in fair agreement. At the moment there appears to be no conclusive evidence on the relative sign, although there are some indications that it may be positive.[13]

In conclusion, we note that, as in our previous work,[1,2] no singularity such as described by Barton[3] appears in the mass splittings as the feedback is turned off (thus, as γ and $\eta\rightarrow 0$, the functions D and D' remain finite).

ACKNOWLEDGMENT

The author would like to thank Douglas Hofstadter for his assistance in computing the integrals.

PHYSICAL REVIEW VOLUME 174, NUMBER 5 25 OCTOBER 1968

Application of Modified Dispersion Relations to the Forward KN Crossing-Even Scattering Amplitude*

DAVID J. GEORGE AND ARNOLD TUBIS
Department of Physics, Purdue University, Lafayette, Indiana 47907
(Received 25 March 1968)

It has recently been shown that a knowledge of the zeros of a forward elastic scattering amplitude could be used to derive new modified dispersion relations. Using the phase representation, we show that the forward crossing-even KN amplitude probably has six zeros in the complex ω (kaon lab energy) plane. Two of these zeros can be very accurately determined from low-energy scattering data. The modified dispersion relations derived using the knowledge of these zeros yield information on the high-energy parameters, and in general provide a consistency test of the presently available data. The infinite-energy total cross section estimated from a dispersion sum rule is about 15.5 mb, in fair agreement with the experimental total cross section of about 17.3 mb at 20 BeV.

I. INTRODUCTION

IN a recent paper,[1] we derived a new class of modified dispersion relations which depended on a knowledge of the zeros of the forward elastic scattering amplitude. In particular, we derived an expression for the infinite-energy cross section for πN scattering. We have now applied this method to KN scattering in order to test the data of Kim[2] and to gain some information on the real part of the scattering amplitude at high energies.

We show in this paper that, according to the phase representation,[3] the forward crossing-even KN amplitude probably has six zeros in the complex ω (kaon laboratory energy) plane. There seem to be two possible arrangements for the zeros, with the presently available data being not precise enough to distinguish between the two possibilities.

In order to test the data on the real part of the scattering amplitude, we have calculated the infinite-energy cross section using the two accurately determined zeros and find fairly good agreement with experiment. We also have calculated the infinite-energy cross section using subtractions at the points on the imaginary axis where the scattering amplitude is a minimum. These points may or may not turn out to correspond to zeros when more accurate data become available.

* Work supported by the U. S. Atomic Energy Commission.
[1] D. J. George, B. Hale, and A. Tubis, Phys. Rev. **168**, 1924 (1968), hereafter referred to as I.
[2] J. K. Kim, Phys. Rev. Letters **19**, 1074 (1967); **19**, 1079 (1967).
[3] M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).

II. ZEROS OF $T(\omega)$

We normalize the KN forward scattering amplitude $T(\omega)$ by writing the optical theorem in the form

$$\mathrm{Im}T(\omega)=\tfrac{1}{2}(\omega^2-m_K{}^2)^{1/2}[\sigma_{K^+p}(\omega)+\sigma_{K^-p}(\omega)],\quad (1)$$

where ω is the kaon lab energy, m_K is the kaon mass, and $\sigma_{K^\pm p}(\omega)$ is the $K^\pm p$ total cross sections. Natural units ($\hbar=c=1$) are used throughout this work.

From the phase representation[3] we find that, for large ω,

$$T(\omega)\propto\omega^{N-M-2\delta(\infty)/\pi},\quad (2)$$

where N and M are, respectively, the number of zeros and poles of $T(\omega)$, and $\delta(\infty)$ is the phase of $T(\omega)$ at infinity. Now, from the available data, we find $\mathrm{Im}T(\omega)>0$ for ω on the real axis above the anomalous $(\Lambda\pi)$ threshold and $\mathrm{Re}T(\omega)>0$ at the $\Lambda\pi$ threshold. We must therefore have $0<\delta(\omega)<\pi$. If we assume that $T(\omega)$ becomes pure imaginary in the infinite-energy limit, we have $\delta(\infty)=\frac{1}{2}\pi$ and thus

$$T(\omega)\propto\omega^{N-M-1}.\quad (3)$$

Since $T(\omega)$ has a Pomeranchon-exchange contribution, it has the high-energy behavior

$$T(\omega)\propto\omega\,,\quad (4)$$

and so we deduce

$$N=M+2.\quad (5)$$

Finally, since $T(\omega)$ has two sets of poles (Λ and Σ) on the real axis, it has six zeros.

