Coherent Soft-Photon States and Infrared Divergences. III. Asymptotic States and Reduction Formulas*

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In paper II of this series, the mass-shell singularities of the Green's functions of quantum electrodynamics were investigated. In this paper, the relationship between these singularities and the asymptotic states of the theory is studied by considering the nature of the intermediate states that can contribute to the corresponding discontinuity functions. The basic principle underlying this work is that the asymptotic states of the theory should not be specified *a priori* but should be determined from the structure of the Green's functions themselves. The pure soft-photon asymptotic states, which can be created from the vacuum by operators constructed from the soft-photon part of the electromagnetic field, are studied first. These states are defined by appropriate weak limits and are shown to span a space with the same structure as in the noninteracting case. Next, states containing a single particle (massive particle or hard photon), together with soft photons, are investigated. These states can appear as intermediate states in the two-point function. They are again defined by weak limits, and are shown to be stable in the absence of external currents. It is demonstrated that the near-mass-shell components of the field operator, acting on the vacuum or on a soft-photon coherent state, yield a state containing one particle and a soft-photon coherent state. Finally, the analysis is extended to two-particle and multiparticle states. The only essentially new feature here is the appearance of factors related to the "Coulomb phases." General reduction formulas are obtained that permit matrix elements between arbitrary asymptotic states to be extracted from the Green's functions. In effect, these matrix elements may be identified with the coefficients not of poles but of branch-point singularities.

1. INTRODUCTION

THIS is the third in a series of four papers[1] devoted to the development of a new field-theoretic approach to the problem of the infrared divergences of quantum electrodynamics.

In paper II, we investigated the mass-shell singularities of the Green's functions of the theory. Here we shall study the asymptotic states implied by this singularity structure. As in a conventional field theory without massless particles, the Green's functions must contain complete information about the theory. It should be possible by studying them to determine, firstly, the nature of the asymptotic states and, secondly, the scattering matrix elements between these states. For example, in the usual case, the presence of stable particles is implied by the existence of mass-shell poles in the Green's functions, and the technique for extracting the scattering matrix elements from the Green's functions is provided by the reduction formulas of Lehmann, Symanzik, and Zimmermann.[2] The scattering matrix elements are in effect identified with the residues of mass-shell poles. In the present case, the Green's functions do not in general have poles, but rather branch points. What we are seeking is a generalization of these methods appropriate to the case of massless particles.

Sections 2 and 3 are preliminary in character. In Sec. 2, we examine certain properties of the generalized coherent states defined in I. Then, in Sec. 3, we recall briefly the definition of asymptotic states in a conventional field theory without massless particles.

In Sec. 4, we discuss the special case of the pure soft-photon asymptotic states, that is to say, the states that can be obtained from the vacuum by the action of operators constructed from the soft-photon parts of the electromagnetic field operators. Since soft photons interact effectively only with external charged lines, it turns out that the structure of these soft-photon asymptotic states is exactly the same as in the noninteracting theory. Thus, in particular, we can define soft-photon generalized coherent states analogous to those introduced in I. In the absence of an external current these states are stable.

Next, in Sec. 5, we examine the two-point function, and, in particular, the nature of the states that can contribute to its discontinuity function in the neighborhood of the mass shell. This discussion is mainly heuristic and is intended to show that the discontinuity function can be correctly reproduced if it is assumed that the contributing states contain a single particle (by which we mean a massive particle or hard photon) in addition to soft photons. The proper definition of such states is given in Sec. 6 in terms of an appropriate weak limit, and used to obtain a one-particle reduction formula. The Lorentz covariance of the definition, which is somewhat complicated by the conventional separation between hard and soft photons, is established in Sec. 7. In Sec. 8, we derive an expression for the matrix elements of time-ordered products between two such states.

The definitions are extended to the case of two-particle states in Sec. 9. The extension is straightforward except for the appearance of additional factors related to the formally divergent Coulomb phase factors that

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[1] T. W. B. Kibble, J. Math. Phys. 9, 315 (1968); Phys. Rev. 173, 1527 (1968). These papers will be referred to as I and II.
[2] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955).

appear whenever there is more than one charged particle in either the initial or the final state. The extension to general multiparticle states encounters no further problems and is indicated briefly in Sec. 10, where the conclusions are also discussed.

Let us recall here the essential results of II that we shall need in our discussion. We considered the Green's functions in the presence of an external current J whose Fourier transform vanishes outside the soft-photon region Ω^s,

$$G(x_1\cdots x_n|J)=\langle 0,\text{out}|T[\phi_1(x_1)\cdots\phi_n(x_n)]|0,\text{in}\rangle J\,, \quad (1.1)$$

where the right-hand side is a formal expression that must be modified by the addition of extra terms in the case of fields ϕ_j of spin greater than $\frac{1}{2}$. We showed that the soft-photon contribution to the Fourier transform of (1.1) could be isolated in a single function Δ^s. For simplicity, let us assume that the only "core" diagrams (those obtained by removing all soft-photon lines) that can contribute significantly in the region of interest have r straight-through lines, joining $p_1\cdots p_r$ to $p_{r+1}\cdots p_{2r}$, and are otherwise connected. (See Fig. 1.) Then the structure of the Green's function is given by

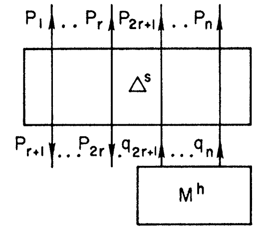

FIG. 1. Structure of the Green's function. The function M^h represents the connected part of the core diagrams. The soft-photon contribution is contained in the function Δ^s.

$$G(p_1\cdots p_n|J)=\prod_{j=1}^{r}[Z^h{}_j(p_j)\Lambda_j(p_j)C_{j,j+r}]$$
$$\times\prod_{j=2r+1}^{n}\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\}\int\frac{dq_{2r+1}}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}$$
$$\times\Delta^s(p_1\cdots p_rp_{2r+1}\cdots p_n;\,-p_{r+1}\cdots-p_{2r}q_{2r+1}\cdots q_n|J)$$
$$\times(2\pi)^4\delta(\sum_{j=2r+1}^{n}q_j)M^h(q_{2r+1}\cdots q_n)\,. \quad (1.2)$$

Here M^h denotes the contribution of the connected part of the core diagrams (with external lines removed), $Z^h{}_j$ is the wave-function renormalization "constant" with soft-photon contributions removed, Λ_j is a spin matrix, and $C_{j,j+r}$ is a charge-conjugation matrix, one of which is required for each straight-through line.

All soft-photon contributions are contained in the function Δ^s that modifies the external lines. We may write the expression [II, (3.19)] for this function in the form

$$\Delta^s(p_1\cdots p_n;\,q_1\cdots q_n|J)=\int dy_1\cdots dy_n$$
$$\times\int_0^\infty d\sigma_1\cdots d\sigma_n\,\exp[-i\sum_{j=1}^{n}(p_j-q_j)\cdot y_j$$
$$-i\sum_{j=1}^{n}\sigma_j(m_j{}^2+p_j{}^2)]\,\exp X\,, \quad (1.3)$$

where

$$X=\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}I^\mu{}_{1\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{1\cdots n}(k) \quad (1.4)$$

and $I^\mu{}_{1\cdots n}$ is the function given by [II, (3.20)], namely,

$$I^\mu{}_{1\cdots n}(k)=J^\mu(k)+i\sum_{j=1}^{n}\frac{e_jp_j{}^\mu}{p_j\cdot k}\exp(-ik\cdot y_j)$$
$$\times[\exp(-2i\sigma_jp_j\cdot k)-1]\,. \quad (1.5)$$

2. GENERALIZED COHERENT STATES

For later convenience, we summarize in this section some of the results of I, and derive certain additional relations.

In this paper, we shall restrict our considerations to *physical* gauges, such as the radiation gauge. Each physical gauge is characterized by a real, symmetric gauge function $\gamma_{\mu\nu}(k)$ that for $k^2=0$ reduces to the projector on the two-dimensional space of allowed polarization vectors. For example, the radiation gauge is characterized by the matrix function whose only nonvanishing components are

$$\gamma_{00}=-k^2/\mathbf{k}^2\,,\quad \gamma_{ij}=\delta_{ij}-k_ik_j/\mathbf{k}^2\,.$$

The generalized coherent states of the free electromagnetic field are denoted by $|f,\lambda\rangle$, where f stands for the photon wave function $f^\mu(\mathbf{k})$ and λ stands for a real function $\lambda(\mathbf{k})$ that specifies the generalized phase of the state. Formally, it represents a phase factor

$$e^{i\lambda}=\exp\left(i\int\frac{d\mathbf{k}}{(2\pi)^32k^0}\lambda(\mathbf{k})\right)\,, \quad (2.1)$$

where $k^0=|\mathbf{k}|$. However, this integral may diverge, and must then be interpreted according to the rules given in I and summarized below. (For the precise conditions on the functions f and λ, see I.) Two functions $f^\mu(\mathbf{k})$ and $g^\mu(\mathbf{k})$ whose difference is proportional to k^μ define the same coherent state. In a specific (physical) gauge, the functions are restricted by the condition

$$\gamma_{\mu\nu}(k)f^\nu(\mathbf{k})=f_\mu(\mathbf{k})\,. \quad (2.2)$$

We shall frequently use the notation

$$f^*g=\int\frac{d\mathbf{k}}{(2\pi)^32k^0}f_\mu{}^*(\mathbf{k})g^\mu(\mathbf{k}) \quad (2.3)$$

without the parentheses used in I. Again, however, the integral may diverge.

The scalar product of two generalized coherent states $|f,\lambda\rangle$ and $|g,\mu\rangle$ is represented by the formal expression

$$\langle f,\lambda|g,\mu\rangle=\exp(f^*g-\tfrac{1}{2}f^*f-\tfrac{1}{2}g^*g-i\lambda+i\mu)\,, \quad (2.4)$$

using the notation of (2.1) and (2.3). The scalar product is equal to (2.4) provided that this integral converges. It is zero by definition if the integral diverges. We shall use this notation frequently in what follows. In each case, a similar interpretation is to be applied. We must collect together all terms in the exponent before performing the integration over **k**, and assign the value zero to the exponential if the integral diverges.

We also recall the definition of the unitary operators $U(f)$ and $V(\lambda)$. (We now drop the subscript $\otimes$ used in I.) They may be defined by their action on the vacuum state $|0\rangle=|0,0\rangle$,

$$U(f)V(\lambda)|0\rangle=|f,\lambda\rangle\,, \quad (2.5)$$

together with the multiplication laws

$$V(\lambda)V(\mu)=V(\lambda+\mu)\,,\quad V(\lambda)U(f)=U(f)V(\lambda)\,,$$
$$U(f)U(g)=U(f+g)V[\tfrac{1}{2}i(f^*g-g^*f)]\,, \quad (2.6)$$

where, of course, $\frac{1}{2}i(f^*g-g^*f)$ denotes the real function

$$\tfrac{1}{2}i[f_\mu{}^*(\mathbf{k})g^\mu(\mathbf{k})-g_\mu{}^*(\mathbf{k})f^\mu(\mathbf{k})].$$

The operators $V(\lambda)$ play the role of generalized phase operators, and commute with all physical observables, so that states differing only in the generalized phase label λ are physically indistinguishable.

For later use, we shall have to evaluate the vacuum expectation value of a product of these unitary operators, which will play an important role in our discussion of asymptotic states. Using (2.6) repeatedly, we can reduce a product of any number of these operators to a single pair $U(f)V(\lambda)$. Hence, using (2.4) and (2.5), we find

$$\langle 0|U(h_1)\cdots U(h_n)|0\rangle =\exp(-\tfrac{1}{2}\sum_i h_i{}^*h_i-\sum_{i<j}h_i{}^*h_j)\,, \quad (2.7)$$

where, as usual, the convention involved in the interpretation of (2.4) is implied. We note that any matrix element between coherent states can be reduced to the form (2.7) by using (2.5). Thus we obtain more generally

$$\langle f,\lambda|U(h_1)\cdots U(h_n)|g,\mu\rangle=\langle f,\lambda|g,\mu\rangle \times\exp(\sum_i(f^*h_i-h_i{}^*g)-\tfrac{1}{2}\sum_i h_i{}^*h_i-\sum_{i<j}h_i{}^*h_j). \quad (2.8)$$

Here, of course, the factor $\langle f,\lambda|g,\mu\rangle$ is to be written in the form (2.4) and the exponents combined before the integration over **k** is performed.

It will be useful also to note the Lorentz-transformation properties of these states. Let $U(a,\Lambda)$ be the unitary operator that represents the transformation $x\to a+\Lambda x$ of the Poincaré group. Its action on a coherent state is given by

$$U(a,\Lambda)|f,\lambda\rangle=|(a,\Lambda)f,\lambda\rangle\,, \quad (2.9)$$

where $(a,\Lambda)f$ denotes the transformed photon wave function

$$(a,\Lambda)f^\mu(k)=\Lambda^\mu{}_\nu f^\nu(\Lambda^{-1}k)e^{-ik\cdot a}. \quad (2.10)$$

Of course, if we want to stick to a specific gauge, we may have to make an additional gauge transformation. For example, in the radiation gauge, we require

$$(a,\Lambda)f^i(k)=(\Lambda^i{}_j-\hat{k}^i\Lambda^0{}_j)f^j(\Lambda^{-1}k)e^{-ik\cdot a}. \quad (2.11)$$

A special case of (2.9) is the relation [I, (21)], namely,

$$e^{-iP\cdot a}|f,\lambda\rangle=|(a)f,\lambda\rangle\,, \quad (2.12)$$

where $(a)f=(a,1)f$. [In I, $(a)f$ was denoted by f_{-a}.]

The separation between Ω^h and Ω^s, introduced in II, defines a corresponding separation of the space of real photon momenta, characterized by the conditions $k^2=0$ and $k^0>0$, into regions where $|\mathbf{k}|>K$ and $|\mathbf{k}|<K$, respectively. If $f^\mu(\mathbf{k})$ is any photon wave function, we may write $f=f^h+f^s$, where f^h is nonvanishing only in Ω^h and f^s is nonvanishing only in Ω^s. Because of the exponential structure of the unitary operators that generate the coherent states from the vacuum, we may write the coherent state as the product of a hard-photon state and a soft-photon state,

$$|f,\lambda\rangle=|f^h,\lambda^h\rangle|f^s,\lambda^s\rangle\,, \quad (2.13)$$

where $\lambda=\lambda^h+\lambda^s$ is the corresponding decomposition of the real function $\lambda(\mathbf{k})$. Since the integral f^*f diverges, if at all, only at $\mathbf{k}=\mathbf{0}$, $f^{h*}f^h$ is certainly finite. Moreover, the integral of λ^h is also finite and contributes only a finite phase factor, so that there is no loss of generality in setting $\lambda^h=0$. It follows that the hard-photon coherent states belong to the photon Fock space and have a convergent expansion in terms of states containing n hard photons. The decomposition (2.13) then corresponds to the tensor-product decomposition of the nonseparable Hilbert space defined in I,

$$\mathcal{H}_{\mathrm{IR}}=\mathcal{H}^h\otimes\mathcal{H}^s{}_{\mathrm{IR}}\,, \quad (2.14)$$

where $\mathcal{H}^h$ is the hard-photon Fock space, and is separable. In $\mathcal{H}^h$ we can, and generally will, use a basis consisting of states containing definite numbers of hard photons. In general, in what follows, "coherent state" will mean a soft-photon generalized coherent state, unless otherwise specified.

