial_\mu a^\mu{}_c)+\lambda_{\pi\rho A}(\partial_\mu\varphi_{\lambda b})H^{\mu\lambda}{}_c+\lambda_{\pi\rho A}\varphi_{\lambda b}(\partial_\mu H^{\mu\lambda}{}_c)]$$

or

$$m_\rho{}^2\partial_\mu v^\mu{}_a=\epsilon_{abc}(g_{\pi\rho A}+m_A{}^2\lambda_{\pi\rho A})\varphi_{\lambda b}a^\lambda{}_c\quad(4.7)$$

to first order in the coupling constants. The vanishing of $\partial_\mu V^\mu{}_a=g_\rho\partial_\mu v^\mu{}_a$ thus requires

$$g_{\pi\rho A}+m_A{}^2\lambda_{\pi\rho A}=0.\quad(4.8)$$

Similarly, to examine Eq. (4.3) (PCAC), we use Eqs. (4.4a), (4.4b), and (4.6b). Again, satisfaction to first order in the coupling constants yields the three equations

$$-\tfrac{1}{2}(g_A/m_A{}^2)g_{\rho AA}+\tfrac{1}{2}(g_A/m_A{}^2)\lambda_{\rho AA}-F_\pi\mu_{\pi\rho A}+\tfrac{1}{2}F_\pi\lambda_{\pi\rho A}-\tfrac{1}{2}F_\pi\bar{\lambda}_{\pi\rho A}=0,\quad(4.9)$$

$$-g_Ag_{\rho AA}+(g_A/m_A{}^2)m_\rho{}^2\lambda_{\rho AA}+F_\pi(g_{\pi\rho A}+m_A{}^2\lambda_{\pi\rho A})-F_\pi m_\rho{}^2\bar{\lambda}_{\pi\rho A}=0,\quad(4.10)$$

$$(g_A/m_A{}^2)g_{\pi\rho A}-(g_Am_\rho{}^2/m_A{}^2)\bar{\lambda}_{\pi\rho A}-2F_\pi g_{\pi\pi\rho}+F_\pi m_\rho{}^2\lambda_{\pi\pi\rho}=0.\quad(4.11)$$

We next turn to the task of expressing the currents in terms of the Heisenberg canonical variables in order to carry out the CCR [Eqs. (2.11)]. Making use of the fact that v_{ia}, a_{ia}, and φ_a are canonical variables ($i=1,2,3$), we find from Eqs. (3.5) that $A^i{}_a$ and $V^i{}_a$ are linear in the canonical variables. Thus,

$$V^i{}_a(x)=g_\rho v^i{}_a(x)\quad(4.12\text{a})$$

and

$$A^i{}_a(x)=g_Aa^i{}_a(x)+F_\pi\partial^i\varphi_a(x).\quad(4.12\text{b})$$

The quantities φ_{ia}, G_{ija}, H_{ija}, v_{0a}, and a_{0a} are constraint variables and may be eliminated by repeated applications of Eqs. (4.4a), (4.5a), (4.6a), (4.5b), and (4.6b), respectively, which will result in their being expressed as an infinite series in powers of the Heisenberg canonical variables. Since we are interested in carrying out the commutation relations only to first order in the coupling constants, we need carry out these expansions only to that order. In particular, we shall need to know $v^0{}_a$ and $a^0{}_a$, and also the time derivative $\partial^0\varphi_a$ in terms of the canonical variables. Inserting these results into $V^0{}_a=g_\rho v^0{}_a$ and $A^0{}_a=g_Aa^0{}_a+F_\pi\partial^0\varphi_a$, we obtain, for the time components of the currents,

$$V^0{}_a=\frac{g_\rho}{m_\rho{}^2}\partial_iG^{i0}{}_a+\frac{g_\rho}{m_\rho{}^2}\epsilon_{abc}\left(g_{\rho\rho\rho}v_{ib}G^{i0}{}_c+g_{\rho AA}a_{ib}H^{i0}{}_c+g_{\pi\pi\rho}\varphi_b\varphi_{c0}+\frac{g_{\pi\rho A}}{m_A{}^2}\varphi_b\partial_iH^{i0}{}_c+\lambda_{\pi\rho A}(\partial_i\varphi_b)H^{0i}{}_c\right),\quad(4.13\text{a})$$