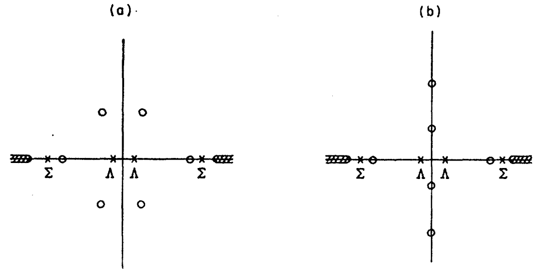

FIG. 1. Singularities and zeros of $T(\omega)$. Poles are marked with an ✕ and zeros with an ○. The two possible arrangements of zeros are indicated in (a) and (b).

It is easily seen that two of the zeros lie on the real axis between the poles. The other four may be the symmetric partners of a single complex zero or else they may form two pairs of zeros on the imaginary axis. The two possibilities are shown in Fig. 1.

We now turn to the explicit calculation of the zeros. The dispersion relation for $\mathrm{Re}T(\omega)$ is[4]

$$\mathrm{Re}T(\omega)=\mathrm{Re}T(M_K)+\sum_{Y=\Lambda,\Sigma}\frac{\omega_Y X_Y}{\omega_Y^2-\omega^2}\frac{\omega^2-M_K^2}{\omega_Y^2-M_K^2}+\frac{2}{\pi}(\omega^2-M_K^2)\int_{\omega_{\Lambda\pi}}^{\infty}\frac{\omega' d\omega'\,\mathrm{Im}T(\omega')}{(\omega'^2-\omega^2)(\omega'^2-M_K^2)},\quad(6)$$

where

$$\omega_Y=(M_Y^2-M_N^2-M_K^2)/2M_N,$$
$$X_Y=g_Y^2[(M_Y-M_N)^2-M_K^2]/4M_N^2.\quad(7)$$

In these expressions, M_N, M_Λ, and M_Σ are, respectively, the nucleon, Σ, and Λ masses and $\omega_{\Lambda\pi}$ is given by the expression for ω_Y with $M_Y=M_\Lambda+M_\pi$. Explicitly we find

$$\omega_\Lambda=0.126M_K,$$
$$\omega_\Sigma=0.313M_K,\quad(8)$$
$$\omega_{\Lambda\pi}=0.482M_K.$$

The expressions for the X_Y differ slightly from those of Kim.[2] The difference has been discussed by Chan and Meiere.[5] We have adjusted the values of the coupling constants so that our X_Y have the same values as 4π times those of Kim. Thus,

$$g_\Lambda^2/4\pi=13.5,$$
$$g_\Sigma^2/4\pi=0.2,\quad(9)$$
$$X_\Lambda=-10.23,$$
$$X_\Sigma=-0.153.$$

The $\mathrm{Re}T(M_K)$ is given in terms of the S-wave scattering lengths $A_I^\pm$ for $K^\pm p$ scattering in isospin channel I by

$$\mathrm{Re}T(M_K)=4\pi(1+M_K/M_N)\tfrac{1}{4}(A_0^-+A_1^-+2A_1^+).\quad(10)$$

[4] P. T. Matthews and A. Salam, Phys. Rev. **110**, 565 (1968); **110**, 569 (1968).
[5] C. H. Chan and F. T. Meiere, Phys. Rev. Letters (to be published).

Using the values[6,7]

$$A_1^+=-0.29\text{ F},$$
$$A_0^-=-1.674\text{ F},\quad(11)$$
$$A_1^-=-0.003\text{ F},$$

we find $\mathrm{Re}T(M_K)=-27.1M_K^{-1}$.