It should be remarked that the decomposition (2.13) is not Lorentz-invariant. Under a Lorentz transformation, a soft-photon coherent state will go over into a state containing some hard photons.

3. ASYMPTOTIC STATES IN A CONVENTIONAL FIELD THEORY

Let us begin by recalling briefly the situation in a conventional theory without massless particles. First, we look for stable single-particle states, whose presence is indicated by the appearance of mass-shell poles in the Green's functions. To do this we examine the propagator function

$$G(p)=\int dx\; e^{-ip\cdot x}\langle 0|T[\phi(x),\bar{\phi}(0)]|0\rangle\,, \quad (3.1)$$

where $\bar{\phi}(x)=\phi^c(x)C^{-1}$ and ϕ^c is the charge-conjugate field to ϕ. (More generally, it may be necessary to consider propagator functions for composite fields if there are bound states in the theory, but the general principles are the same.) According to the general principles of quantum field theory, $G(p)$ is the boundary value of an analytic function of p whose singularities are located at the positions of real intermediate states. For $p^0>0$ the discontinuity function

$$\mathrm{disc}G(p)=G(p+i\epsilon)-G(p-i\epsilon)\,, \tag{3.2}$$

where $p\pm i\epsilon$ denotes the four-vector $(p^0\pm i\epsilon,\,\mathbf{p})$, may be written in the form

$$\mathrm{disc}G(p)=\sum_n \langle 0|\phi(0)|n\rangle(2\pi)^4\delta(p-p_n)\langle n|\bar{\phi}(0)|0\rangle\,, \tag{3.3}$$

where the sum is over a complete set of intermediate states.

If $G(p)$ has a pole at $p^2=-m^2$, then in that vicinity

$$\mathrm{disc}G(p)=Z\Lambda(p)2\pi\delta(p^2+m^2)\,. \tag{3.4}$$

The only states that can contribute to the sum in (3.3) and yield (3.4) are stable single-particle states, whose amplitude is determined by the relations

$$\begin{aligned}\langle 0|\phi(0)|\mathbf{p}\rangle&=Z^{1/2}u(\mathbf{p})\,,\\ \langle \mathbf{p}|\bar{\phi}(0)|0\rangle&=Z^{1/2}\bar{u}(\mathbf{p})\,,\end{aligned} \tag{3.5}$$

where $u(\mathbf{p})$ and $\bar{u}(\mathbf{p})$ are spin wave functions. Here we use a covariant normalization

$$\langle \mathbf{p}\alpha|\mathbf{p}'\beta\rangle=(2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}')\delta_{\alpha\beta}\,, \tag{3.6}$$

with $p^0=(\mathbf{p}^2+m^2)^{1/2}$. The corresponding normalization of the spin wave functions is given by

$$\begin{aligned}\bar{u}_\alpha(\mathbf{p})u_\beta(\mathbf{p})&=\delta_{\alpha\beta}N\,,\\ \sum_\alpha u_\alpha(\mathbf{p})\bar{u}_\alpha(\mathbf{p})&=\Lambda(p)\,,\end{aligned} \tag{3.7}$$

where N is the normalization factor introduced in [II, 3.10)].

Although we have written these formulas in the notation familiar for the Dirac-field case, they are in a form applicable to fields of any spin.

If the propagator function has a pole at $p^2=-m^2$, then any other Green's function involving the same field ϕ will also have a pole there. The reduction formulas of Lehmann, Symanzik, and Zimmermann[2] essentially provide a method of isolating the residues at these poles.

Let us choose any spin and momentum-space wave function $\bar{\psi}(\mathbf{p})$, and any smooth function $\sigma(p)$ satisfying the conditions[3]

$$\begin{aligned}\sigma(p)&=1 \quad\text{for}\quad p^0=\omega_{\mathbf{p}}\,,\\ \sigma(p)&=0 \quad\text{for}\quad p^0=-\omega_{\mathbf{p}}\,,\end{aligned} \tag{3.8}$$

where $\omega_{\mathbf{p}}=(\mathbf{p}^2+m^2)^{1/2}$. The usual choice for this function is

$$\sigma(p)=(p^0+\omega_{\mathbf{p}})/2\omega_{\mathbf{p}}\,, \tag{3.9}$$

but we shall find the added generality useful, because (3.9) is inconvenient for the case of massless particles, for which the denominator can vanish.

Then we consider the limit for large positive t of the matrix element

$$N^{-1}\int \frac{dp}{(2\pi)^4}\bar{\psi}(\mathbf{p})\sigma(p)\,\exp[-i(p^0-\omega_{\mathbf{p}})t]$$
$$\times Z^{-1/2}\int dx\; e^{-ip\cdot x}\langle 0|T[\phi(x)\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle\,,$$

where N is the normalization factor of (3.7). As $t\to\infty$, the values of x^0 that contribute also tend to $+\infty$, so that $\phi(x)$ may be taken outside the time-ordering sign. Moreover, because of the exponential factor depending on t the only momentum components that survive in the limit are those with $p^0=\omega_{\mathbf{p}}$, which come from stable single-particle intermediate states. Thus, using (3.5), we find that

$$\lim_{t\to\infty} N^{-1}\int \frac{dp}{(2\pi)^4}\bar{\psi}(\mathbf{p})\sigma(p)\,\exp[-i(p^0-\omega_{\mathbf{p}})t]$$
$$\times Z^{-1/2}\int dx\; e^{-ip\cdot x}\langle 0|T[\phi(x)\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle$$
$$=\langle\psi|T[\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle\,, \tag{3.10}$$

where $\langle\psi|$ denotes the single-particle state

$$\langle\psi|=N^{-1}\int\frac{d\mathbf{p}}{(2\pi)^3 2p^0}[\bar{\psi}(\mathbf{p})u(\mathbf{p})]\langle\mathbf{p}|\,. \tag{3.11}$$

Here a sum over spins is implied, but will not generally be indicated explicitly. Clearly, the effect of the limit in (3.10) is to isolate the residue of the pole at $p^2=-m^2$. It may be expressed more formally by the relation

$$\langle\mathbf{p}|T[\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle$$
$$=Z^{-1/2}\bar{u}(p)\lim_{p^0\to\omega_{\mathbf{p}}+i\epsilon} i(m^2+p^2)\int dx\; e^{-ip\cdot x}$$
$$\times\langle 0|T[\phi(x)\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle\,. \tag{3.12}$$

The relation (3.10) or (3.12) effectively defines the one-particle states as (weak) limits. Because the mass-shell singularities in the remaining variables are unaffected by going to this limit, we can proceed to define multiparticle states by the same method. The formula (3.10) remains valid if arbitrary states $\langle f;\,\text{out}|$ and $|i;\,\text{in}\rangle$ are written in place of the vacuum state, and then defines the state $\langle f,\psi;\,\text{out}|$ containing one additional particle.

4. SOFT-PHOTON STATES

We now wish to examine the pure soft-photon asymptotic states, that is to say, the states that can be created from the vacuum by the action of soft-photon operators.

[3] The second condition is not strictly necessary here but will be convenient later.

We consider first the special case $n=0$ of the Green's function (1.1), namely, the functional

$$G(J)=\langle 0;\text{out}|0;\text{in}\rangle_J. \tag{4.1}$$

In this case, there are, of course, no core diagrams, so that (1.2) and (1.3) yield simply

$$G(J)=\exp\left(\tfrac{1}{2}i\int dxdy\ J^\mu(x)D^s{}_{\mu\nu}(x-y)J^\nu(y)\right). \tag{4.2}$$

This is precisely the expression for the vacuum-to-vacuum transition amplitude for a free field interacting only with the external current J, as it must be, since the soft photons cannot interact directly with each other, but only with external charged-particle lines, of which there are none in (4.1).

This result suggests that the structure of the states obtained from the vacuum by the action of soft-photon operators must be just the same as in the free-field case, so that, in particular, we should be able to define asymptotic coherent states $\langle f,\lambda;\text{out}|$ and $|f,\lambda;\text{in}\rangle$ that span the space of asymptotic soft-photon states.

To define such states, let us choose a photon wave function f and introduce an external current $J_f{}^\mu(t,x)$ depending on the parameter t as well as on the space-time variable x, and defined in momentum space by

$$J_f{}^\mu(t,k)=-i\sigma(k)f^\mu(\mathbf{k})\exp[i(k^0-|\mathbf{k}|)t] +i\sigma^*(-k)f^{\mu*}(-\mathbf{k})\exp[i(k^0+|\mathbf{k}|)t], \tag{4.3}$$

where $\sigma(k)$ is now required to vanish outside Ω^s, and within this region to satisfy the conditions (3.8) with $\omega_{\mathbf{k}}=|\mathbf{k}|$.

In the radiation gauge, $J_f{}^\mu$ has no time component, and thus its effect in an arbitrary Green's function may be represented by the appearance of a time-ordered exponential factor:

$$G(x_1\cdots x_n|J+J_f(t))=\langle 0;\text{out}|T\left\{\phi_1(x_1)\cdots\phi_n(x_n)\times\exp\left[i\int dx\ A_\mu(x)J_f{}^\mu(t,x)\right]\right\}|0;\text{in}\rangle_J. \tag{4.4}$$

[If $J_f{}^\mu(t)$ had a time component, we would need an extra factor representing the direct Coulomb interaction.] Because of the structure of (4.3), it is easy to see that, as $t\to\pm\infty$, the values of x^0 that contribute in (4.4) also tend to $\pm\infty$, so that the exponential factor may be removed from the time ordering and allowed to act directly on the vacuum state. Provided that the current $J^\mu(x)$ falls off in a suitable manner as $x^0\to\pm\infty$ (which is a necessary condition for the existence of asymptotic states, and is assured by our assumption that its singularity at $k=0$ is no worse than $1/k$), we should thus expect to obtain in the limit

$$\lim_{t\to+\infty}G(x_1\cdots x_n|J+J_f(t)) =\sum_\alpha\lim_{t\to+\infty}\langle 0|T\exp\left[i\int dx\ A_\mu(x)J_f{}^\mu(t,x)\right]|\alpha\rangle \times\langle\alpha;\text{out}|T\{\phi_1(x_1)\cdots\phi_n(x_n)\}|0;\text{in}\rangle_J \tag{4.5a}$$

and

$$\lim_{t\to-\infty}G(x_1\cdots x_n|J+J_f(t)) =\sum_\alpha\lim_{t\to-\infty}\langle 0;\text{out}|T\{\phi_1(x_1)\cdots\phi_n(x_n)\}|\alpha;\text{in}\rangle_J \times\langle\alpha|T\exp\left[i\int dx\ A_\mu(x)J_f{}^\mu(t,x)\right]|0\rangle, \tag{4.5b}$$

where $|\alpha\rangle$ denotes a complete orthonormal set of soft-photon states, and where the matrix elements involving $J_f(t)$ are the same as in the free-field case. (The sum over $|\alpha\rangle$ contains uncountably many terms, but only a countable subset will, in fact, give nonzero contributions.)

Because of the exponential structure of the operators involving $J_f(t)$, it is easy to compute these free-field matrix elements, and we obtain

$$\lim_{t\to+\infty}\langle 0|T\exp\left[i\int dx\ A_\mu(x)J_f{}^\mu(t,x)\right]|\alpha\rangle =\langle -f,-\lambda|\alpha\rangle, \tag{4.6a}$$

$$\lim_{t\to-\infty}\langle\alpha|T\exp\left[i\int dx\ A_\mu(x)J_f{}^\mu(t,x)\right]|0\rangle =\langle\alpha|f,\lambda\rangle, \tag{4.6b}$$

where the phase λ arises from the phase factor relating the time-ordered operator to the corresponding unordered operator and is given by the formal relation

$$e^{i\lambda}=\lim_{t\to\pm\infty}\exp\left(\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}J_f{}^\mu(t,k)^*\frac{\gamma_{\mu\nu}(k)}{k^2}J_f{}^\nu(t,k)\right) =\exp\left(i\int\frac{dk}{(2\pi)^4}\frac{|\sigma(k)|^2}{k^2}f_\mu{}^*(\mathbf{k})f^\mu(\mathbf{k})\right), \tag{4.7}$$

where the integral is a principal-value integral.

Because of (4.6) the effect of the sums over α in (4.5) is to replace $\langle\alpha|$ with $\langle -f,-\lambda|$ and $|\alpha\rangle$ with $|f,\lambda\rangle$. Thus we should obtain

$$\lim_{t\to+\infty}G(x_1\cdots x_n|J+J_f(t)) =\langle -f,-\lambda;\text{out}|T[\phi_1(x_1)\cdots\phi_n(x_n)]|0;\text{in}\rangle_J, \tag{4.8a}$$

$$\lim_{t\to-\infty}G(x_1\cdots x_n|J+J_f(t)) =\langle 0;\text{out}|T[\phi_1(x_1)\cdots\phi_n(x_n)]|f,\lambda;\text{in}\rangle_J. \tag{4.8b}$$

Hence, provided that we can establish the existence of these limits, these equations define the asymptotic states $\langle -f,-\lambda;\text{out}|$ and $|f,\lambda;\text{in}\rangle$.

To prove the existence of the limits we examine the structure of the Green's function (4.4). It is given by (1.2) and (1.3), but with the exponent X replaced with

$$X(t)=\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}[I^\mu{}_{1\ldots n}(k)+J_f{}^\mu(t,k)]^* \times[\gamma_{\mu\nu}(k)/(k^2-i\epsilon)][I^\nu{}_{1\ldots n}(k)+J_f{}^\nu(t,k)]. \tag{4.9}$$

Because of the structure of (4.3), it is easy to see that in the cross term between $I_{1\ldots n}$ and $J_f(t)$ only those momentum components with $k^0=\pm|\mathbf{k}|$ survive, and we obtain

$$\lim_{t\to+\infty} X(t)=X-if^*I_{1\cdots n}-\tfrac{1}{2}f^*f+i\lambda\,,$$

$$\lim_{t\to-\infty} X(t)=X+iI_{1\cdots n}{}^*f-\tfrac{1}{2}f^*f+i\lambda\,,$$

using again the formal notation of (2.3). [Note that we no longer distinguish, as we did in I, between $I_{1\cdots n}(k)$ and its mass-shell restriction obtained by setting $k^0=|\mathbf{k}|$. Its appearance in a combination like $f^*I_{1\cdots n}$ implies this restriction.]