$$A^0{}_a=\frac{g_A}{m_A{}^2}\partial_iH^{i0}{}_a+F_\pi\varphi^0{}_a+\frac{g_A}{m_A{}^2}\epsilon_{abc}\left(g_{\rho AA}v_{ib}H^{i0}{}_c+\lambda_{\rho AA}a_{ib}G^{i0}{}_c+\frac{g_{\pi AA}}{m_\rho{}^2}(\partial_iG^{i0}{}_b)\varphi_c+\bar{\lambda}_{\pi\rho A}(\partial_i\varphi_b)G^{0i}{}_c\right)$$
$$-F_\pi\epsilon_{abc}\left(\frac{g_{\pi\pi\rho}}{m_\rho{}^2}(\partial_iG^{i0}{}_b)\varphi_c-\lambda_{\pi\pi\rho}(\partial_i\varphi_b)G^{i0}{}_c+\lambda_{\pi\rho A}v_{ib}H^{i0}{}_c+\bar{\lambda}_{\pi\rho A}a_{ib}G^{i0}{}_c\right).\quad(4.13\text{b})$$

Equation (4.13a) may be simplified by making use of Eq. (4.8). Thus

$$V^0{}_a=\frac{g_\rho}{m_\rho{}^2}\partial_iG^{i0}{}_a+\frac{g_\rho}{m_\rho{}^2}\epsilon_{abc}[g_{\rho\rho\rho}v_{ib}G^{i0}{}_c+g_{\rho AA}a_{ib}H^{i0}{}_c+g_{\pi\pi\rho}\varphi_b\varphi_{c0}-\lambda_{\pi\rho A}\partial_i(\varphi_bH^{i0}{}_c)].\quad(4.14)$$

Similarly, using Eqs. (4.8), (4.10), and (4.11), one finds

$$A^0{}_a=\frac{g_A}{m_A{}^2}\partial_i H^{i0}{}_a+F_\pi\varphi^0{}_a+\epsilon_{abc}\bigg[\bigg(\frac{g_A}{m_A{}^2}g_{\rho AA}-F_\pi\lambda_{\pi\rho A}\bigg)v_{ib}H^{i0}{}_c+\frac{g_A}{m_\rho{}^2}g_{\rho AA}a_{ib}G^{i0}{}_c+\frac{F_\pi}{m_\rho{}^2}g_{\pi\pi\rho}(\partial_i\varphi_b)G^{i0}{}_c$$
$$+\bigg(F_\pi\frac{g_{\pi\pi\rho}}{m_\rho{}^2}+g_A\frac{\lambda_{\pi\rho A}}{m_\rho{}^2}\bigg)\partial_i(\varphi_b G^{i0}{}_c)\bigg]. \quad (4.15)$$

It is easier to satisfy Eqs. (2.11) by first considering the once-spatially-integrated commutation relations. Having done this, we shall then go back and satisfy Eq. (2.11) completely, by requiring that there be *no* q-number Schwinger terms. Thus we first investigate the following equal-time commutation relations:

$$\bigg[\int V^0{}_a(x)d^3x, V^\mu{}_b(y)\bigg]=i\epsilon_{abc}V^\mu{}_c(y), \quad (4.16a)$$

$$\bigg[\int V^0{}_a(x)d^3x, A^\mu{}_b(y)\bigg]=i\epsilon_{abc}A^\mu{}_c(y), \quad (4.16b)$$

$$\bigg[\int A^0{}_a(x)d^3x, V^\mu{}_b(y)\bigg]=i\epsilon_{abc}A^\mu{}_c(y), \quad (4.16c)$$

$$\bigg[\int A^0{}_a(x)d^3x, A^\mu{}_b(y)\bigg]=i\epsilon_{abc}V^\mu{}_c(y). \quad (4.16d)$$