Below $\omega=1.4M_K$ we use Kim's data[2] for $\mathrm{Im}T(\omega)$ and above $\omega=1.4M_K$ we use experimental cross sections[8] in the integral. These values are displayed in Fig. 2. We then calculate $\mathrm{Re}T(\omega)$ for ω in the unphysical region ($|\omega|<M_K$ and ω on the imaginary axis). Above $37M_K$

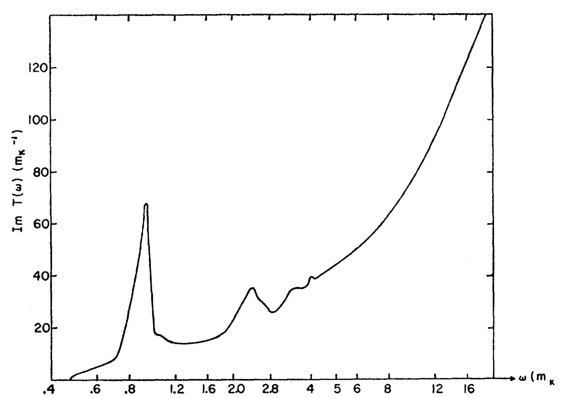

FIG. 2. The imaginary part of $T(\omega)$ as a function of the kaon lab energy ω. Note the logarithmic scale.

we assume $\sigma(\omega)$ to have a constant value of 19 mb. The contribution to the integral from this energy range is very small [about $-0.2M_K^{-1}$ compared with $\mathrm{Re}T(M_K)=-27.1M_K^{-1}$, for example].

The calculated values of $\mathrm{Re}T(\omega)$ may be compared with Kim's values for $\omega_{\Lambda\pi}<\omega<1.4M_K$ and are found to be in agreement except in the interval $\omega_{\Lambda\pi}<\omega<0.7M_K$. This disagreement is probably due to the influence of the Σ pole which lies close to $\omega_{\Lambda\pi}$. In Fig. 3 we plot $\mathrm{Re}T(\omega)$ for ω on the real axis as calculated from dispersion relations for $\omega<1.4M_K$, and from experiment[9] above $1.4M_K$. For comparison we have also plotted Kim's values for K^-p to which we have added K^+p real parts as calculated from an effective-range expansion

$$p\cot\delta=(A_1^+)^{-1}+\tfrac{1}{2}R_1^+p^2,\quad(12)$$

where p is the c.m. momentum and R_1^+ has the value 0.5 F.[6]

[6] S. Goldhaber, W. Chinowsky, G. Goldhaber, W. Lee, T. O'Halloran, T. F. Stubbo, G. M. Pjerrow, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **9**, 135 (1962).
[7] J. K. Kim, Phys. Rev. Letters **14**, 29 (1965).
[8] R. J. Abrams *et al.*, Phys. Rev. Letters **19**, 259 (1967); **19**, 678 (1967); J. D. Davies *et al.*, *ibid.* **18**, 62 (1967); R. L. Cool *et al.*, *ibid.* **16**, 1228 (1966); **17**, 102 (1966); W. Galbraith *et al.*, Phys. Rev. **138**, B913 (1965); A. Fridman and A. Michalon, Nuovo Cimento **48**, 344 (1967).
[9] A compilation of the results from several experiments can be found in N. Zovko, Z. Physik **196**, 16 (1966).