Hence the function (4.4) does possess well-defined limits as $t\to\pm\infty$, which define the asymptotic soft-photon states by (4.8).

To evaluate a general matrix element between soft-photon states, we have to apply both formulas (4.8). We thus replace $J_f(t)$ with $J_{-f}(t)+J_g(t')$ and consider the limits $t\to\infty$, $t'\to-\infty$. The limit of the exponent $X(t,t')$ corresponding to (4.9) is then

$$\lim_{t\to\infty}\lim_{t'\to-\infty} X(t,t')=X+(f^*g-\tfrac{1}{2}f^*f-\tfrac{1}{2}g^*g-i\lambda+i\mu)+i(f^*I_{1\cdots n}+I_{1\cdots n}{}^*g)=X_{f\lambda,g\mu}\,,\quad(4.10)$$

say, where $-\lambda$ and μ are related to f and g, respectively, by (4.7).

Hence, applying both parts of (4.8), we see that the matrix element

$$G_{f\lambda,g\mu}(x_1\cdots x_n|J)=\langle f,\lambda;\text{out}|T[\phi_1(x_1)\cdots\phi_n(x_n)]|g,\mu;\text{in}\rangle_J\quad(4.11)$$

is again given by (1.2) and (1.3), but with X replaced with the function $X_{f\lambda,g\mu}$ defined in (4.10). We note that, formally,

$$\exp X_{f\lambda,g\mu}=\langle f,\lambda|g,\mu\rangle\times\exp[X+i(f^*I_{1\cdots n}+I_{1\cdots n}{}^*g)].\quad(4.12)$$

This structure for the matrix element (4.11) is precisely what we should have obtained by treating the external soft photons in the asymptotic states according to the usual rules of perturbation theory, as may easily be verified. However, our derivation does not assume the existence of these asymptotic states *a priori*, but defines them by the limits (4.8), which may be regarded as soft-photon reduction formulas. Note that, since the phase λ can be adjusted at will by a suitable choice of the function $\sigma(k)$, we can define *any* asymptotic soft-photon state in this way.

We note that for $n=0$ our formula for (4.11) reduces to

$$G_{f\lambda,g\mu}(J)=\langle f,\lambda;\text{out}|g,\mu;\text{in}\rangle_J=\langle f,\lambda|g,\mu\rangle\exp\biggl(i(f^*J+J^*g)+\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}J^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}J^\nu(k)\biggr),\quad(4.13)$$

using (4.12) with $I^\mu(k)=J^\mu(k)$. This agrees precisely with Eq. (59) of I. Thus, when there are no external charged particles, the scattering in a soft-photon state is the same as in the free-field case, as we should expect, since the soft photons then interact only with the external current. In particular, for $J=0$,

$$\langle f,\lambda;\text{out}|g,\mu;\text{in}\rangle=\langle f,\lambda|g,\mu\rangle\,,\quad(4.14)$$

so that we need not distinguish the in and out states. When there is no external current, the pure soft-photon states are stable.

5. TWO-POINT FUNCTION

The next step is to consider the nature of the states that can be created from the vacuum, or, more generally, from a pure soft-photon state, by the action of field operators with momentum components close to the mass shell. We should expect these states to contain a single particle together with some soft photons, and therefore that a suitable basis spanning the subspace of such states would be $\langle\mathbf{l};f,\lambda;\text{out}|$ or $|\mathbf{l};f,\lambda;\text{in}\rangle$, where $\mathbf{l}$ denotes the particle momentum (and suppressed spin index) and f,λ are the usual coherent-state labels. Here and in what follows, we use the word "particle" to mean any particle other than a soft photon, including charged particles and hard photons.

What we have to do is to define these states (or, rather, wave packets formed from them) by appropriate limiting formulas analogous to (3.10) or (4.8). To guide our choice, we begin by examining the two-point function

$$G_{f\lambda,g\mu}(p,-q)C^{-1}=\int dxdy\times e^{-ip\cdot x+iq\cdot y}\langle f,\lambda|T[\phi(x),\bar\phi(y)]|g,\mu\rangle\,,\quad(5.1)$$

which is the special case $n=2$ of the function defined in (4.11). (Since we have set $J=0$, we may omit the "in" and "out" labels on the coherent states.)

In this case, the only core diagrams that can contribute are those with a single straight-through line. Hence (1.2) reduces to

$$G_{f\lambda,g\mu}(p,-q)C^{-1}=Z^h(p)\Lambda(p)\Delta^s{}_{f\lambda,g\mu}(p,q)\,.\quad(5.2)$$

The exponent X defined by (1.3) now has only a single term, which may be recognized as the term $\tfrac{1}{2}X_{11}$ of [II, (4.4)]. As we saw in II, the contribution to it from the pole at $p\cdot k=0$, namely, [II, (4.19)], is linear in K and therefore negligible. Thus we need only retain the contribution [II, (4.20)] from the pole at $k^2=0$. Using again the formal notation introduced in (2.3) and (2.12), this term may be written (for $p^0>0$) in the form

$$X=(2p\sigma)s_p{}^*s_p-s_p{}^*s_p\,,\quad(5.3)$$

where s_p denotes the photon wave function

$$s_{p\mu}(k)=\gamma_{\mu\nu}(k)ep^\nu/p\cdot k\,.\quad(5.4)$$

The other terms in the expression (4.12) for $X_{f\lambda,g\mu}$ may be written, in the same notation, as

$$i(f^*I_1+I_1^*g)=-f^*(y+2p\sigma)s_p+f^*(y)s_p +(y+2p\sigma)s_p^*g-(y)s_p^*g.$$

Thus we obtain

$$\Delta^s{}_{f\lambda,g\mu}(p,q)=\int dy\, e^{-i(p-q)\cdot y}\int_0^\infty d\sigma \times e^{-i\sigma(m^2+p^2)}\langle f,\lambda|g,\mu\rangle \exp[(2p\sigma)s_p^*s_p-s_p^*s_p -f^*(y+2p\sigma)s_p+f^*(y)s_p +(y+2p\sigma)s_p^*g-(y)s_p^*g]. \quad (5.5)$$

As we have already remarked at the end of II, there is, in the radiation gauge, one special point at which the Green's functions do have a pole on the mass shell, namely, $\mathbf{p}=\mathbf{0}$. This is clear from (5.5). For in that gauge s_p vanishes when $\mathbf{p}=\mathbf{0}$, so that (5.2) and (5.5) yield

$$G_{f\lambda,g\mu}(p,-q)C^{-1}|_{\mathbf{p}=0} =Z^h(\mathbf{0})\Lambda(\mathbf{0})2\pi\delta(p^0-q^0)(2\pi)^3\delta(\mathbf{q}) \times[-i/(m^2-p_0{}^2-i\epsilon)]\langle f,\lambda|g,\mu\rangle. \quad (5.6)$$

Thus the $\mathbf{p}=\mathbf{0}$ component of ϕ, with p^0 close to m, must have the effect on a soft-photon state of simply adding a single particle, with no change in the soft-photon state. The generalization of (3.5) that correctly reproduces the discontinuity of (5.6) is

$$\langle\alpha|\phi(0)|\mathbf{0};\beta\rangle=[Z^h(\mathbf{0})]^{1/2}u(\mathbf{0})\langle\alpha|\beta\rangle, \quad \langle\mathbf{0};\alpha|\bar\phi(0)|\beta\rangle=[Z^h(\mathbf{0})]^{1/2}\bar u(\mathbf{0})\langle\alpha|\beta\rangle, \quad (5.7)$$

where $\langle\alpha|$ and $|\beta\rangle$ are arbitrary soft-photon states, linear combinations of the coherent states.

It should be remarked that we have not written "in" or "out" labels on the states in (5.7), although, unlike the pure single-particle states, they are not obviously stable. In fact, it will be shown later that these states are actually stable. Physically, this is because, in a state containing only a single charged particle and soft photons, the soft photons cannot transfer a significant amount of momentum to the particle, which is therefore effectively unaccelerated and does not radiate. (An argument to this effect has also been presented in a different context by Storrow.[4]) For the moment, however, we must regard the Eqs. (5.7) merely as indications of the kind of structure we may expect rather than as proven.

We could obtain from (5.7) corresponding equations for arbitrary momentum by applying a Lorentz transformation, but this is difficult to do directly for two reasons. Firstly, the transformation properties of ϕ are rather complicated in the radiation gauge, particularly in the case of finite transformations,[5] and, secondly, a Lorentz transformation applied to a soft-photon state in general yields a state containing some hard photons. It will be more convenient, therefore, to obtain corresponding formulas for other momenta independently by a heuristic argument based on the structure of the Green's functions, and later to verify their consistency with Lorentz transformations.

To do this it will be useful to make a transformation of variables in (5.5). We first introduce a new variable of integration x, and an associated δ function with the Fourier representation

$$\int\frac{dk}{(2\pi)^4}e^{-ik\cdot(x+y+2p\sigma)}.$$

Then we can replace $2p\sigma$ where it occurs in the exponent with $x-y$ and perform the integration over σ. Using the fact that $(x-y)s^*s=(x)s^*(y)s$, we may then write (5.5) as

$$\Delta^s{}_{f\lambda,g\mu}(p,q)=-i\int dxdy\int\frac{dk}{(2\pi)^4}\frac{e^{-ik\cdot x-i(p-q-k)\cdot y}}{m^2+p^2-2p\cdot k-i\epsilon} \times\langle f,\lambda|g,\mu\rangle \exp[(x)s_p^*(y)s_p-\tfrac12(x)s_p^*(x)s_p -\tfrac12(y)s_p^*(y)s_p-f^*(x)s_p+f^*(y)s_p +(x)s_p^*g-(y)s_p^*g]. \quad (5.8)$$

Next, we observe that k may be treated as small. For, if we were to expand the exponential, we would find in each term that the x integration yields a δ function that sets k equal to a sum of soft-photon momenta. Thus we may add a term k^2 to the denominator, and transform to the new variable $l=p-k$ so that it becomes $(m^2+l^2-i\epsilon)^{-1}$. [Recall that such terms were dropped in obtaining the original formula (1.3).] Moreover, s_p is a slowly varying function of p, as are $Z^h(p)$ and $\Lambda(p)$, so that in these it is legitimate to replace p with l. Finally, we note that the exponential factor in (5.8) has just the structure of the right-hand side of (2.8) with $h_1=-(x)s_l$ and $h_2=(y)s_l$. Thus we obtain

$$\Delta^s{}_{f\lambda,g\mu}(p,q)=-i\int dxdy\int\frac{dl}{(2\pi)^4}\frac{e^{-i(p-l)\cdot x-i(l-q)\cdot y}}{m^2+l^2-i\epsilon} \times\langle f,\lambda|U[-(x)s_l]U[(y)s_l]|g,\mu\rangle. \quad (5.9)$$

Now let us substitute in (5.2) and evaluate the discontinuity function in the variable p. This is clearly given by the δ-function part of the denominator in (5.9), and thus, inserting a complete set of soft-photon states, we obtain

$$\mathrm{disc}G_{f\lambda,g\mu}(p,-q)C^{-1}=\int dxdy \times e^{-ip\cdot x+iq\cdot y}\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}e^{il\cdot(x-y)}Z^h(l)\Lambda(l) \times\sum_\alpha\langle f,\lambda|U[-(x)s_l]|\alpha\rangle\langle\alpha|U[(y)s_l]|g,\mu\rangle. \quad (5.10)$$

On the other hand, if we insert a complete set of states directly in the matrix element (5.1) with the time-ordering symbol removed and restrict this sum to the

[4] J. K. Storrow, Nuovo Cimento **54**, 15 (1968).

[5] D. G. Boulware, thesis, Harvard University, 1962 (unpublished).

states $|\mathbf{l};\alpha\rangle$ containing a single particle and soft photons, then we obtain

$$\mathrm{disc}G_{f\lambda,g\mu}(p,-q)C^{-1}=\int dxdy \times e^{-ip\cdot x+iq\cdot y}\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}\sum_\alpha\langle f,\lambda|\phi(x)|\mathbf{l};\alpha\rangle \times\langle\mathbf{l};\alpha|\bar{\phi}(y)|g,\mu\rangle. \quad (5.11)$$

Thus, comparing (5.10) and (5.11), it is natural to suppose that the correct generalization of (5.7) is

$$\langle\alpha|\phi(x)|\mathbf{l};\beta\rangle =[Z^h(l)]^{1/2}u(\mathbf{l})e^{il\cdot x}\langle\alpha|U[-(x)s_l]|\beta\rangle, \quad (5.12a)$$

$$\langle\mathbf{l};\alpha|\bar{\phi}(x)|\beta\rangle =[Z^h(l)]^{1/2}\bar{u}(\mathbf{l})e^{-il\cdot x}\langle\alpha|U[(x)s_l]|\beta\rangle. \quad (5.12b)$$

It is easy to verify that these relations are consistent with translational invariance, using the relation

$$e^{-iP\cdot x}|\mathbf{l};f,\lambda\rangle=|\mathbf{l};(x)f,\lambda\rangle e^{-il\cdot x}. \quad (5.13)$$

They also clearly reproduce (5.7).

These relations have a very interesting physical interpretation. They show that when the field operator $\phi(0)$ creates a particle of momentum $\mathbf{l}$, the change in the soft-photon state is just that corresponding to the creation of additional soft photons in the coherent state $|s_l,0\rangle$. The behavior of $\phi(x)$ is similar, but with the translated function $(x)s_l$ in place of s_l.

One point of arbitrariness should be noted. The correspondence between (5.10) and (5.11) would be unaffected if we multiplied the two relations (5.12) by any generalized phase factor depending on l and by its complex conjugate, respectively. There could, of course, be analogous phase factors even in (5.7), but these can be removed by appropriate choice of the phase of the state $|\mathbf{0};\beta\rangle$. However, once this phase is fixed, the phases in (5.12) are also fixed, in principle, if we want the states to transform correctly under Lorentz transformations. That the correct choice of phase is in fact that given in (5.12) will be demonstrated later, in Sec. 7.

6. ONE-PARTICLE REDUCTION FORMULAS

Now that we know the structure of the two-point function, we are in a position to give a proper limiting definition of the states containing a single particle and soft photons, which are defined in a purely formal sense by (5.12).