In examining these relations, we recall that the left-hand side need only be evaluated to first order in the coupling constants, the right-hand side to zeroth order. We first consider Eq. (4.16a). This gives (for all μ) the condition

$$g_{\rho\rho\rho}=m_\rho{}^2/g_\rho. \quad (4.17)$$

From Eq. (4.16b) one finds (for all μ)

$$g_{\rho AA}=m_\rho{}^2/g_\rho \quad (4.18)$$

and

$$g_{\pi\pi\rho}=m_\rho{}^2/g_\rho. \quad (4.19)$$

Equation (4.16c) and the $\mu=i$ components of Eq. (4.16d) produce no new relations. From Eq. (4.16d), for $\mu=0$, we find

$$\frac{g_A{}^2}{g_\rho{}^2}\frac{m_\rho{}^2}{m_A{}^2}-\frac{g_A}{g_\rho}F_\pi\lambda_{\pi\rho A}=1. \quad (4.20)$$

Equations (4.8)–(4.11) and (4.17)–(4.20) constitute eight equations restricting the values of the coupling constants. They may be solved and all the remaining constants (except $\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$) evaluated in terms of g_A, g_ρ, F_π, m_ρ, m_A, and the constant $\lambda_{\rho AA}$, which is undetermined. The latter is related to the anomalous magnetic moment of the A_1 meson λ_A by

$$\lambda_{\rho AA}=m_\rho{}^2 g_\rho{}^{-1}\lambda_A. \quad (4.21)$$

We note that $\mu_{\rho\rho\rho}$ and $\mu_{\rho AA}$ do not appear in our conditions, and in this sense are orthogonal to the current algebra.

We now return to the CCR for the current densities [Eqs. (2.11)] and evaluate them with the requirement that there be *no* q-number Schwinger terms.[10] This gives rise to only one additional condition: the first Weinberg sum rule.[6]

$$F_\pi{}^2=g_\rho{}^2m_\rho{}^{-2}-g_A{}^2m_A{}^{-2}. \quad (4.22)$$

The solution to this system of algebraic equations is most clearly expressed in terms of the three parameters x, y, and z which are defined as follows:

$$x\equiv\sqrt{2}m_\rho/m_A, \quad y\equiv g_A/g_\rho, \quad z\equiv g_\rho/\sqrt{2}m_\rho F_\pi. \quad (4.23)$$

The coupling constants are then found to be

$$\begin{aligned} &g_{\rho\rho\rho}=g_{\rho AA}=g_{\pi\pi\rho}=m_\rho{}^2/g_\rho,\\ &g_{\pi\rho A}=-m_A{}^2\lambda_{\pi\rho A}=m_\rho{}^2(F_\pi x^2yz^2)^{-1},\\ &g_\rho\lambda_{\pi\pi\rho}=x^4y^2z^2\tfrac{1}{2}\lambda_A+2(1-z^2),\\ &F_\pi\bar{\lambda}_{\pi\rho A}=-y(1-x^2\tfrac{1}{2}\lambda_A),\\ &2F_\pi\mu_{\pi\rho A}=y-y^{-1}, \end{aligned} \quad (4.24)$$

while the first Weinberg sum rule may be rewritten as

$$x^2y^2z^2-2z^2+1=0. \quad (4.25)$$

Using the result given in Eq. (4.24) permits one to write the currents $V^\mu{}_a$ and $A^\mu{}_a$ in a more transparent form. Thus, we have

$$V^i{}_a=g_\rho v^i{}_a$$

and

$$A^i{}_a=g_A a^i{}_a+F_\pi\partial^i\varphi_a \quad (4.26)$$

as before, while

$$V^0{}_a=g_\rho m_\rho{}^{-2}\partial_i G^{i0}{}_a+\epsilon_{abc}[a_{ib}H^{i0}{}_c+v_{bi}G^{i0}{}_c+\varphi_b\varphi_{c0}+F_\pi g_A{}^{-1}\partial_i(\varphi_b H^{i0}{}_c)] \quad (4.27)$$

and

$$A^0{}_a=g_A m_A{}^{-2}\partial_i H^{i0}{}_a-F_\pi\varphi_{a0}+\epsilon_{abc}[F_\pi g_\rho{}^{-1}(\partial_i\varphi_b)G^{i0}{}_c+g_A g_\rho{}^{-1}a_{ib}G^{i0}{}_c+g_\rho g_A{}^{-1}v_{ib}H^{i0}{}_c]. \quad (4.28)$$

One may now proceed to compute the T products of the currents using the effective Lagrangian with the values of the coupling constants as given by

[10] Note that one need not be concerned with the possible existence of "implicit" q-number Schwinger terms due to the singular nature of the current operators. Any such Schwinger terms would be dynamical in origin and hence of higher order in the corresponding constants than the first and therefore are to be ignored. That is to say, such Schwinger terms can only arise if one considers two-particle and higher intermediate states.