In Fig. 4 we have plotted the dispersion-relation prediction for ω on the imaginary axis. Between ω_Λ and ω_Σ there is the expected zero at about $\omega^2=0.095M_K^2$. In order to determine this exactly we set $\mathrm{Re}T(A)=0$ (A being the assumed zero position) and rewrite (6) in the form

$$\frac{\mathrm{Re}T(M_K)}{A^2-M_K^2}+\sum_{Y=\Lambda,\Sigma}\frac{Q_Y}{\omega_Y^2-A^2}+\frac{2}{\pi}\int_{\omega_{\Lambda\pi}}^{\infty}\frac{\omega' d\omega'\,\mathrm{Im}T(\omega')}{(\omega'^2-A^2)(\omega'^2-M_K^2)}=0, \quad (13)$$

where

$$Q_Y=\omega_Y X_Y/(\omega_Y^2-M_K^2).$$

Then, with the definition

$$J(A^2)=\frac{-\mathrm{Re}T(M_K)}{M_K^2-A^2}+\frac{2}{\pi}\int_{\omega_{\Lambda\pi}}^{\infty}\frac{\omega' d\omega'\,\mathrm{Im}T(\omega')}{(\omega'^2-A^2)(\omega'^2-M_K^2)}, \quad (14)$$

Eq. (13) becomes

$$\frac{Q_\Lambda}{\omega_\Lambda^2-A^2}+\frac{Q_\Sigma}{\omega_\Sigma^2-A^2}+J(A^2)=0 \quad (15)$$

and

$$A^2=[D-(D^2-JC)]^{1/2}/J, \quad (16)$$

where

$$\begin{aligned}2D&=Q_\Lambda+Q_\Sigma+J(\omega_\Lambda^2+\omega_\Sigma^2),\\ C&=Q_\Lambda\omega_\Sigma^2+Q_\Sigma\omega_\Lambda^2+J\omega_\Lambda^2\omega_\Sigma^2.\end{aligned} \quad (17)$$

As a first approximation we substitute $J(0.095M_K^2)$ and

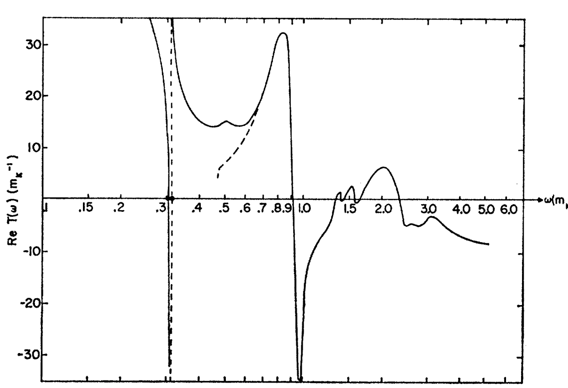

FIG. 3. The real part of $T(\omega)$ as a function of ω. Note the logarithmic scale. The broken line is Kim's prediction for the K^-p contribution plus the K^+p contribution calculated from the effective-range expansion (12).

the iterate until the input and output values of A^2 differ by less than 0.00001. The final value of A^2 is then $A^2=(0.09551\pm0.00001)M_K^2$. The accuracy just stated was obtained in three iterations and is required because of the closeness of A^2 to the Σ pole position.

On the imaginary axis, there are no zeros in evidence but there is a minimum at about $\omega^2=-0.35M_K^2$. Although we have only plotted $T(\omega)$ for $\omega^2\leq-1.55M_K^2$, a few values were calculated further from the origin and $\mathrm{Re}T(\omega)$ was found to be decreasing monotonically to $-\infty$.

TABLE I. Contributions to $T(A)$ and $T(iB)$ in Eq. (6); $A^2=0.09551M_K^2$, $B^2=0.35M_K^2$.

	$T(A)$	$T(iB)$
$\mathrm{Re}T(M_K)$	-27.1	-27.1
Λ pole	15.0	-4.8
Σ pole	-19.7	-0.2
Integral ($\omega_{\Lambda\pi}$ to $1.41M_K$)	39.2	34.6
Integral ($1.41M_K$ to $37M_K$)	-7.2	-9.9
Integral ($37M_K$ to ∞)	-0.2	-0.3
Total	0.0	-7.7