We begin by examining the singularity at $p_1^2=-m_1^2$ of the general function $G_{f\lambda,g\mu}$ defined in (4.11). For simplicity, let us again suppose that we are interested in a region of momenta in which only completely connected core diagrams can contribute significantly. Then we may set $r=0$ in (1.2) and obtain

$$G_{f\lambda,g\mu}(p_1\cdots p_n|J)=\prod_{j=1}^n\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\} \times\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s{}_{f\lambda,g\mu}(p_1\cdots p_n;q_1\cdots q_n|J) \times(2\pi)^4\delta(q_1+\cdots+q_n)M^h(q_1\cdots q_n). \quad (6.1)$$

It will be convenient to apply to the variables associated with p_1 the same transformation as that which led to (5.9). We begin by adding a small imaginary part to the denominator $p_1\cdot k$ in (1.5), so that the two terms involving p_1 may be separated. Then, as before, we introduce new variables x_1 $(=y_1+2p_1\sigma_1)$ and l_1 and perform the integration over σ_1, to obtain

$$\Delta^s{}_{f\lambda,g\mu}(p_1\cdots p_n;q_1\cdots q_n|J)=-i\int dx_1dy_1 \times\int\frac{dl_1}{(2\pi)^4}\frac{\exp[-i(p_1-l_1)\cdot x_1-i(l_1-q_1)\cdot y_1]}{m_1^2+l_1^2-i\epsilon} \times\int dy_2\cdots dy_n\int_0^\infty d\sigma_2\cdots d\sigma_n\exp\Big(-i\sum_{j=2}^n(p_j-q_j)\cdot y_j -i\sum_{j=2}^n\sigma_j(m_j^2+p_j^2)\Big)\exp X'_{f\lambda,g\mu}, \quad (6.2)$$

where $X'_{f\lambda,g\mu}$ is obtained from $X_{f\lambda,g\mu}$ by the replacement

$$I^\mu{}_{1\cdots n}(k)\to i[e_1l_1{}^\mu/(l_1\cdot k-i\epsilon)] \times[\exp(-ik\cdot x_1)-\exp(-ik\cdot y_1)]+I^\mu{}_{2\cdots n}(k). \quad (6.3)$$

Now let us examine the various terms in $X'_{f\lambda,g\mu}$. Those involving only the variables with subscript 1 are of the same form as in (5.8), but with l_1 in place of p and x_1, y_1 for x, y. The cross term between the two parts (6.3) is

$$\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{e_1l_1{}^\mu}{l_1\cdot k+i\epsilon}[\exp(ik\cdot x_1)-\exp(ik\cdot y_1)] \times\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{2\cdots n}(k). \quad (6.4)$$

The mass-shell singularity of (6.1) at $p_1^2=-m_1^2$ with $p_1^0>0$ is governed by the asymptotic behavior of (6.2) for large x_1 with $x_1^0>0$. Hence in the first of the two terms here (but not in the second) we may complete the k^0 contour in the lower half-plane. Since significant contributions to (6.2) come only from the region where l_1 is close to p_1 and $l_1^0>0$, this contour excludes the pole at $l_1\cdot k=0$, and so, provided that $J^\mu(k)$ has no singularities in this half-plane, we obtain simply $i(x_1)s_1{}^*I_{2\cdots n}$, where, of course, s_1 is s_l evaluated for $l=l_1$. Actually, the condition on J here may be considerably relaxed, since singularities a finite distance below the real axis do not matter. What is excluded is, for example, a pole on the real axis, corresponding to a charge persisting to infinite times. We could deal with such cases too, but it will be unnecessary to do so. Indeed, for most applications we could already set $J=0$ at this stage, since it has fulfilled its major role of creating soft-photon states.

Clearly, if we were interested in the region $p_1{}^0<0$, we should have to close the contour in the upper half-plane, again excluding the pole at $l_1\cdot k=0$ (since now $l_1{}^0<0$), and would obtain instead $-iI_{2\cdots n}{}^*(x_1)s_1$.

Next, we shall regroup these terms, collecting together all those that involve either x_1 or the soft-photon states. Denoting the remaining terms, independent of these variables, by $X_{(1)}$, we have

$$\begin{aligned}\exp X'_{f\lambda,g\mu}=&\langle f,\lambda|g,\mu\rangle\exp\{[-f^*(x_1)s_1+f^*(y_1)s_1\\&+(x_1)s_1{}^*g-(y_1)s_1{}^*g]+[(x_1)s_1{}^*(y_1)s_1-\tfrac{1}{2}(x_1)s_1{}^*(x_1)s_1]\\&+i[f^*I_{2\cdots n}+I_{2\cdots n}{}^*g]+i[(x_1)s_1{}^*I_{2\cdots n}]\}\exp X_{(1)}\\=&\langle f,\lambda|U[-(x_1)s_1]|g,\mu\rangle\exp\{i[f+(x_1)s_1]^*I_{(1)2\cdots n}\\&+iI_{(1)2\cdots n}{}^*g\}\exp X_{(1)},\quad(6.5)\end{aligned}$$

where we have introduced the function

$$I^\mu{}_{(1)2\cdots n}(k)=-i[e_1l_1{}^\mu/(l_1\cdot k-i\epsilon)]\times\exp(-ik\cdot y_1)+I^\mu{}_{2\cdots n}(k).\quad(6.6)$$

The remaining terms which constitute $X_{(1)}$ are easily found to be

$$\begin{aligned}X_{(1)}=&-\tfrac{1}{2}(y_1)s_1{}^*(y_1)s_1-\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{e_1l_1{}^\mu}{l_1\cdot k+i\epsilon}\\&\times\exp(ik\cdot y_1)\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{2\cdots n}(k)+\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}\\&\times I^\mu{}_{2\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{2\cdots n}(k).\quad(6.7)\end{aligned}$$

This expression may also be written more compactly in terms of the function (6.6) as

$$X_{(1)}=\tfrac{1}{2}i\int'_{\Omega^s}\frac{dk}{(2\pi)^4}I^\mu{}_{(1)2\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{(1)2\cdots n}(k),\quad(6.8)$$

where the prime on the integral indicates that in the term involving only the variables with label 1 the contribution of the pole at $l_1\cdot k=0$ is to be discarded, leaving only the first term in (6.7). [Such an interpretation is needed to give (6.8) a meaning even within the context of the rules that we have introduced for dealing with infrared-divergent integrals. For, otherwise, the k^0 integral does not exist even for $\mathbf{k}\neq\mathbf{0}$, because the contour is pinched between a coincident pair of poles.]

The interesting and important feature of the formula (6.5) is the fact that the soft-photon state $\langle f,\lambda|$ and the variable x_1 appear *only* in the combination $\langle f,\lambda|\times U[-(x_1)s_1]$. [Note that this is again a coherent state, with the photon wave function $f+(x_1)s_1$.]

When we substitute (6.2) into (6.1), we may replace p_1 with l_1 in the slowly varying functions $Z^h{}_1$ and Λ_1. It is clear that the discontinuity function in the variable p_1 is given simply by replacing the denominator in (6.2) with its δ-function part. Hence the structure that we obtain is of the form

$$\begin{aligned}\mathrm{disc}_1G_{f\lambda,g\mu}(p_1\cdots p_n|J)=&\int dx_1\exp(-ip_1\cdot x_1)\\&\times\int\frac{d\mathbf{l}_1}{(2\pi)^3 2l_1{}^0}[Z^h{}_1(l_1)]^{1/2}\Lambda_1(l_1)\exp(il_1\cdot x_1)\\&\times\langle f,\lambda|U[-(x_1)s_1]\cdots,\quad(6.9)\end{aligned}$$

where the remainder of the expression, indicated by the dots, is independent of x_1.

On the other hand, this discontinuity function should be related to the appearance of real intermediate states in (4.11). If we remove $\phi_1(x_1)$ from the time-ordering symbol, insert a complete set of states of the form $|\mathbf{l}_1;\alpha\rangle$, and use (5.12), we obtain

$$\begin{aligned}\mathrm{disc}_1G_{f\lambda,g\mu}(p_1\cdots p_n|J)=&\int dx_1\cdots dx_n\\&\times\exp(-i\sum_{j=1}^n p_j\cdot x_j)\int\frac{d\mathbf{l}_1}{(2\pi)^3 2l_1{}^0}\\&\times\exp(il_1\cdot x_1)\sum_\alpha\langle f,\lambda|U[-(x_1)s_1]|\alpha\rangle\\&\times\langle\mathbf{l}_1;\alpha;\mathrm{out}|T[\phi_2(x_2)\cdots\phi_n(x_n)]|g,\mu;\mathrm{in}\rangle_J.\quad(6.10)\end{aligned}$$

Clearly, the effect of the sum over α is to replace $\langle\alpha|$ in the expression for the second matrix element with $\langle f,\lambda|U[-(x_1)s_1]$. Hence (6.9) and (6.10) have indeed the same structure, and, comparing them, we find an expression for the matrix element in (6.10), namely,

$$\begin{aligned}\int dx_2\cdots dx_n&\exp(-i\sum_{j=2}^n p_j\cdot x_j)\langle\mathbf{l}_1;f,\lambda;\mathrm{out}|T[\phi_2(x_2)\cdots\phi_n(x_n)]|g,\mu;\mathrm{in}\rangle_J\\&=\bar{u}(\mathbf{l}_1)\prod_{j=2}^n\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\}\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s{}_{f\lambda,g\mu}(l_1,p_2\cdots p_n;q_1\cdots q_n|J)\\&\qquad\times(2\pi)^4\delta(q_1+\cdots+q_n)M^h(q_1\cdots q_n),\quad(6.11)\end{aligned}$$

where we have introduced a new function Δ^s with one variable l_1 restricted to its mass shell, defined by

$$\begin{aligned}\Delta^s{}_{f\lambda,g\mu}(l_1,p_2\cdots p_n;q_1\cdots q_n|J)=&\int dy_1\exp[-i(l_1-q_1)\cdot y_1]\int dy_2\cdots dy_n\int_0^\infty d\sigma_2\cdots d\sigma_n\\&\times\exp(-i\sum_{j=2}^n(p_j-q_j)\cdot y_j-i\sum_{j=2}^n\sigma_j(m_j{}^2+p_j{}^2))\langle f,\lambda|g,\mu\rangle\exp[i(f^*I_{(1)2\cdots n}+I_{(1)2\cdots n}{}^*g)+X_{(1)}],\quad(6.12)\end{aligned}$$

with $X_{(1)}$ given by (6.8).

For certain special cases, e.g., $n=1$, the notation for this function might be ambiguous. However, we shall avoid this by always using l_j to denote a variable on its mass shell. Thus $\Delta^s(p;\,q)$ denotes the special case $n=1$ of (1.3), while $\Delta^s(l;\,q)$ is a special case of (6.12). We may regard (6.12) as a "truncated" form of (1.3).

Now let us return to the question of defining the asymptotic states $\langle \mathbf{l};\,f,\lambda;\,\text{out}|$ as weak limits. To do this we must, of course, work with wave packets formed out of these states, of the form

$$N^{-1}\int \frac{d\mathbf{l}}{(2\pi)^3 2l^0}[\bar{\psi}(\mathbf{l})u(\mathbf{l})]\langle \mathbf{l};\,\alpha_l;\,\text{out}|\,, \tag{6.13}$$

where $\bar{\psi}(\mathbf{l})$ is a prescribed spin and momentum-space wave function and where, for each $\mathbf{l}$, $\langle\alpha_l|$ is some soft-photon state—in general, a linear combination of coherent states.

The limit (3.10) yields a nonzero result only because of the pole in the Green's function. In our case, the Green's functions have no pole, in general, so that the same technique fails. What we can do, however, is to modify the soft-photon state in such a way as to construct a function that does have a pole.

Let us define

$$\langle\alpha_{l,x}| = \langle\alpha_l|\,U[(x)s_l]. \tag{6.14}$$

Then we choose to define the state (6.13) by the limit

$$\begin{aligned} N^{-1}\int \frac{d\mathbf{l}}{(2\pi)^3 2l^0}&[\bar{\psi}(\mathbf{l})u(\mathbf{l})]\langle \mathbf{l};\,\alpha_l;\,\text{out}|\,T[\phi_2(x_2)\cdots\phi_n(x_n)]\,|\beta;\,\text{in}\rangle_J \\ &= \lim_{t\to\infty} N^{-1}\int \frac{dl}{(2\pi)^4}\bar{\psi}(\mathbf{l})\sigma(l)\,\exp[-i(l^0-\omega_1)t][Z^h{}_1(l)]^{-1/2} \\ &\quad\times\int dx\, e^{-il\cdot x}\langle\alpha_{l,x};\,\text{out}|\,T[\phi_1(x)\phi_2(x_2)\cdots\phi_n(x_n)]\,|\beta;\,\text{in}\rangle_J. \end{aligned} \tag{6.15}$$

We now must show that this limit exists. To do this we insert our expression for the Green's function on the right-hand side. As far as its dependence on the variable p_1 is concerned, the Green's function has the structure of (6.9), but with a four-dimensional integration over l_1 and a denominator $(m_1^2+l_1^2-i\epsilon)^{-1}$. Hence, taking the Fourier transform, we set $x_1=x$ and obtain for the right-hand side of (6.15) the structure

$$\begin{aligned} \lim_{t\to\infty}(-i)N^{-1}\int \frac{dl}{(2\pi)^4}\bar{\psi}(\mathbf{l})\sigma(l)\,\exp[-i(l^0-\omega_1)t][Z^h{}_1(l)]^{-1/2}\int dx\int \frac{dl_1}{(2\pi)^4}[Z^h{}_1(l_1)]^{1/2}\Lambda_1(l_1) \\ \times\frac{\exp[-i(l-l_1)\cdot x]}{m_1^2+l_1^2-i\epsilon}\langle\alpha_l|\,U[(x)s_l]U[-(x)s_1]\cdots\,, \end{aligned} \tag{6.16}$$

were, as before, the remaining factors are independent of x. But now from the structure of the x integration it is clear that $l-l_1$ is small. Hence we can replace l_1 with l in the slowly varying function s_l. But then the two soft-photon unitary operators cancel, and the x integral yields simply $(2\pi)^4\delta(l-l_1)$. Thus we obtain

$$\lim_{t\to\infty}(-i)\int\frac{dl}{(2\pi)^4}\bar{\psi}(\mathbf{l})\sigma(l)\frac{\exp[-i(l^0-\omega_1)t]}{m_1^2+l^2-i\epsilon}\langle\alpha_l|\cdots. \tag{6.17}$$

In this form, it is clear that the limit exists and has the effect of picking out the residue at the pole, yielding

$$\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}\bar{\psi}(\mathbf{l})\langle\alpha_l|\cdots. \tag{6.18}$$

The result that we obtain in this way for the matrix element (6.15) agrees precisely with (6.11). Hence we have shown that the states (6.13) can be defined as weak limits. We still must show that our labeling of these states is consistent with Lorentz covariance, but we shall do this later.

Of course, this limiting formula (6.15) is not a complete reduction formula for the state (6.13), in that it does not define it directly in terms of the action of field operators on the vacuum. However, we already know how to create soft-photon states from the vacuum, by using (4.8). So we could, if we wished, combine this formula with (6.15) to obtain a complete reduction formula for (6.13). However, we shall not write out the result explicitly.