Eqs. (4.21), (4.23), and (4.24). These are given in the Appendix. We see that although q-number Schwinger terms have been eliminated from the current commutators they do arise in some of the T products, e.g., $\langle 0|T(\varphi^{\alpha}{}_a(x)a^{\beta}{}_b(y)v^{\mu}{}_c(z))|0\rangle$ [just as Schwinger terms appear in $\langle 0|T(V^{\mu}{}_a(x)V^{\nu}{}_b(y))|0\rangle$]. However, all T products involving three *current* operators are in fact free of such terms.

In computing an actual physical process it is of course not necessary to go through the intermediary of the T product. One may now use the effective Lagrangian directly.

V. σ MESON

In this section, we should like to anticipate the extensions that will be needed to calculate four- and higher-point functions.[11] The present data concerning π-π scattering[12] seem to indicate the existence of a scalar, isotopic-spin-zero, positive-G-parity particle somewhere between 700 MeV and 1 GeV, which we will call the σ meson. We should thus like to include it in our system on the same footing as one includes the π, ρ, and A_1 mesons. However, unlike these latter particles neither the vector, nor the axial-vector currents act as an interpolating field for it. Thus, when one considers the evaluation of the T product of three currents by the use of single-particle saturation, the σ meson can never appear in an intermediate state. It is for this reason that it has not entered into our considerations up to this point.

However, if one evaluates the T product of four currents (or more) by single-particle saturation, the σ meson can now appear in an intermediate state. For example, let

$$F^{\alpha\beta\gamma\nu}=\sum_{n,m,p}\langle 0|A^{\alpha}{}_a(x)|n\rangle\langle n|A^{\beta}{}_b(y)|m\rangle \times\langle m|A^{\gamma}{}_c(z)|0\rangle\langle p|A^{\nu}{}_d(\omega)|0\rangle. \qquad (5.1)$$

While neither the states $|n\rangle$ nor $|p\rangle$ may be the σ meson, the state $|m\rangle$ can include it. We should therefore like to introduce additional vertex functions when considering the matrix elements of the currents between two single-particle states. Thus, in Eqs. (2.8) we should like to allow the indices B and C to range over the σ meson as well as the π, ρ, and A_1 mesons.

This modification will not cause any change in our current-field identification as given by Eqs. (3.5); however, it does mean that additional terms have to be added to our effective Lagrangian to simulate the new vertex functions. We thus introduce an additional free-field term $\mathfrak{L}_{0\sigma}$ as well as a term describing the interactions of this σ meson with the others, $\mathfrak{L}_{(3)\sigma}$. Thus, introducing the Heisenberg fields σ and σ^{μ}, we have

$$\mathfrak{L}_{0\sigma}=-\sigma^{\mu}\partial_{\mu}\sigma+\tfrac{1}{2}(\sigma^{\mu}\sigma_{\mu}-m_{\sigma}^2\sigma^2) \qquad (5.2)$$