If our $\mathrm{Im}T(\omega)$ values and the values of g_Λ^2 and g_Σ^2 are correct, we can put a lower limit on the value of $\mathrm{Re}T(M_K)$ required to produce zeros on the imaginary axis:

$$\mathrm{Re}T(M_K)>-19.5M_K^{-1} \quad (18)$$

or

$$A_0^- - A_1^- + 2A_1^+ > -1.62\text{ F}. \quad (19)$$

If we take $A_1^+=-0.29$ F we get

$$A_0^- + A_1^- > -1.04\text{ F}. \quad (20)$$

This is quite a lot smaller than Kim's value,[7]

$$A_0^- + A_1^- = -1.67\text{ F}, \quad (21)$$

or Sakitt's value,[10]

$$A_0^- + A_1^- = -1.82\text{ F}. \quad (22)$$

It is important to note that the value of $\mathrm{Re}T(M_K)$ indicated in (18) is not very dependent on the KNY coupling constants nor on the uncertainties in the high-energy scattering. Therefore, it seems that if Kim's scattering lengths are correct, $T(\omega)$ has four complex zeros which are symmetric partners of one another.

Two particular values of $\mathrm{Re}T(\omega)$ are of interest to us, the ones at

$$\omega^2=A^2=0.09551M_K^2$$

and at

$$\omega^2=-B^2=-0.35M_K^2.$$

The various contributions in (6) to these values are shown in Table I.

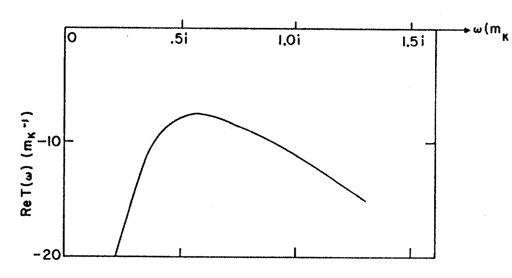

FIG. 4. $T(\omega)$ on the imaginary ω axis.

[10] M. Sakitt, T. B. Day, R. G. Glasser, N. Seeman, J. Friedman, W. Humphrey, and R. R. Ross, Phys. Rev. **139**, B719 (1965).

TABLE II. Contributions to $\sigma(\infty)$ in Eqs. (24) and (26).

Term	Subtraction at A	Subtraction at iB
$T(\omega_{\Lambda\pi})/(\omega_{\Lambda\pi}^2-A^2)$	63.6 ± 6.4	$\cdots$
$[T(\omega_{\Lambda\pi})-T(iB)]/(\omega_{\Lambda\pi}^2+B^2)$	$\cdots$	46.6 ± 4.6
Λ pole	55.8	-12.1
Σ pole	-85.2	-0.5
Integral $\omega_{\Lambda\pi}$ to M_K	3.3 ± 0.3	1.2 ± 0.1
Integral M_K to $5M_K$	-16.4 ± 1.6	-13.6 ± 1.4
Integral above $5M_K$	-5.7 ± 0.6	-5.7 ± 0.6
Total	15.4 ± 8.9	15.9 ± 6.8

III. INFINITE-ENERGY CROSS SECTIONS

In I, we derived an expression for the infinite-energy πN cross section. In KN scattering we proceed in an analogous fashion, starting from

$$T(\omega)=\frac{\omega[T(\omega)-T(\omega_{\Lambda\pi})]e^{i\pi\beta}}{(\omega^2-A^2)(\omega^2-\omega_{\Lambda\pi}^2)^\beta},\tag{23}$$

with $\beta=\frac{1}{2}$. In (23), we must use $(\omega^2-\omega^2{}_{\Lambda\pi})^\beta$ instead of $(\omega^2-M_K^2)^\beta$ because the cut of $T(\omega)$ starts at $\omega_{\Lambda\pi}$. The infinite-energy cross section is then given by

$$\sigma(\infty)=\frac{T(\omega_{\Lambda\pi})}{(\omega_{\Lambda\pi}^2-A^2)^{1/2}}+\sum_{Y=\Lambda,K}\frac{\omega_Y X_Y}{(\omega_Y^2-A^2)(\omega_{\Lambda\pi}^2-\omega_Y^2)^{1/2}}+\frac{2}{\pi}\int_{\omega_{\Lambda\pi}}^{\infty}\frac{\omega d\omega\,\mathrm{Re}[T(\omega)-T(\omega_{\Lambda\pi})]}{(\omega^2-A^2)(\omega^2-\omega_{\Lambda\pi}^2)^{1/2}}.\tag{24}$$