It is clear that the in states containing a single particle and soft photons could be defined in a similar way, by examining the mass-shell singularity of (6.1) for $p_1{}^0<0$. Then we should have to complete the k^0 contour in the opposite half-plane, so that $(x_1)s_1$ would not appear in (6.5) as an addition to f, but subtracted from g, and thus in the combination $U[-(x_1)s_1]|g,\mu\rangle$. The net result is that, in the formula corresponding to (6.11) with an antiparticle of momentum $-\mathbf{l}_1$ in the initial state, we have only to replace $\bar{u}(\mathbf{l}_1)$ with $\bar{u}^c(-\mathbf{l}_1)$ and to evaluate (6.12) for $l_1^0<0$.

The modification of (6.11) required to accommodate disconnected core diagrams is obvious, and we shall not write it down explicitly. However, it will be useful to give an explicit formula for the special case $n=2$, for which the only core diagrams are those with a single straight-through line. For this case the appropriate form of (6.11) is

$$\int dy\, e^{iq\cdot y}\langle \mathbf{l};\, f,\lambda;\, \text{out}|\bar{\phi}(y)|g,\mu;\, \text{in}\rangle_J = \bar{u}(\mathbf{l})[Z^h(l)]^{1/2}\Delta^s{}_{f\lambda,g\mu}(l;q|J)\,, \quad (6.19)$$

where we have used the fact that $l-q$ is small to permit the replacement of q with l in $Z^h(q)$ and $\Lambda(q)$. Thus, using (6.12) and taking the Fourier transform on q, we obtain

$$\langle \mathbf{l};\, f,\lambda;\, \text{out}|\bar{\phi}(y)|g,\mu;\, \text{in}\rangle_J = \bar{u}(\mathbf{l})[Z^h(l)]^{1/2}e^{-il\cdot y}\langle f,\lambda|g,\mu\rangle \times\exp\biggl(f^*(y)s_l-(y)s_l{}^*g+i(f^*J+J^*g)-\tfrac{1}{2}s_l{}^*s_l - \int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{el^\mu e^{ik\cdot y}}{l\cdot k-i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}J^\nu(k) + \tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}J^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}J^\nu(k)\biggr)\,. \quad (6.20)$$

The corresponding formula for the matrix element $\langle f,\lambda;\,\text{out}|\phi(x)|\mathbf{l};\, g,\mu;\,\text{in}\rangle_J$ is obtained by replacing y with x, $\bar{u}(\mathbf{l})$ with $u(\mathbf{l})$, and changing the signs of l^μ and e (and hence of s_l). We note that for $J=0$ these formulas reproduce (5.12), as they must for consistency.

7. LORENTZ INVARIANCE

Although we have shown that the limit (6.15) defines asymptotic states containing a single particle and soft photons, we have yet to show that our labeling of these states is consistent with Lorentz invariance. To do this we have to verify the Lorentz covariance of the relations (5.12).

Since these relations are clearly invariant under translations and spatial rotations, we need only consider pure Lorentz transformations. Moreover, it is clearly sufficient to restrict our discussion to infinitesimal transformations, which for our purpose have the great advantage that they do not affect the property of being a soft-photon state. We therefore take Λ to be of the form $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\epsilon^\mu{}_\nu$, where the only nonvanishing components of $\epsilon_{\mu\nu}$ are $\epsilon_{0j}=-\epsilon_{j0}$.

What we have to show is that for any such Λ

$$\langle f,\lambda|U^{-1}(\Lambda)\phi(0)U(\Lambda)|\mathbf{l};\, g,\mu\rangle = \langle(\Lambda)f,\lambda|\phi(0)|\mathbf{l}';\,(\Lambda)g,\mu\rangle\,, \quad (7.1)$$

where $U(\Lambda)=U(0,\Lambda)$ and $l'^\mu=\Lambda^\mu{}_\nu l^\nu$, and where the left-hand side is to be evaluated using the known transformation properties of ϕ. Since Λ is infinitesimal, the right-hand side is again a matrix element between soft-photon states, and thus, according to (5.12), may be written as

$$[Z^h(l')]^{1/2}u(\mathbf{l}')\langle(\Lambda)f,\lambda|U[-s_{l'}]|(\Lambda)g,\mu\rangle.$$

The soft-photon matrix element itself is invariant under infinitesimal Lorentz transformations, as may easily be verified from (2.11). Hence we may write equivalently

$$[Z^h(l')]^{1/2}u(\mathbf{l}')\langle f,\lambda|U[-(\Lambda^{-1})s_{l'}]|g,\mu\rangle. \quad (7.2)$$

In infinitesimal form, (7.1) reads

$$-i\epsilon^{0j}\langle f,\lambda|[\phi(0),J_{0j}]|\mathbf{l};\, g,\mu\rangle = \delta\{[Z^h(l)]^{1/2}u(\mathbf{l})\langle f,\lambda|U[-s_l]|g,\mu\rangle\}\,, \quad (7.3)$$

where the right-hand side stands for the difference between (7.2) and (5.12). It may be written as a sum of three terms corresponding to the variations of the three factors.

The variation of $u(\mathbf{l})$ is described simply by

$$\delta u(\mathbf{l})=-i\epsilon^{0j}S_{0j}u(\mathbf{l})\,, \quad (7.4)$$

where S_{0j} is the appropriate spin matrix.

To find the variation of the soft-photon factor, we note that

$$\delta s^\mu(\mathbf{k})=(\Lambda^{-1})s_{l'}{}^\mu(\mathbf{k})-s_l{}^\mu(\mathbf{k}) \quad (7.5)$$

is in fact independent of l and given explicitly by

$$\delta s^\mu(\mathbf{k})=-e(k^0)^{-1}\gamma^\mu{}_j(k)\epsilon^{0j},$$

as is easy to verify. [Basically, this is because the right-hand side of (7.5) is nonzero only in virtue of the explicit dependence of s_l on the timelike unit vector n.] Then, because of the exponential structure of the soft-photon matrix element, we have

$$\delta\langle f,\lambda|U[-s_l]|g,\mu\rangle = \langle f,\lambda|U[-s_l]|g,\mu\rangle(-f^*\delta s+\delta s^*g-\tfrac{1}{2}\delta s^*s-\tfrac{1}{2}s^*\delta s) = \langle f,\lambda|U[-s_l]|g,\mu\rangle e\epsilon^{0j}\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0} \times\left[f_j{}^*(\mathbf{k})-g_j(\mathbf{k})-\gamma_{jk}(k)\frac{el^k}{l\cdot k}\right]. \quad (7.6)$$

Finally, the variation of $Z^h(l)$ is given by

$$\delta Z^h(l)=\epsilon^{0j}\left(l_j\frac{\partial}{\partial l^0}-l_0\frac{\partial}{\partial l^j}\right)Z^h(l)\,. \quad (7.7)$$

Now let us examine the left-hand side of (7.3). The Lorentz transformation of ϕ is described by[5]

$$-i[\phi(0),J_{0j}]=-iS_{0j}\phi(0)+ie\lambda_{0j}(0)\cdot\phi(0)\,, \quad (7.8)$$

where λ_{0j} is the parameter of the associated gauge transformation that is required to bring us back to the radiation gauge in the new frame and the dot denotes a symmetrized product. Explicitly, λ_{0j} is given by

$$\lambda_{0j}(0)=(1/\nabla^2)(\partial_0 A_j+\partial_j A_0)$$
$$=i\int \frac{dk}{(2\pi)^4}\frac{k_0A_j(k)+k_jA_0(k)}{\mathbf{k}^2}. \qquad (7.9)$$

When we substitute these relations into (7.3), we find that the first term in (7.8) cancels the variation of $u(\mathbf{l})$ given by (7.4). What we have to prove, therefore, is that

$$ie\langle f,\lambda|\lambda_{0j}(0)\cdot\phi(0)|\mathbf{l};g,\mu\rangle$$
$$=[Z^h(l)]^{1/2}u(\mathbf{l})\langle f,\lambda|U(-s_l)|g,\mu\rangle$$
$$\times\left\{e\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2(k^0)^2}\left[f_j^*(\mathbf{k})-g_j(\mathbf{k})-\gamma_{jk}(k)\frac{el^k}{l\cdot k}\right]\right.$$
$$\left.+\tfrac{1}{2}\left(l_j\frac{\partial}{\partial l^0}-l_0\frac{\partial}{\partial l^j}\right)[\ln Z^h(l)]\right\}. \qquad (7.10)$$

To do this we must evaluate the matrix element that appears on the left-hand side, which may be written in terms of the function

$$\int dxdz\, e^{-ip\cdot x-ik\cdot z}\langle f,\lambda|T[\phi(x),A_\mu(z)]|\mathbf{l};g,\mu\rangle. \qquad (7.11)$$

Let us first consider the soft-photon part, for which $k\in\Omega^s$. This function may be obtained from the conjugate equation to (6.20) by differentiating with respect to $J_\mu(k)$ and then setting $J=0$. The result is to multiply the corresponding formula for the matrix element without A_μ by a sum of three terms, namely,

$$i[f_\mu^*(\mathbf{k})\theta(k^0)+g_\mu(-\mathbf{k})\theta(-k^0)]2\pi\delta(k^2)$$
$$+\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{el^\nu}{l\cdot k+i\epsilon}e^{ik\cdot x}.$$

It follows from (7.9) that

$$ie\langle f,\lambda|\lambda^s{}_{0j}(0)\cdot\phi(0)|\mathbf{l};g,\mu\rangle$$
$$=-e\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{k_0\delta_j{}^\mu+k_j\delta_0{}^\mu}{\mathbf{k}^2}\int dz\, e^{-ik\cdot z}\langle f,\lambda|T[\phi(0),A_\mu(z)]|\mathbf{l};g,\mu\rangle$$
$$=-ie[Z^h(l)]^{1/2}u(\mathbf{l})\langle f,\lambda|U(-s_l)|g,\mu\rangle$$
$$\times\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{k_0\delta_j{}^\mu+k_j\delta_0{}^\mu}{\mathbf{k}^2}\left([f_\mu^*(\mathbf{k})\theta(k^0)+g_\mu(-\mathbf{k})\theta(-k^0)]2\pi\delta(k^2)-i\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{el^\nu}{l\cdot k+i\epsilon}\right). \qquad (7.12)$$

Now in the last term of the integrand no contribution can come from the term in $\delta(l\cdot k)$, since this is odd in k. Hence we may close the contour in either half-plane and obtain a contribution only from the pole at $k^2=0$. The k integral thus reduces to the same form as the $\mathbf{k}$ integral in (7.6), so that the soft-photon contribution to the left-hand side of (7.10) cancels the $\mathbf{k}$ integral on the right.

Next, we must consider the hard-photon part of (7.11). This function is, of course, just a special case of the function (6.11). However, in several respects it is different from cases that we have considered before. Firstly, since the electromagnetic field is neutral, the corresponding part of the function Δ^s would be simply a momentum δ function, so that we may omit it and regard the hard-photon line as being attached directly to the core of the diagram. Secondly, since we have chosen to work with a renormalized electromagnetic field, there should be no corresponding factor $[Z^h(k)]^{1/2}$. The most important new feature, however, is that we are no longer interested only in the region where p is close to its mass shell, so that our earlier formulas in which p^2+m^2 was assumed to be small are inapplicable. It is no longer possible to replace the propagator function $G^h(p)$ with its pole term alone. However, this circumstance also introduces a corresponding simplification. When p^2+m^2 is not small, there is no significant interaction between the line of momentum p and the soft photons, so that this line too may be attached directly to the core of the diagram.

At first sight one might think that the neighborhood of the mass shell would require special treatment, but, in fact, this is not necessary. We are dealing in effect with an internal line, and have to integrate over all values of p. Provided that the dependence on p of the remaining factors in the diagram is smooth near the mass shell (as it is here), the soft-photon contributions do not affect the value of the integral. Were this not so, we should have had to consider soft-photon corrections to internal lines in our original analysis. To see that it is so, take an internal line with soft-photon corrections whose ends have momenta p_1 and q_1, and consider its contribution near the mass shell written in terms of (6.2). If we multiply this function by slowly varying functions of p_1 and q_1 and integrate, we obtain an integral over x_1 and y_1 restricted to small values. This means that the corresponding terms in (6.3), which represent the soft-photon contributions, drop out. Only if the other functions of p_1 and q_1 vary appreciably in a region of the size of Ω^s can the soft-photon contributions be important.

In the region of interest we thus find that (7.11) is given by

$$\int dxdz\, e^{-ip\cdot x-ik\cdot z}\langle f,\lambda|T[\phi(x),A_\mu(z)]|\mathbf{l};\,g,\mu\rangle$$
$$=\int\frac{dq}{(2\pi)^4}G^h(p)D^h{}_{\mu\nu}(k)(2\pi)^4\delta(p+k-q)e\Gamma^{h\nu}(p,q)\times[Z^h(q)]^{1/2}\Delta^s{}_{f\lambda,g\mu}(q;\,l)u(\mathbf{l}).\quad(7.13)$$

Here $G^h(p)$ is the complete propagator function with the (irrelevant) soft-photon contributions removed, $D^h(k)$ is the corresponding function for the electromagnetic field, and $e\Gamma^h(p,q)$, the vertex function with soft-photon contributions removed, plays essentially the role of M^h in (6.11).

In the non-soft-photon parts of (7.13) we may replace q with l. Thus, as in (7.12), we obtain

$$ie\langle f,\lambda|\lambda^h{}_{0j}(0)\cdot\phi(0)|\mathbf{l};\,g,\mu\rangle$$
$$=-e\int_{\Omega^h}\frac{dk}{(2\pi)^4}\frac{k_0\delta_j{}^\mu+k_j\delta_0{}^\mu}{\mathbf{k}^2}\int dz e^{-ik\cdot z}\times\langle f,\lambda|T[\phi(0),A_\mu(z)]|\mathbf{l};\,g,\mu\rangle$$
$$=-e[Z^h(l)]^{1/2}\langle f,\lambda|U(-s_l)|g,\mu\rangle\times\int_{\Omega^h}\frac{dk}{(2\pi)^4}\frac{k_0\delta_j{}^\mu+k_j\delta_0{}^\mu}{\mathbf{k}^2}G^h(l-k)D_{\mu\nu}{}^h(k)\times e\Gamma^{h\nu}(l-k,\,l)u(\mathbf{l}).\quad(7.14)$$

The dependence on the soft-photon states here is the same as in (7.10). Hence the two sides of (7.10) are equal provided only that

$$\tfrac{1}{2}u(\mathbf{l})\left(l_j\frac{\partial}{\partial l^0}-l_0\frac{\partial}{\partial l^j}\right)[\ln Z^h(l)]=-e^2\int_{\Omega^h}\frac{dk}{(2\pi)^4}\frac{k_0\delta_j{}^\mu+k_j\delta_0{}^\mu}{\mathbf{k}^2}G^h(l-k)D^h{}_{\mu\nu}(k)\times\Gamma^{h\nu}(l-k,\,l)u(\mathbf{l}).\quad(7.15)$$

In order to prove this, we clearly have to assume, as we have done before, that the theory without soft photons has the desired property. It is easy to see that this assumption does indeed lead to (7.15). For $Z^h(l)$ may be defined formally by the relation

$$\langle 0|\phi(0)|\mathbf{l}\rangle^h=[Z^h(l)]^{1/2}u(\mathbf{l}),\quad(7.16)$$

where the superscript h on the left-hand side indicates that this expression is to be evaluated with the neglect of all soft-photon contributions. But then, if we apply the same argument as before to the Lorentz transformation properties of (7.16), we get exactly (7.14), but without the soft-photon matrix element. This then leads to (7.15). Of course, it should also be possible to establish (7.15) directly from the definition of $Z^h(l)$ in terms of the residue of G^h at its mass-shell pole. However, we shall not attempt such a derivation here.