and

$$\begin{aligned}\mathfrak{L}_{(3)\sigma}=&\tfrac{1}{2}g_{\sigma\pi\pi}\varphi_a\varphi_a\sigma+\tfrac{1}{2}\lambda_{\sigma\pi\pi}\varphi^{\mu}{}_a\varphi_{\mu a}\sigma+\tfrac{1}{2}g_{\sigma\rho\rho}v^{\mu}{}_a v_{\mu a}\sigma\\ &+\tfrac{1}{4}\lambda_{\sigma\rho\rho}G^{\mu\nu}{}_a G_{\mu\nu a}\sigma+\tfrac{1}{2}g_{\sigma AA}a^{\mu}{}_a a_{\mu a}\sigma+\tfrac{1}{4}\lambda_{\sigma AA}H^{\mu\nu}{}_a H_{\mu\nu a}\sigma\\ &+\lambda_{\sigma\pi A}\varphi_a a^{\mu}{}_a\sigma_{\mu}+\tilde{\lambda}_{\sigma\pi A}a^{\mu}{}_a\varphi_{\mu a}\sigma+\mu_{\sigma\rho\rho}v_{\mu a}G^{\mu\nu}{}_a\sigma_{\nu}\\ &+\mu_{\sigma AA}a_{\mu a}H^{\mu\nu}{}_a\sigma_{\nu}+\mu_{\sigma\pi A}\varphi_{a\mu}H^{\mu\nu}{}_a\sigma_{\nu}+\mu_{\sigma\pi\pi}\varphi_a\varphi^{\nu}{}_a\sigma_{\nu}\\ &+g_{\sigma\sigma\sigma}\sigma\sigma\sigma+\lambda_{\sigma\sigma\sigma}\sigma\sigma_{\mu}\sigma^{\mu}. \end{aligned} \qquad (5.3)$$

The canonical conjugate pair of dynamical variables are σ and σ_0, while the σ_i are constraint variables.

The variables σ and σ^{μ} along with the coupling constants appearing in Eq. (5.3) will now appear in the equations of motion for the ρ, π, and A_1 meson fields. Thus these constants will be restricted by the requirements of CVC and PCAC. Furthermore, when the time components of the currents are now expanded in Heisenberg canonical variables, σ and σ^0 will appear, once again along with their coupling constants. Following the same procedure as in Sec. IV, we find that six relations emerge among ten of the fourteen coupling constants of $\mathfrak{L}_{(3)\sigma}$. They are

$$\begin{aligned}F_{\pi}g_{\sigma\pi\pi}&=m_{\sigma}^2(\lambda_3-\epsilon_{\sigma}\lambda_1)\,,\\ F_{\pi}\lambda_{\sigma\pi\pi}&=-(\lambda_1+\lambda_2)\,,\\ F_{\pi}g_{\sigma AA}&=(x^2yz)^{-2}2m_{\rho}^2(\lambda_1-\lambda_2)\,,\\ \sqrt{2}m_{\rho}\mu_{\sigma\pi A}&=-x^2yz\mu_{\sigma AA}\,,\\ g_{\sigma\rho\rho}&=0=\mu_{\sigma\rho\rho}\,,\end{aligned} \qquad (5.4)$$

where $\epsilon_{\sigma}\equiv(m_{\pi}/m_{\sigma})^2$ and

$$\lambda_1\equiv(g_A m_A^{-2})\lambda_{\sigma\pi A}\,,\quad \lambda_2\equiv(g_A m_A^{-2})\tilde{\lambda}_{\sigma\pi A}\,,\quad \lambda_3\equiv\lambda_1+F_{\pi}\mu_{\sigma\pi\pi}. \qquad (5.5)$$

The quantities x, y, z have been defined in Eq. (4.23). The remaining four constants $\lambda_{\sigma\rho\rho}$, $\lambda_{\sigma AA}$, $g_{\sigma\sigma\sigma}$, and $\lambda_{\sigma\sigma\sigma}$ are totally unconstrained. The final result for the time components of the currents is

$$V^0{}_a=\epsilon_{abc}(v_{bi}G^{i0}{}_c+a_{bi}H^{i0}{}_c+\varphi_b\varphi_{c0}) +\partial_i(g_{\rho}m_{\rho}^{-2}G^{i0}{}_a+F_{\pi}g_A^{-1}\epsilon_{abc}\varphi_b H^{i0}{}_c) \qquad (5.6)$$

and

$$\begin{aligned}A^0{}_a=&g_A m_A^{-2}\partial_i H^{i0}{}_a-F_{\pi}\varphi_{a0}+\epsilon_{abc}[g_{\rho}g_A^{-1}v_{ib}H^{i0}{}_c\\ &+g_A g_{\rho}^{-1}a_{ib}G^{i0}{}_c+F_{\pi}g_{\rho}^{-1}(\partial_i\varphi_b)G^{i0}{}_c]\\ &+F_{\pi}g_A^{-1}\lambda_1\varphi_{0a}\sigma-\lambda_3\varphi_a\sigma_0. \end{aligned} \qquad (5.7)$$

We note that $V^0{}_a$ differs from the usual isotopic current density by a divergence, so that $\int d^3x\, V^0{}_a(x)$ is the total isotopic spin. The divergence in Eq. (5.6) arises as a result of the demand that there be no q-number Schwinger terms in current-density commutation relations.