A similar analysis of the expression

$$\frac{\omega[T(\omega)-T(\omega_{\Lambda\pi})]e^{i\pi\beta}}{(\omega^2+B^2)(\omega^2-\omega_{\Lambda\pi})^\beta},\tag{25}$$

with $\beta=\frac{1}{2}$, leads to

$$\sigma(\infty)=\frac{T(\omega_{\Lambda\pi})-T(iB)}{(\omega_{\Lambda\pi}^2+B^2)^{1/2}}+\sum_{Y=\Lambda,K}\frac{\omega_Y X_Y}{(\omega_Y^2+B^2)(\omega_{\Lambda\pi}^2-\omega_Y^2)^{1/2}}+\frac{2}{\pi}\int_{\omega_{\Lambda\pi}}^{\infty}\frac{\omega d\omega\,\mathrm{Re}[T(\omega)-T(\omega_{\Lambda\pi})]}{(\omega^2+B^2)(\omega^2-\omega_{\Lambda\pi}^2)^{1/2}}.\tag{26}$$

We have evaluated (24) and (26) using the data displayed in Fig. 3. The contributions from the various terms are shown in Table II. The 10% errors associated with the nonpole contributions are based on a crude estimate of the experimental uncertainty in the KN data.[2,6–9] Changes in g_Λ^2 and g_Σ^2 would, of course, lead to changes in A^2 and B^2. It was found that 15% changes in the coupling constants, along with the corresponding changes in A^2 and B^2, did not significantly change the final values for $\sigma(\infty)$.

For $\omega>5M_K$ we have assumed that $\mathrm{Re}T$ is given entirely by the P' trajectory, so that

$$\mathrm{Re}T(\omega)=\frac{\gamma_{P'}}{\tan\frac{1}{2}\pi\alpha_{P'}}\left(\frac{\omega}{1\text{ BeV}}\right)^{\alpha_{P'}}.\tag{27}$$

To fix $\gamma_{P'}$ we use

$$\mathrm{Re}T(5M_K)=-8.0M_K^{-1}\tag{28}$$

and assume $\alpha_{P'}=0.39$ as in I.

We then find

$$\gamma_{P'}=3.9M_K^{-1}=3.0\text{ mb BeV},\tag{29}$$

which is of the same order as $\gamma_{P'}$ in the πN case,

$$\gamma_{P'}{}^{(\pi N)}=3.0\,\mu^{-1}=8.4\text{ mb BeV}.\tag{30}$$

Our final results are then

$$\sigma(\infty)=15.4\pm8.9\text{ mb}\tag{31}$$

from (24) and

$$\sigma(\infty)=15.9\pm6.8\text{ mb}\tag{32}$$

from (26).

The present experimental data[8] indicate

$$\sigma(20\text{ BeV})\approx17.3\text{ mb},$$

in fair agreement with the estimates (31) and (32).

IV. SUMMARY AND CONCLUSIONS

We have determined the number and approximate location of the zeros of the forward crossing-even KN amplitude in the complex ω (kaon lab energy) plane.

Using the knowledge of these zeros and the data of Kim[2,7] and of others,[6–9] we have derived dispersion sum-rule estimates of the infinite-energy KN cross section. Reasonable agreement was found between the calculated infinite-energy cross section and the experimental cross section at 20 BeV.

As a byproduct of our analysis, we have compared Kim's data[2] for $\mathrm{Re}T(\omega)$ in the interval $\omega_{\Lambda\pi}<\omega<1.4M_K$ with that given by the ordinary dispersion relation. Disagreement was found in the interval $\omega_{\Lambda\pi}<\omega<0.7M_K$. It is probably due to the influence of the Σ pole which lies close to $\omega_{\Lambda\pi}$.