Thus we have established the Lorentz covariance of the relations (5.12).

8. MATRIX ELEMENTS BETWEEN ONE-PARTICLE STATES

Starting from (6.11), we may apply the one-particle reduction formula (6.15) again to obtain a matrix element between initial and final states each containing one particle.

The mass-shell singularity of (6.11) in the variable p_2 for $p_2{}^0<0$ has essentially the same form as before. As in the derivation of (5.9) and (6.2), we introduce new variables x_2 $(=y_2+2p_2\sigma_2)$ and l_2, eliminate σ_2, and separate the terms in $I_{2\ldots n}$ depending on x_2 and y_2. The only new feature is in the term in $X_{(1)}$, given by (6.7), involving y_1 and x_2, namely,

$$-i\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{e_1l_1{}^\mu}{l_1\cdot k+i\epsilon}\times\exp(ik\cdot y_1)\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{e_2l_2{}^\nu}{l_2\cdot k-i\epsilon}\exp(-ik\cdot x_2).\quad(8.1)$$

Now we are interested in the behavior of this function for large negative $x_2{}^0$. Hence we may complete the k^0 contour in the lower half-plane. Since $l_1{}^0>0$ and $l_2{}^0<0$, this contour excludes both the poles at $l_1\cdot k=0$ and $l_2\cdot k=0$. Thus (8.1) reduces simply to $(y_1)s_1{}^*(x_2)s_2$, and combines with the term $-(y_1)s_1{}^*g$ in (6.5) to yield

$$-(y_1)s_1{}^*[g-(x_2)s_2].\quad(8.2)$$

Thus the conclusion that x_2 and the soft-photon state $|g,\mu\rangle$ appear only in the combination $U[-(x_2)s_2]|g,\mu\rangle$ is unchanged.

The rest of the analysis goes through exactly as before, and we obtain

$$\int dx_3\cdots dx_n\exp(-i\sum_{j=3}^n p_j\cdot x_j)\langle\mathbf{l}_1;\,f,\lambda;\,\text{out}|T[\phi_3(x_3)\cdots\phi_n(x_n)]|-\mathbf{l}_2;\,g,\mu;\,\text{in}\rangle_J$$
$$=\bar u(\mathbf{l}_1)\bar u^c(-\mathbf{l}_2)\prod_{j=3}^n\{[Z_j{}^h(p_j)]^{1/2}\Lambda_j(p_j)\}\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s{}_{f\lambda,g\mu}(l_1l_2,p_3\cdots p_n;\,q_1\cdots q_n|J)\times(2\pi)^4\delta(q_1+\cdots+q_n)M^h(q_1\cdots q_n),\quad(8.3)$$

where

$$\Delta^s{}_{f\lambda,g\mu}(l_1l_2,p_3\cdots p_n;\,q_1\cdots q_n|J)$$

$$=\int dy_1dy_2\exp\Big(-i\sum_{j=1}^{2}(l_j-q_j)\cdot y_j\Big)\int dy_3\cdots dy_n\int_0^\infty d\sigma_3\cdots d\sigma_n\exp\Big(-i\sum_{j=3}^{n}(p_j-q_j)\cdot y_j-i\sum_{j=3}^{n}\sigma_j(m_j^2+p_j^2)\Big)$$

$$\times\langle f,\lambda|g,\mu\rangle\exp\bigg(i(f^*I_{(12)3\cdots n}+I_{(12)3\cdots n}{}^*g)+\tfrac{1}{2}i\int'_{\Omega^s}\frac{dk}{(2\pi)^4}I^\mu{}_{(12)3\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{(12)3\cdots n}(k)\bigg),\quad(8.4)$$

in the same notation as (6.8), with

$$I^\mu{}_{(12)3\cdots n}(k)=-i\sum_{j=1}^{2}\frac{e_jl_j{}^\mu}{l_j\cdot k-i\epsilon}\times\exp(-ik\cdot y_j)+I^\mu{}_{3\cdots n}(k).\quad(8.5)$$

Here, of course, the prime signifies that in both the terms 11 and 22 the contribution of the double pole is to be dropped, leaving only $-\frac{1}{2}s_1{}^*s_1-\frac{1}{2}s_2{}^*s_2$.

The inclusion of disconnected core diagrams again poses no difficulty, except in one special case, that of a straight-through line joining the two mass-shell lines 1 and 2. This case requires a rather different treatment.

It will be sufficient to illustrate the method by examining the simplest case $n=2$. Thus we start with (6.20) and examine the asymptotic behavior for large negative y^0, so that we may apply the appropriate limiting formula to get a particle in the initial state. The only term in the exponent of (6.20) that may give trouble is the one linear in J. For large negative y^0, we must complete the k^0 contour in the upper half k^0 plane. Since $l^0>0$, the pole at $l\cdot k=0$ is thus included within the contour. The reason why this case is special is now clear. Whenever we consider a new line, we may choose the sign of the imaginary part in the denominator $l\cdot k$ at will, and we choose it so that this pole is excluded from the contour. But in this case, we have already chosen the sign of the imaginary part, in such a way that, when we considered the limit of large positive x^0, it would not contribute. Because for y^0 we are interested in the opposite limit, we now get a contribution.

The pole at $k^0=-|\mathbf{k}|$ yields $iJ^*(y)s_l$, which combines correctly with iJ^*g to give an expression involving only the combination $U[(y)s_l]|g,\mu\rangle$. However, there remains the contribution from the pole at $l\cdot k=0$, namely,

$$i\int_{\Omega^s}\frac{dk}{(2\pi)^4}el^\mu 2\pi\delta(l\cdot k)e^{ik\cdot y}\frac{\gamma_{\mu\nu}(k)}{k^2}J^\nu(k).\quad(8.6)$$

Here we have omitted the imaginary part from the denominator k^2, because, when $l\cdot k=0$, k is necessarily spacelike. It follows from the reality of $J(x)$ that $J^*(k)=J(-k)$, and hence that (8.6) is purely imaginary.

Now, if we transcribe the limiting definition (6.15) of the one-particle states into a form appropriate to the present case, we obtain

$$N^{-1}\int\frac{d\mathbf{l}'}{(2\pi)^3 2l'^0}\langle\mathbf{l};\alpha;\text{out}|\mathbf{l}';\beta_{l'};\text{in}\rangle_J[\bar{u}(\mathbf{l}')\psi(\mathbf{l}')]$$

$$=\lim_{t\to-\infty}N^{-1}\int\frac{dl'}{(2\pi)^4}\sigma(l')\exp[i(l'^0-\omega_{l'})t][Z^h(l')]^{-1/2}\times\int dy\,e^{il'\cdot y}\langle\mathbf{l};\alpha;\text{out}|\bar{\phi}(y)|\beta_{l',y};\text{in}\rangle_J\psi(\mathbf{l}').\quad(8.7)$$

where

$$|\beta_{l',y}\rangle=U[-(y)s_{l'}]|\beta_{l'}\rangle.\quad(8.8)$$

As in our discussion of (6.16), we may argue that $l-l'$ is necessarily small. Thus, when we substitute the asymptotic form of (6.20) into (8.7), the soft-photon unitary operators $U[(y)s_l]$ and $U[-(y)s_{l'}]$ cancel. Moreover, the renormalization functions $[Z^h(l)]^{1/2}$ and $[Z^h(l')]^{-1/2}$ also cancel. However, it is not true any longer that the y integral then yields a δ function, because of the extra term (8.5). The remaining factors of (6.20), independent of y, may be recognized as being just those in the expression (4.13) for $\langle f,\lambda;\text{out}|g,\mu;\text{in}\rangle_J$. Hence the right-hand side of (8.7) reduces to

$$\lim_{t\to-\infty}N^{-1}\int\frac{dl'}{(2\pi)^4}\sigma(l')\exp[i(l'^0-\omega_{l'})t]\times[\bar{u}(\mathbf{l})\psi(\mathbf{l}')]\langle\alpha;\text{out}|\beta_{l'};\text{in}\rangle_J\int dy\,e^{-i(l-l')\cdot y}\times\exp\bigg(i\int_{\Omega^s}\frac{dk}{(2\pi)^4}el^\mu 2\pi\delta(l\cdot k)e^{ik\cdot y}\frac{\gamma_{\mu\nu}(k)}{k^2}J^\nu(k)\bigg).\quad(8.9)$$

Now, because of the factor $\delta(l\cdot k)$ in its integrand, the exponent is independent of the component of y in the direction of l, so that the integral over this component yields a δ function $\delta[l\cdot(l-l')]$. But in view of the fact $l-l'$ is small in the region of interest, this δ function is equivalent to a mass-shell δ function $\delta(m^2+l'^2)$. We may define a function $\Delta^s(l;l'|J)$ with both variables l and l' on their mass shells by

$$\Delta^s(l;l'|J)2\pi\delta(m^2+l'^2)=\int dy\,e^{-i(l-l')\cdot y}\exp\bigg(i\int_{\Omega^s}\frac{dk}{(2\pi)^4}el^\mu 2\pi\delta(l\cdot k)\times e^{ik\cdot y}\frac{\gamma_{\mu\nu}(k)}{k^2}J^\nu(k)\bigg).\quad(8.10)$$

This is precisely what is required for the limit $t\to-\infty$ in (8.9) to yield a finite result. Isolating the coefficient of $\psi(\mathbf{l}')$, we may write the result in the form

$$\langle \mathbf{l};\alpha;\text{out}|\mathbf{l}';\beta;\text{in}\rangle_J =\langle\alpha;\text{out}|\beta;\text{in}\rangle_J N^{-1}[\bar u(\mathbf{l})u(\mathbf{l}')]\Delta^s(l;l'|J). \quad (8.11)$$

This is a very interesting result, because it shows that, in a state containing a single particle and soft photons, in the presence of a soft external current J the particle and soft photons scatter independently of one another. We note that, since $l-l'$ is small, the factor $N^{-1}[\bar u(\mathbf{l})u(\mathbf{l}')]$ is just a spin δ symbol. Clearly, the factor $\Delta^s(l;l'|J)$ describes the small-angle scattering of the particle produced by the external current.

It will be convenient here to note certain symmetry properties of $\Delta^s(l;l'|J)$. First, we remark that, since $l\cdot k=0$ in the region of integration, and since J^μ is conserved (i.e., $k_\mu J^\mu=0$), the factor $\gamma_{\mu\nu}(k)$ may be dropped. Thus this function is actually gauge-invariant. We also note that, because the exponent is purely imaginary, it satisfies the relation

$$\Delta^s(l;l'|J)^*=\Delta^s(-l;-l'|J).$$

Also, it is unaffected by changing the signs of l and J, with no change in $l-l'$, so that, using the freedom to replace l with l' in the exponent, we obtain

$$\Delta^s(l;l'|-J)=\Delta^s(-l';-l|J) =\Delta^s(l';l|J)^*. \quad (8.12)$$

Finally, unitarity is assured by the relation

$$\int\frac{dl}{(2\pi)^4}\Delta^s(l;l''|J)^*2\pi\delta(m^2+l^2)\Delta^s(l;l'|J) \times2\pi\delta(m^2+l'^2)=(2\pi)^4\delta(l''-l')2\pi\delta(m^2+l'^2),$$

which may also be written

$$\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}\Delta^s(l;l''|J)^*\Delta^s(l;l'|J) =(2\pi)^3 2l'^0\delta(\mathbf{l}''-\mathbf{l}'). \quad (8.13)$$

It is also interesting to note that, in the rest frame of l', (8.10) reduces to

$$\Delta^s(l;l'|J)=2m\int d\mathbf{y}\, e^{-i\mathbf{l}\cdot\mathbf{y}} \times\exp\left(-ie\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3}\frac{J^0(0,\mathbf{k})}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{y}}\right), \quad (8.14)$$

which may be recognized as being essentially the non-relativistic Coulomb scattering amplitude. We note that, if the integral in the exponent diverges, it does so only at $\mathbf{k}=\mathbf{0}$, and the divergence may be isolated in a single constant factor $e^{i\sigma}$ that has the effect of changing the phase label of the coherent state by an amount σ.

Exactly the same analysis may be applied to the general case in which a straight-through line connects the two particles on their mass shells, with $I_{2\cdots n}$ now playing the role of J. We therefore merely quote the result, which is

$$\int dx_2\cdots dx_n\exp(-i\sum_{j=2}^n p_j\cdot x_j)\langle\mathbf{l}_1;f,\lambda;\text{out}|T[\phi_2(x_2)\cdots\phi_n(x_n)]|\mathbf{l}_1';g,\mu;\text{in}\rangle_J$$
$$=N_1^{-1}[\bar u(\mathbf{l}_1)u(\mathbf{l}_1')]\prod_{j=2}^n\{[Z^h{}_j(p_j)]^{1/2}\Delta_j(p_j)\}\int\frac{dq_2}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s{}_{f\lambda,g\mu}(l_1,p_2\cdots p_n;l_1',q_2\cdots q_n|J)$$
$$\times(2\pi)^4\delta(q_2+\cdots+q_n)M^h(q_2\cdots q_n), \quad (8.15)$$

where now

$$\Delta^s{}_{f\lambda,g\mu}(l_1,p_2\cdots p_n;l_1',q_2\cdots q_n|J)2\pi\delta(m_1^2+l_1'^2)$$
$$=\int dy_1\exp[-i(l_1-l_1')\cdot y_1]\int dy_2\cdots dy_n\int_0^\infty d\sigma_2\cdots d\sigma_n\exp(-i\sum_{j=2}^n(p_j-q_j)\cdot y_j-i\sum_{j=2}^n\sigma_j(m_j^2+p_j^2))\langle f,\lambda|g,\mu\rangle$$
$$\times\exp\Bigg(i(f^*I_{2\cdots n}+I_{2\cdots n}{}^*g)+\tfrac12 i\int_{\Omega^s}\frac{dk}{(2\pi)^4}I^\mu{}_{2\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{2\cdots n}(k)$$
$$+i\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_1l_1^\mu 2\pi\delta(l_1\cdot k)\exp(ik\cdot y_1)\frac{\gamma_{\mu\nu}(k)}{k^2}I^\nu{}_{2\cdots n}(k)\Bigg). \quad (8.16)$$

It is worth noting that the structures of (8.4) and (8.16) are not as different as might appear at first sight. Since l_1' corresponds to an incoming rather than an outgoing particle, it should contribute a term to the current I^μ in which the signs of e and l are changed. Thus, by analogy with (8.5), it is natural to define

$$I^\mu{}_{(11')2\cdots n}(k)=-i\frac{e_1l_1^\mu}{l_1\cdot k-i\epsilon}\exp(-ik\cdot y_1)+i\frac{e_1l_1^\mu}{l_1\cdot k+i\epsilon} \times\exp(-ik\cdot y_1)+I^\mu{}_{2\cdots n}(k)$$
$$=e_1l_1^\mu 2\pi\delta(l_1\cdot k) \times\exp(-ik\cdot y_1)+I^\mu{}_{2\cdots n}(k). \quad (8.17)$$

Now, since the first term here vanishes for $k^2=0$, because then $l_1\cdot k$ is never zero, it does not contribute to the terms involving the soft-photon states. Thus we may write the exponent in (8.16) as

$$i[f^*I_{(11')2\cdots n}+I_{(11')2\cdots n}{}^*g]$$
$$+\tfrac{1}{2}i\int_{\Omega^s}{}' \frac{dk}{(2\pi)^4}I^\mu{}_{(11')2\cdots n}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{(11')2\cdots n}(k). \quad (8.18)$$

At first sight, it might seem that we would have to introduce a new interpretation of the prime on this integral, since the exponent in (8.16) contains no quadratic term in the part of the current associated with particle 1. Actually, however, this is just what our rule prescribes. The quadratic terms arising separately from the 11 and 1′1′ terms are each taken to be $-\frac{1}{2}s_1{}^*s_1$. But the cross term 11′ is

$$-i\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{e_1l_1{}^\mu}{l_1\cdot k+i\epsilon}\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}\frac{e_1l_1{}^\nu}{l_1\cdot k+i\epsilon}.$$

Here we may close the contour in the lower half-plane and exclude the double pole. Thus we obtain only the contribution of the pole at $k^0=|\mathbf{k}|$, which is $s_1{}^*s_1$. The sum of all three terms is therefore zero.