[11] R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. (to be published).

[12] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters 18, 630 (1967); E. Malamud and P. E. Schlein, Phys. Rev. Letters 19, 1056 (1967).

VI. CONCLUSION

In the preceding sections, a hard-pion algebra method has been described for calculating T products of three vector (axial-vector) current operators. The analysis was based on the assumptions of π, ρ, and A_1 meson saturation of intermediate sums, the chiral $SU(2)\times SU(2)$ current commutation relations, CVC, and PCAC. The single-meson-saturation hypothesis implied that T products could be calculated from an effective Lagrangian, this Lagrangian to be used only to first order in the coupling constants. It is to be emphasized that *a priori* the only symmetry conditions imposed on the T products, and hence the Lagrangian, were the conservation of isotopic spin and G parity. The coupling constants were then determined by the constraints imposed by the CCR, CVC, and PCAC. This then leads to a final effective Lagrangian that in fact is not symmetric under chiral $SU(2)\times SU(2)$ transformations, with the breaking of the symmetry occurring in several different ways, e.g., PCAC, g_A and g_ρ unrelated, m_A and m_ρ unrelated, etc.

In Sec. V, a σ meson was introduced and its couplings to the π, ρ, and A_1 system examined. The motivation for this lies in the fact that such a resonance appears to play an important role in π-π scattering.[12,13]

The coupling constants that appear in the effective Lagrangian fall into three categories: (1) those that are determined by the current-algebra conditions (e.g., $g_{\rho\rho\rho}$; $g_{\sigma\rho\rho}$), (2) those that are related but not completely determined by the current algebra (e.g., $\lambda_{\pi\pi\rho}$, $\lambda_{\rho AA}$; $g_{\sigma\pi\pi}$, $\lambda_{\sigma\pi A}$, $\mu_{\sigma\pi A}$) and (3) finally there are the coupling constants that do not enter into any of the current-algebra conditions (e.g., $\mu_{\rho\rho\rho}$ and $\lambda_{\sigma\rho\rho}$). The last set of coupling constants appear to be completely "orthogonal" to the current-algebra conditions. It is thus tempting simply to set them to zero, and this has in fact been due for analogous structures found in other treatments.[5] However, it may be that additional principles other than the current-algebra conditions are needed to evaluate these undetermined constants. It is also interesting to note that the hard-pion current-algebra analysis does *not* determine the value of g_ρ. Thus, the KSRF relation[14]

$$g_\rho^2 = 2m_\rho^2 F_\pi^2 \tag{6.1}$$

is not a consequence of the current-algebra constraints on three-point functions. One can show[11] that this is also the case for N-point functions.[15]

In the following paper,[7] a comparison of the three-point functions calculated here is made with experiment. Existing data appear to be consistent with the hard-pion $SU(2)\times SU(2)$ current-algebra conditions.

APPENDIX

We give below the result for the T product of three field operators. For the T product involving two A_1 fields and one ρ field, one finds

$$\int d^4x d^4y\, e^{ip\cdot x} e^{-iq\cdot y} \langle T(a^\alpha{}_a(x) a^\beta{}_b(y) v^\mu{}_c(0))\rangle = i\epsilon_{abc}\, {}_\rho\Delta^{\mu\nu}(k)\, {}_{A_1}\Delta^{\lambda\alpha}(p)\, {}_{A_1}\Delta^{\beta\sigma}(q) \left(\frac{m_\rho^2}{g_\rho} [(g_{\sigma\nu} q_\lambda - q_\nu g_{\lambda\sigma}) + (p_\sigma g_{\lambda\nu} - p_\nu g_{\sigma\lambda})] + \frac{m_\rho^2}{g_\rho} \lambda_A (g_{\nu\sigma} k_\lambda - k_\sigma g_{\nu\lambda}) \right), \tag{A1}$$

where $k\equiv q-p$, and ${}_\rho\Delta^{\mu\nu}(k)$, etc., are the ρ meson, etc., propagators and have no c-number Schwinger terms. For the T product involving two pion fields and one ρ-meson field, we find