9. TWO-PARTICLE STATES

We now proceed to examine the nature of the asymptotic states containing two particles and some soft phtotons. As in the case of one-particle states, our aim is to give a definition of these states as weak limits. To do this we begin by examining the singularity of the function (6.11) at $p_2{}^2=-m_2{}^2$, as we did to obtain matrix elements between one-particle states, but now for the case $p_2{}^0>0$ rather than for $p_2{}^0<0$.

The corresponding discontinuity function should be related to the intermediate states that can contribute to the matrix element with $\phi_2(x_2)$ taken outside the time-ordering symbol, to the left. Near the mass shell we may expect that the states that can contribute will be out states containing two particles, with momenta close to, but not necessarily equal to, l_1 and p_2, respectively, and some soft photons. Thus the structure that we expect to obtain for the discontinuity function is

$$\int dx_2\cdots dx_n \exp(-i\sum_{j=2}^n p_j\cdot x_j)\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\frac{d\mathbf{l}_2}{(2\pi)^3 2l_2{}^0}$$
$$\times\sum_\beta\langle \mathbf{l}_1;f,\lambda;\text{out}|\phi_2(x_2)|\mathbf{l}_1'\mathbf{l}_2;\beta;\text{out}\rangle$$
$$\times\langle\mathbf{l}_1'\mathbf{l}_2;\beta;\text{out}|T[\phi_3(x_3)\cdots\phi_n(x_n)]|g,\mu;\text{in}\rangle_J. \quad (9.1)$$

Hence we seek to isolate the dependence on the variables $\mathbf{l}_1$ and x_2 that appear only in the first matrix element, and on the soft-photon state $\langle f,\lambda|$.

As in the preceding discussion of the case $p_2{}^0<0$, it is convenient to begin by applying the same transformation as before to the variables associated with p_2. We introduce new variables x_2 and l_2, and eliminate σ_2, as we did in obtaining (8.3).

The only difference from the previous analysis is in the treatment of the term (8.1). We are now interested in the behavior of this function for large positive values of $x_2{}^0$. So we must complete the k^0 contour in the upper half k^0 plane. As before, the pole at $l_2\cdot k=0$ is excluded, since now $l_2{}^0>0$. However, the pole at $l_1\cdot k=0$ now lies inside the contour. Thus we obtain two terms. The first from the pole at $k^0=-|\mathbf{k}|$ is $(x_2)s_2{}^*(y_1)s_1$. It combines with the term $f^*(y_1)s_1$ in the exponent of (6.12) to give

$$[f+(x_2)s_2]^*(y_1)s_1.$$

Once again, therefore, we find that $\langle f,\lambda|$ appears in the combination $\langle f,\lambda|U[-(x_2)s_2]$. However, this is no longer the only dependence on x_2, for we have also the second term, from the pole at $l_1\cdot k=0$, which is

$$-e_1l_1{}^\mu\int_{\Omega^s}\frac{dk}{(2\pi)^4}2\pi\delta(l_1\cdot k)\frac{\gamma_{\mu\nu}(k)}{k^2}\frac{e_2l_2{}^\nu}{l_2\cdot k-i\epsilon}$$
$$\times\exp[ik\cdot(y_1-x_2)]. \quad (9.2)$$

This term may be recognized as one contribution to the term $X_{21}{}^{(2)}(\sigma_2,\infty)$ of [II, (4.8)]. (The other contribution is the corresponding term involving y_2 rather than x_2.) It is easy to see, by making the transformation $k\to -k$, that (9.2) is purely imaginary. Indeed, it is just the function (8.6) with J^μ replaced with the nonconserved current $-I^\mu{}_{(2)}$, where

$$I^\mu{}_{(2)}(k)=-i[e_2l_2{}^\mu/(l_2\cdot k-i\epsilon)]\exp(-ik\cdot x_2). \quad (9.3)$$

Thus the exponential of (9.2) may be expressed as the Fourier transform with respect to l_1-l_1' of $\Delta^s(l_1;l_1'|-I_{(2)})2\pi\delta(m_1{}^2+l_1'^2)$, that is, as

$$\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\exp[i(l_1-l_1')\cdot y_1]\Delta^s(l_1';l_1|I_{(2)})^*, \quad (9.4)$$

in which we have used the symmetry relation (8.12).

We now take the function defined by the right-hand side of (6.11), perform the indicated transformations, and go over to the discontinuity function by keeping only the δ-function part of the denominator $(m_2{}^2+l_2{}^2-i\epsilon)^{-1}$. The structure that we obtain in this way is similar to (6.9), except for the appearance of the extra factor (9.4). Explicitly, it is

$$\int dx_2\exp(-ip_2\cdot x_2)\int\frac{d\mathbf{l}_2}{(2\pi)^3 2l_2{}^0}[Z^h{}_2(l_2)]^{1/2}\Lambda_2(l_2)$$
$$\times\exp(il_2\cdot x_2)\int dy_1\exp(-il_1\cdot y_1)\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}$$
$$\times\exp[i(l_1-l_1')\cdot y_1]\Delta^s(l_1';l_1|I_{(2)})^*$$
$$\times\langle f,\lambda|U[-(x_2)s_2]\cdots, \quad (9.5)$$

in which the remaining factors indicated by dots are independent of the variables x_2 and l_1. Hence, comparing with (9.1), we see that the natural generalization of

(5.12) is

$$\langle \mathbf{l}_1;\alpha;\text{out}|\phi_2(x_2)|\mathbf{l}_1',\mathbf{l}_2;\beta;\text{out}\rangle$$
$$=[Z^h{}_2(l_2)]^{1/2}N_1^{-1}[\bar{u}(\mathbf{l}_1)u(\mathbf{l}_1')]u(\mathbf{l}_2)$$
$$\times\exp(il_2\cdot x_2)\langle\alpha|U[-(x_2)s_2]|\beta\rangle$$
$$\times\Delta^s(l_1';l_1|I_{(2)})^*. \quad (9.6)$$

Clearly, the factor $\Delta^s(l_1';l_1|I_{(2)})^*$ represents the effect on particle 1 of the current (9.3), just as the factor $U[-(x_2)s_2]$ represents its effect on the soft-photon state.

Identifying the remaining factors of (9.5) with the second matrix element in (9.1), we obtain

$$\int dx_3\cdots dx_n \exp(-i\sum_{j=3}^{n}p_j\cdot x_j)\langle\mathbf{l}_1\mathbf{l}_2;f,\lambda;\text{out}|T[\phi_3(x_3)\cdots\phi_n(x_n)]|g,\mu;\text{in}\rangle_J$$
$$=\bar{u}(\mathbf{l}_1)\bar{u}(\mathbf{l}_2)\prod_{j=3}^{n}\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\}\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s{}_{f\lambda,g\mu}(l_1l_2,p_3\cdots p_n;q_1\cdots q_n|J)$$
$$\times(2\pi)^4\delta(q_1+\cdots+q_n)M^h(q_1\cdots q_n), \quad (9.7)$$

where once again the function Δ^s is given precisely by (8.4) together with (8.5), but is now to be evaluated for $l_1^0>0$ and $l_2^0>0$. The fact that (9.7) is obtained from (8.3) simply by applying the crossing transformation to l_2 suggests that the arbitrary phase factor that could be present in (9.6) has been correctly chosen. Indeed, since we have proved the Lorentz covariance of our definition of one-particle states, (8.3) must be Lorentz-covariant, and thus the covariance of (9.7) follows.

We can now proceed to construct a proper limiting definition of the states containing two particles, analogous to (6.15) for the one-particle states. There we made a modification of the asymptotic soft-photon state depending on l and x in such a way as to yield a function that has a pole on the mass shell. In the present case, we adopt a similar procedure, but now it is necessary to modify also the state of the single particle already present. We adopt the definition

$$N_2^{-1}\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}[\bar{\psi}(\mathbf{l})u(\mathbf{l})]\langle\mathbf{l}_1\mathbf{l};\alpha_l;\text{out}|T[\phi_3(x_3)\cdots\phi_n(x_n)]|\beta;\text{in}\rangle_J$$
$$=\lim_{t\to\infty}N_2^{-1}\int\frac{dl}{(2\pi)^4}\bar{\psi}(\mathbf{l})\sigma(l)\exp[-i(l^0-\omega_1)t][Z^h{}_2(l)]^{-1/2}\int dx\,e^{-il\cdot x}\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\Delta^s(l_1;l_1'|I_{(2)})$$
$$\times\langle\mathbf{l}_1';\alpha_{l,x};\text{out}|T[\phi_2(x)\phi_3(x_3)\cdots\phi_n(x_n)]|\beta;\text{in}\rangle_J, \quad (9.8)$$

where $\langle\alpha_{l,x}|$ is again defined by (6.14).

The proof that this limit exists goes through very much as before. Substituting the expression that we have already obtained for the matrix element on the right, we obtain a structure similar to (6.16) with an integral over l_2 and a denominator function $(m_2^2+l_2^2-i\epsilon)^{-1}$. As before, $l-l_2$ is necessarily small, and therefore the soft-photon unitary operators $U[(x)s_l]$ and $U[-(x)s_2]$ cancel, as do the renormalization functions $[Z^h{}_2(l)]^{-1/2}$ and $[Z^h{}_2(l_2)]^{1/2}$. The part of this expression containing the function Δ^s is

$$\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\Delta^s(l_1;l_1'|I_{(2)})\int\frac{d\mathbf{l}_1''}{(2\pi)^3 2l_1''^0}$$
$$\times\Delta^s(l_1'';l_1'|I_{(2)})^*\cdots,$$

and in view of (8.13) this simply has the effect of setting $l_1''=l_1$. Thus the limiting formula (9.8) reproduces (9.7).

It is easy to construct a similar definition for two-particle in states. The formulas are obtained simply by crossing, so that we shall not write them down explicitly. We note one special case, however. The matrix element of $\bar{\phi}(x_2)$ between a two-particle in state and a one-particle in state is obtained from (9.6) by the substitutions $l_2\to -l_2$ and $e_2\to -e_2$, which by (9.3) implies

$$I_{(2)}{}^\mu(k)\to I_{(2')}{}^\mu(k)=i[e_2l_2^\mu/(l_2\cdot k+i\epsilon)]\exp(-ik\cdot x_2).$$

Thus we find

$$\langle\mathbf{l}_1\mathbf{l}_2;\alpha;\text{in}|\bar{\phi}_2(x_2)|\mathbf{l}_1';\beta;\text{in}\rangle$$
$$=[Z^h{}_2(l_2)]^{1/2}N_1^{-1}[\bar{u}(\mathbf{l}_1)u(\mathbf{l}_1')]\bar{u}(\mathbf{l}_2)$$
$$\times\exp(-il_2\cdot x_2)\langle\alpha|U[(x_2)s_2]|\beta\rangle$$
$$\times\Delta^s(l_1';l_1|I_{(2')})^*. \quad (9.9)$$

Pairs of Straight-Through Lines

As before, we may proceed to calculate the matrix elements between two-particle states. These are given by obvious generalization of the corresponding formulas for one-particle matrix elements, and we shall not write down a general formula for them at this stage. There is, however, one case in which we do encounter something new, namely, that in which there is a pair of straight-through lines connecting the particles on their mass shells.

Suppose, for example, that we wish to calculate the scattering matrix element

$$\langle\mathbf{l}_1\mathbf{l}_2;f,\lambda;\text{out}|\mathbf{l}_1'\mathbf{l}_2';g,\mu;\text{in}\rangle_J \quad (9.10)$$

in a region of momenta in which only core diagrams with two straight-through lines contribute.

We may begin with the function

$$\int dy_1 dy_2 \exp(iq_1\cdot y_1+iq_2\cdot y_2) \times\langle \mathbf{l}_1\mathbf{l}_2;\, f,\lambda;\, \mathrm{out}|T[\bar\phi_1(y_1)\bar\phi_2(y_2)]|g,\mu;\, \mathrm{in}\rangle_J ,$$

and apply the reduction formula successively to the two in-state particles.

This function has a structure similar to (6.19), namely,

$$\bar u(\mathbf{l}_1)\bar u(\mathbf{l}_2)[Z^h{}_1(q_1)Z^h{}_2(q_2)]^{1/2}\Delta^s{}_{f\lambda,g\mu}(l_1l_2;\, q_1q_2|J) ,$$

where Δ^s is given by (8.4). Now, when we apply the reduction formula in the form (8.7) to q_1, we obtain an expression similar to (8.11), namely,

$$\int dy_2 \exp(iq_2\cdot y_2)\langle \mathbf{l}_1\mathbf{l}_2;\, f,\lambda;\, \mathrm{out}|\bar\phi_2(y_2)|\mathbf{l}_1';\, g,\mu;\, \mathrm{in}\rangle_J = N_1^{-1}[\bar u(\mathbf{l}_1)u(\mathbf{l}_1')]\bar u(\mathbf{l}_2)[Z^h{}_2(q_2)]^{1/2}\Delta^s{}_{f\lambda,g\mu}(l_1l_2;\, l_1',q_2|J) , \quad (9.11)$$

where

$$\Delta^s{}_{f\lambda,g\mu}(l_1l_2;\, l_1',q_2|J)2\pi\delta(m_1^2+l_1'^2) = \int dy_1dy_2 \exp[-i(l_1-l_1')\cdot y_1-i(l_2-q_2)\cdot y_2]\langle f,\lambda|g,\mu\rangle \exp\Big(i(f^*I_{(2)}+I_{(2)}{}^*g)+\tfrac12 i\int'_{\Omega^s}\frac{dk}{(2\pi)^4}I_{(2)}{}^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu{}_{(2)}(k) + i\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_1l_1{}^\mu 2\pi\delta(l_1\cdot k)\exp(ik\cdot y_1)\frac{\gamma_{\mu\nu}(k)}{k^2}I^\nu{}_{(2)}(k)\Big) , \quad (9.12)$$

with

$$I^\mu{}_{(2)}(k)=J^\mu(k)-i[e_2l_2{}^\mu/(l_2\cdot k-i\epsilon)]\exp(-ik\cdot y_2) . \quad (9.13)$$

This form may be compared with (8.4) and (8.16). We recall that the last term in the exponent of (9.12), which is essentially the term (8.6), came from the pole at $l_1\cdot k=0$.

Next, we must again apply the reduction formula to the variable q_2. To do this, we must examine the asymptotic behavior of the integrand of (9.12) for large negative $y_2{}^0$. As before, we shall find a term analogous to (8.6) from the pole at $l_2\cdot k=0$ in the cross term between the two terms of (9.13), The new feature that arises in the present case, however, is the term involving y_2 in the last term of the exponent of (9.12), namely,

$$\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_1l_1{}^\mu 2\pi\delta(l_1\cdot k)\frac{\gamma_{\mu\nu}(k)}{k^2}\frac{e_2l_2{}^\nu}{l_2\cdot k-i\epsilon}\exp[ik\cdot(y_1-y_2)] . \quad (9.14)$$

Once again the small imaginary part in the denominator here has the "wrong" sign. We cannot, of course, any longer apply an argument based on the completion of the k^0 contour, because of the factor $\delta(l_1\cdot k)$. However, it is easy to see that what we are interested in is the behavior of the integrand as the component of y_2 in the direction of l_2 becomes infinite. Now, if we were to take the limit $-l_2\cdot y_2\rightarrow+\infty$, we should find that (9.14) would not contribute. However, the limit with which we are concerned is $-l_2\cdot y_2\rightarrow-\infty$, and in this limit we do obtain a contribution, which may be regarded as coming from the pole at $l_2\cdot k=0$, namely,

$$i\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_1l_1{}^\mu 2\pi\delta(l_1\cdot k)\frac{\gamma_{\mu\nu}(k)}{k^2}e_2l_2{}^\nu 2\pi\delta(l_2\cdot k)\exp[ik\cdot(y_1-y_2)] . \quad (9.15)$$

The result is to give a cross term between particles 1 and 2 in the expression for the matrix element (9.10). We obtain, finally,

$$\langle \mathbf{l}_1\mathbf{l}_2;\, f,\lambda;\, \mathrm{out}|\mathbf{l}_1'\mathbf{l}_2';\, g,\mu;\, \mathrm{in}\rangle_J = N_1^{-1}[\bar u(\mathbf{l}_1)u(\mathbf{l}_1')]N_2^{-1}[\bar u(\mathbf{l}_2)u(\mathbf{l}_2')]\Delta^s{}_{f\lambda,g\mu}(l_1l_2;\, l_1'l_2'|J) , \quad (9.16)$$

where

$$\Delta^s{}_{f\lambda,g\mu}(l_1l_2;\, l_1'l_2'|J)2\pi\delta(m_1^2+l_1'^2)2\pi\delta(m_2^2+l_2'^2) = \int dy_1dy_2 \exp\Big(-i\sum_{j=1}^{2}(l_j-l_j')\cdot y_j\Big)\langle f,\lambda|g,\mu\rangle \exp\Big(i(f^*J+J^*g)+\tfrac12 i\int_{\Omega^s}\frac{dk}{(2\pi)^4}J^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}J^\nu(k) + i\sum_{j=1}^{2}\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_jl_j{}^\mu 2\pi\delta(k\cdot l_j)\frac{\gamma_{\mu\nu}(k)}{k^2}J^\nu(k)\exp(ik\cdot y_j) + i\int_{\Omega^s}\frac{dk}{(2\pi)^4}e_1l_1{}^\mu 2\pi\delta(l_1\cdot k)\frac{\gamma_{\mu\nu}(k)}{k^2}e_2l_2{}^\nu 2\pi\delta(l_2\cdot k)\exp[ik\cdot(y_1-y_2)]\Big) . \quad (9.17)$$

As in the single-particle case, the charged particles and soft photons scatter independently.

Note that the exponent may again be written in a form similar to (8.18) with the current

$$I^{\mu}{}_{(11'22')}(k)=J^{\mu}(k)+\sum_{j=1}^{2} e_j l_j{}^{\mu} 2\pi\delta(l_j\cdot k)\exp(-ik\cdot y_j). \tag{9.18}$$

10. MULTIPARTICLE STATES; CONCLUSION

The generalization of the preceding discussion to multiparticle states is now almost obvious, and we shall merely write down the results.

The general formula corresponding to (5.12) and (9.6) is

$$\langle \mathbf{l}_1\cdots\mathbf{l}_n;\alpha;\text{out}|\phi_{n+1}(x)|\mathbf{l}_1'\cdots\mathbf{l}_n'\mathbf{l};\beta;\text{out}\rangle$$
$$=[Z^h{}_{n+1}(l)]^{1/2}u(\mathbf{l})\langle\alpha|U[-(x)s_l]|\beta\rangle e^{il\cdot x}\prod_{j=1}^{n}\{N_j^{-1}[\bar{u}(\mathbf{l}_j)u(\mathbf{l}_j')]\Delta^s(l_j';l_j|I_{(n+1)})^*\}. \tag{10.1}$$

The general reduction formula, corresponding to (6.15) and (9.8), that defines the multiparticle states recursively, is

$$N_{m+1}{}^{-1}\int\frac{d\mathbf{l}}{(2\pi)^3 2l^0}[\bar{\psi}(\mathbf{l})u(\mathbf{l})]\langle\mathbf{l}_1\cdots\mathbf{l}_m\mathbf{l};\alpha_l;\text{out}|T[\phi_{m+2}(x_{m+2})\cdots\phi_n(x_n)]|\beta;\text{in}\rangle_J$$
$$=\lim_{t\to\infty}N_{m+1}{}^{-1}\int\frac{dl}{(2\pi)^4}\bar{\psi}(\mathbf{l})\sigma(l)\exp[-i(l^0-\omega_1)t][Z^h{}_{m+1}(l)]^{-1/2}\int dx\, e^{-il\cdot x}\int\frac{d\mathbf{l}_1'}{(2\pi)^3 2l_1'^0}\cdots\frac{d\mathbf{l}_m'}{(2\pi)^3 2l_m'^0}$$
$$\times\prod_{j=1}^{m}\Delta^s(l_j;l_j'|I_{(m+1)})\langle\mathbf{l}_1'\cdots\mathbf{l}_m';\alpha_{l,x};\text{out}|T[\phi_{m+1}(x)\phi_{m+2}(x_{m+2})\cdots\phi_n(x_n)]|\beta;\text{in}\rangle_J. \tag{10.2}$$

Using these formulas, we may derive an expression for the matrix element of any time-ordered product between arbitrary multiparticle states. It is not hard to see what kind of structure is obtained thereby, but instead of writing down a general formula, we shall find it more useful to give the rules for constructing it.

We start with the expression for the corresponding Green's function, which, in general, is a sum of terms of the form (1.2), each corresponding to core diagrams of one particular connectivity structure. Then for each line on its mass shell we replace p_j with l_j, drop the factor $[Z^h{}_j(p_j)]^{1/2}$, and replace $\Lambda_j(p_j)$ with $\bar{u}(\mathbf{l}_j)$ or $\bar{u}^c(-\mathbf{l}_j)$. [In the case of a straight-through line with both ends on the mass shell, this introduces a factor N_j^{-1}, since there is in (1.2) only one factor $\Lambda_j(p_j)$ for each straight-through line, and not two.]

The most general function Δ^s that we encounter in doing this is one with n lines of which r have both ends on the mass shell and $m-r$ have one end on the mass shell. It is given by

$$\Delta^s{}_{f\lambda,g\mu}(l_1\cdots l_m,p_{m+1}\cdots p_n;l_1'\cdots l_r',q_{r+1}\cdots q_n)\prod_{j=1}^{r}2\pi\delta(m_j^2+l_j'^2)$$
$$=\int dy_1\cdots dy_n\int_0^\infty d\sigma_{m+1}\cdots d\sigma_n\exp\Bigl(-i\sum_{j=1}^{r}(l_j-l_j')\cdot y_j-i\sum_{j=r+1}^{m}(l_j-q_j)\cdot y_j$$
$$-i\sum_{j=m+1}^{n}(p_j-q_j)\cdot y_j-i\sum_{j=m+1}^{n}\sigma_j(m_j^2+p_j^2)\Bigr)\langle f,\lambda|g,\mu\rangle\exp\left(i(f^*I+I^*g)+\tfrac{1}{2}i\int_{\Omega^s}'\frac{dk}{(2\pi)^4}I^{\mu}(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^{\nu}(k)\right), \tag{10.3}$$

where $I^{\mu}(k)$ stands for the function that we have denoted by $I^{\mu}{}_{(11'\cdots rr'r+1\cdots m)m+1\cdots n}(k)$, namely,

$$I^{\mu}(k)=J^{\mu}(k)+\sum_{j=1}^{r}e_j l_j{}^{\mu}2\pi\delta(l_j\cdot k)\exp(-ik\cdot y_j)$$
$$-i\sum_{j=r+1}^{m}\frac{e_j l_j{}^{\mu}}{l_j\cdot k-i\epsilon}\exp(-ik\cdot y_j)-i\sum_{j=m+1}^{n}\frac{e_j p_j{}^{\mu}}{p_j\cdot k}[\exp(-2ik\cdot p_j\sigma_j)-1]\exp(-ik\cdot y_j). \tag{10.4}$$

As usual, the prime on the k integral signifies the special treatment to be accorded the diagonal terms $i=j$.

We have now defined the most general multiparticle state, by (10.2), and obtained a set of rules that allow us to write down the matrix element of a time-ordered product between any two such states. To do this, we have had to develop reduction formulas that can be used to extract such matrix elements from the Green's functions with which we began. In the case of a conventional field theory without massless particles, the reduction formulas essentially isolate the residues at mass-shell poles. In the present case, the Green's functions themselves have branch points rather than poles, but we have been able to modify the soft-photon states, for example, as in (6.15), in such a way as to obtain functions that do have poles and thus yield analogous formulas.

The scattering matrix elements of the theory are special cases of the matrix elements that we have discussed. Thus our formulas yield, in particular, expressions for these matrix elements, which will be investigated in more detail in the following paper. That the asymptotic states that we have defined form a complete set must be proved by establishing the unitarity of the scattering operator so defined. We must also prove that it possesses the other properties demanded of it, such as gauge invariance, Lorentz invariance, and crossing symmetry. All this will be demonstrated in the next paper of this series.

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Pion Form-Factor Zero and "Dipole" Nucleon Structure

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It is suggested that the dipole parametrization of the isovector-nucleon form factor is a consequence of the behavior of the $\pi\pi$ phase shift above the ρ meson. It is conjectured that the phase shift rises through π, and as a result the pion form factor exhibits a zero in the vicinity of 900 MeV. This yields the dipole parametrization without introducing additional parameters.

Isovector nucleon structure for squared momentum transfer $|t|$ up to 10 $(\mathrm{GeV}/c)^2$ can be parametrized[1] by a dipole expression of the form $(1-t/m^2)^{-2}$. The parameter m^2 has the value 0.71 $(\mathrm{GeV}/c)^2$ so that $m=840$ MeV. The purpose of this paper is to offer some physical explanation for the dipole nature of the fit.

We suggest that the origin of the effect can be attributed to the behavior of the $\pi\pi$ $T=J=1$ phase shift δ above the ρ-meson mass. We assume that δ, as shown in Fig. 1, remains substantially real and rises through π below the $N\bar{N}$ threshold. Given this behavior, general arguments would suggest the validity of the dipole formula without the need for extensive detailed calculation.

The form-factor problem is a coupled-channel problem; it is this fact which, when implemented by our assumption on δ, leads to the effect in question. We consider a coupled model of the form factors of the pion and of the nucleon, and write[2]

$$F=gD^{-1}. \tag{1}$$

F is a row matrix whose elements are the form factors F_π and F_N. D is the D matrix for the coupled scattering problem with channels $\pi\pi$ and $N\bar{N}$. To expedite the illustration we ignore the multiplicity of the $N\bar{N}$ channel. The elements of the row matrix g are entire functions which secure the correct normalization of F_π and F_N at $t=0$. It follows that

$$F_N/F_\pi=(g_2D_{11}-g_1D_{12})(g_1D_{22}-g_2D_{21})^{-1}; \tag{2}$$

both F_N and F_π have the denominator $\det D$. The same D occurs in the scattering matrix. In particular the elastic $\pi\pi$ amplitude is

$$M_{11}=(ND^{-1})_{11}=(N_{11}D_{22}-N_{12}D_{21})/\det D. \tag{3}$$

Below the $N\bar{N}$ threshold the numerator of M_{11} is real as is the factor $(g_1D_{22}-g_2D_{21})$, the numerator of F_π. We are assuming that δ behaves as in Fig. 1 and that this behavior can be obtained from this two-channel model. Given this, there exists a point t_0 between m_ρ^2 and $4M^2$ where $N_{11}D_{22}-N_{12}D_{21}=0$. It is expected that $g_1D_{22}-g_2D_{21}=0$, and thus that $F_\pi=0$, at a nearby point $\hat{t}_0$. Quite apart from our hypothesis about the behavior of δ, there is good reason to expect the numerator of F_π to have a real zero between the $\pi\pi$ and $N\bar{N}$

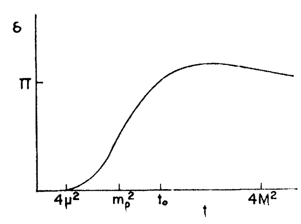

Fig. 1. The $T=J=1$ $\pi\pi$ phase shift versus $(\pi\pi\ \text{mass})^2$; μ is the pion mass, m_ρ is the ρ-meson mass, M is the nucleon mass.

[1] R. E. Taylor, Stanford Linear Accelerator Center Report No. SLAC-PUB-372, 1967 (unpublished).
[2] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).