$$\int d^4x d^4y\, e^{ip\cdot x} e^{-iq\cdot y} \langle T(\varphi_a(x) \varphi_b(y) v^\mu{}_c(0))\rangle = i\epsilon_{abc}\, {}_\pi\Delta(q)\, {}_\pi\Delta(p)\, {}_\rho\Delta^{\mu\nu}(k) g_\rho^{-1} \times[-m_\rho^2(q_\lambda+p_\lambda) - \tfrac{1}{2}\lambda_A(q\cdot k p_\lambda - k\cdot p q_\lambda)]. \tag{A2}$$

The T product involving three ρ-meson fields

$$\int d^4x d^4y\, e^{ip\cdot x} e^{-iq\cdot y} \langle T(v^\alpha{}_a(x) v^\beta{}_b(y) v^\mu{}_c(0))\rangle = i\epsilon_{abc}\, {}_\rho\Delta^{\mu\nu}(k)\, {}_\rho\Delta^{\lambda\alpha}(p)\, {}_\rho\Delta^{\beta\sigma}(q) (m_\rho^2/g_\rho) \times[(g_{\sigma\nu} q_\lambda - q_\nu g_{\lambda\sigma}) + (p_\sigma g_{\lambda\nu} - p_\nu g_{\sigma\lambda}) + (g_{\nu\sigma} k_\lambda - k_\sigma g_{\nu\lambda})]. \tag{A3}$$

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13 A summary of the theoretical analysis and discussion of π-π scattering is given in R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters **20**, 475 (1968). A more detailed account is given in Ref. 11.

14 K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966).

15 A discussion as to why the KSRF relation cannot be derived from hard-pion current-algebra conditions can be found in R. Arnowitt, M. H. Friedman, and P. Nath, Nucl. Phys. **B5**, 115 (1968).

The T products listed above are covariant, i.e., they do not involve any c-number Schwinger terms. In general, the T product of any three field operators are not necessarily free of noncovariant terms. We list below two T products that involve such terms:

$$\int d^4x d^4y\, e^{ip\cdot x} e^{-iq\cdot y} \langle T(\partial^\alpha \varphi_a(x) \partial^\beta \varphi_b(y) v^\mu{}_c(0))\rangle = i\epsilon_{abc}\, {}_\pi\Delta(q)\, {}_\pi\Delta(p)\, {}_\rho\Delta^{\mu\nu}(k) g_\rho{}^{-1} p^\alpha q^\beta \times[-m_\rho{}^2(q_\lambda+p_\lambda)-\tfrac{1}{2}\lambda_A(q\cdot kp_\lambda-k\cdot pq_\lambda)]-i\epsilon_{abc}g_\rho{}^{-1}[\delta^\alpha{}_0\delta^\mu{}_0 q^\beta\, {}_\pi\Delta(q)+\delta^\beta{}_0\delta^\mu{}_0 p^\alpha\, {}_\pi\Delta(p)], \quad (A4)$$

$$\int d^4x d^4y\, e^{ip\cdot x} e^{-iq\cdot y} \langle T(\varphi_a(x) a^\beta{}_b(y) v^\mu{}_c(0))\rangle = i\epsilon_{abc}\, {}_\pi\Delta(p)\, {}_\rho\Delta^{\mu\nu}(k)\, {}_{A_1}\Delta^{\beta\lambda}(q)(1/2F_\pi) \times[-2m_\rho{}^2 g_{\nu\lambda}+(q_\nu p_\lambda-q\cdot pg_{\nu\lambda})+(\lambda_A-2)(p\cdot kg_{\nu\lambda}-p_\nu k_\lambda)]-i\epsilon_{abc}\delta^\beta{}_0\delta^\mu{}_0(F_\pi m_A{}^2)^{-1}\, {}_\pi\Delta(p). \quad (A5)$$

On the other hand, it is easily verified that no c-number Schwinger terms appear in the T products of three current operators.

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Application of Hard-Pion Three-Point Functions*

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Some comparisons with experiment of the hard-pion three-point functions obtained from $SU(2)\times SU(2)$ current algebra are given. Available experimental data involving π, ρ, A_1, and σ mesons are examined. The hard-pion current-algebra method is used to calculate the decays $\rho\to\pi+\pi$, $A_1\to\pi+\rho$, and $\sigma\to\pi+\pi$, and the electromagnetic form factor of the pion. Peripheral processes such as $\pi+N\to\rho+N$ are also examined as a test of meson-vertex functions for spacelike momentum transfers. Here, to reproduce correctly the momentum-transfer dependence at the nucleon vertex, a new extrapolation for the pionic nucleon form factor is introduced, using the Goldberger-Treiman relation. The results of the above calculations are found to be consistent with the present experimental situation. Current-algebra predictions for the $\gamma+N\to A_1+N$ cross section and the decays $A_1\to\pi+\gamma$ and $A_1\to\pi+\sigma$ are given. A cross section of about 0.1 μb is obtained for the A_1 photoproduction, which is on the verge of being detectable.

I. INTRODUCTION

RECENTLY, it has become apparent that soft-pion methods[1] may lead to erroneous results when applied to processes involving energetic pions. Thus the soft-pion method yields a width of approximately 800 MeV for the $A_1\to\pi+\rho$ decay,[2] in contrast to the experimental width of $\approx$100 MeV. These considerations, coupled with the desire to exploit more fully the content of the current algebras, have motivated interest in extending the analysis beyond the domain of the soft-pion method. An important step in this direction was first taken by Weinberg, who used Ward identities, $SU(2)\times SU(2)$ algebra of currents, and the hypothesis of meson dominance of vector and axail-vector currents to obtain the sum rule[3]

$$g_\rho{}^2/m_\rho{}^2=(g_A{}^2/m_A{}^2)+F_\pi{}^2. \quad (1.1)$$

Here, g_ρ and g_A are the coupling strengths of the vector current to the ρ meson and the axial-vector current to the A_1 meson, and F_π is the usual pion-decay amplitude.[4] This result when supplemented by the second Weinberg sum rule[3]

$$g_A=g_\rho \quad (1.2)$$

and the KSRF relation[5]

$$g_\rho\simeq\sqrt{2}F_\pi m_\rho \quad (1.3)$$

yields the well-known result $m_A=\sqrt{2}m_\rho$ which is borne out experimentally.[3]

In the preceding paper,[6] new techniques were described to obtain current-algebra solutions to vertex

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[1] For a review of the soft-pion calculations, see R. F. Dashen, in *Proceedings of the Thirteenth International Conference in High-Energy Physics* (University of California Press, Berkeley, 1967).

[2] The soft-pion analysis of the π-ρ-A_1 vertex has been carried out by several authors; see e.g., B. Renner, Phys. Letters **21**, 453 (1966); D. Geffen, Phys. Rev. Letters **19**, 770 (1967).

[3] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). Extension of these results to other vector and axial-vector currents are given by T. Das, V. S. Mathur, and S. Okubo, *ibid.* **18**, 761 (1967); S. L. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967).

[4] We define g_ρ by the relation

$$\langle 0|V_{\mu a}(0)|\rho;k,b,\sigma\rangle\equiv g_\rho\delta_{ab}N_\rho\epsilon_\mu{}^\sigma(k),$$

where a, b=1, 2, 3 are $SU(2)$ isotopic indices, $\epsilon_\mu{}^\sigma(k)$ is the ρ polarization vector normalized by $\epsilon_\mu{}^{\sigma *}\epsilon^{\mu\sigma'}=\delta^{\sigma\sigma'}$, and

$$N_\rho\equiv[(2\pi)^3 2\omega_k]^{-1/2}.$$

Similarly, g_A is defined from the A_1 matrix element $\langle 0|A_{\mu a}(0)|A_1;k,b,\sigma\rangle$ and F_π by $\langle 0|A_{\mu a}(0)|\pi,kb\rangle=iF_\pi\delta_{ab}k_\mu N_\pi$. Our currents are normalized such that the experimental value of F_π is 94 MeV.

[5] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[6] R. Arnowitt, M. H. Friedman, and P. Nath, preceding paper, Phys. Rev. **174**, 1999 (1968) (hereafter referred to as I). A brief account of these results were given in R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967).