$F_{i_1 \cdots i_t}{}^t$ to the basis $\Pi_{\lambda_1\lambda_2\lambda_3}{}^t$, each component of which is an eigenstate of the operators

$$I_3=(\sqrt{5})H_1, \quad Y=(\sqrt{5})H_2, \quad N=(2\sqrt{5})H_3=2t+3,$$

with eigenvalues λ_1, λ_2, λ_3, respectively. Figure 5 shows the weight diagram of this representation. As we restrict ourselves to $SO(5)$ transformations, the representation splits into a sum of representations whose dimension, given by Eq. (2.11), is

$$\tfrac{1}{6}(t+1)(t+2)(2t+3)=\sum_{t'=0}^{t}(t'+1)^2,$$

and each of which occurs once. An $SO(5)_\sigma$ representation corresponding to $t=\sigma$ decomposes as follows, if we restrict ourselves to $SO(4)\subset SO(5)$:

$$SO(5)_\sigma=\sum_{t'=0}^{\sigma}\oplus SO(4)_{t'}. \tag{C6}$$

As in the previous case, a physical system described by this representation will be labeled by three quantum numbers, the eigenvalues of the operators I_3, Y, and N, which as far as an elementary particle system is concerned may be taken to be the third component of isospin, the hypercharge, and a new quantum number. The isospin and hypercharge content of the representation is clearly shown in Fig. 5.

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Optimized Polynomial Expansion for Scattering Amplitudes*

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In devising the most efficient way to determine a scattering amplitude from experimental data, it is important to make full use of the analyticity properties of the amplitude. The amplitude $f(x)$ considered here is given by data on a connected part of the real x axis, and $f(x)$ is assumed to be analytic in a simply connected part of the x plane; there are branch cuts on part of the remainder of the real axis. For convenience in practical calculations it is simplest to expand in polynomials, but for greater flexibility one may consider polynomials of some function $z(x)$. The polynomial expansion will converge as rapidly as possible if $z(x)$ maps the domain of analyticity in the x plane onto the interior of a certain ellipse in the z plane. More precisely, the expansion will then have the greatest possible geometric rates of convergence, both to $f(x)$ in the physical region, and also at any arbitrary point away from the physical region to which one may wish to extrapolate. Formulas are given that enable the mapping from a cut plane to an ellipse to be calculated quickly and easily. Some properties of the transformation that are relevant to partial-wave analysis are examined in detail. A method is suggested whereby the requirements of unitarity may be explicitly incorporated.

I. INTRODUCTION

A METHOD of exploiting the analyticity properties of scattering amplitudes has been employed by the present authors in analysis of scattering data.[1] Our method used a conformal transformation to increase the rate of convergence of a polynomial expansion.[2] In this paper, we shall discuss some of the mathematical properties of this transformation. We shall first review the convergence properties of polynomial expansions,[2a] as they have been described by Walsh.[3] In Sec. II, we shall show that an optimum expansion is obtained by mapping as much as possible of the domain of analyticity onto the interior of an ellipse.[3] In the case of a cut plane of analyticity, this mapping turns out to involve elliptic functions, but in a form that is especially convenient for numerical computation, as we show in Sec. III. In Secs. IV and V, we examine the limiting form of the transformation at high and low energies and point out some properties of the transformation that are important in applications. The construction of unitary approximations is discussed in Sec. VI. Section VII is devoted to a brief description of several practical circumstances in which we expect that

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† On leave from Indian Institute of Technology, Kanpur, India.
[1] R. E. Cutkosky and B. B. Deo, Phys. Rev. Letters **20**, 1272 (1968).
[2] J. S. Levinger and R. F. Peierls, Phys. Rev. **134**, B1341 (1964); W. R. Frazer, *ibid.* **123**, 2180 (1961); D. M. Greenberger and B. Margolis, Phys. Rev. Letters **6**, 310 (1961). These authors have used conformal mapping to analyze data on form factors and on the differential scattering cross section, and to parametrize the left-hand cut.
[2a] S. Ciulli, in a recent CERN Report (unpublished) obtained after this paper was submitted for publication, has independently proved the optimality of a mapping which is equivalent to ours and has, in an appendix, sketched the proof of the main convergence theorem.
[3] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain* (American Mathematical Society, Providence, R. I., 1956), 2nd ed., Vol. 20, Chaps. III–VI.

the optimized-expansion theory will be helpful in the analysis of experimental data.

Consider a sequence $p_n(z)$ of polynomial approximations to a function $f(z)$. For each n the polynomial is chosen to minimize some rather general measure of the error of approximation in a domain D of the z plane. The convergence properties are associated with the electrostatics problem in which D forms a conductor at a potential $V=0$ in free space with a unit negative charge.[3] We obtain a potential function $V(z)$ and thus a family of equipotential curves $V(z)=\text{const}$. Let V be any finite potential smaller than each of the potentials $V(z_s)$ at the singular points z_s of f. Then $p_n(z)$ converges uniformly to $f(z)$ inside the closed region that is bounded by the equipotential curve $V(z)=V$. Furthermore, the error in the nth approximation (on D) is bounded by e^{-nV} for some M. These results are essentially independent of the measure of approximation that is used on D. If, however, we use a weighted least-squares approximation, that is, expand in some orthonormal polynomials, then we have the additional result that the coefficient a_n of the nth term satisfies the asymptotic relation $\limsup(a_n)^{1/n}=1/R$, where $\ln R=\min V(z_s)$. The special case in which D is the real interval $(-1,+1)$, leads to a region of convergence that is the interior of a unifocal ellipse (foci at $z=\pm 1$). This elliptical convergence region is a well-known property of the partial-wave expansion; it is usually derived from explicit formulas for the Jacobi polynomials.

Now, let us suppose that $f(x)$ (x is the cosine of the scattering angle) is a scattering amplitude that is analytic in the region enclosed by some curve C, but is singular at some point or points on C. In practice, we shall be concerned with analyticity in a cut plane, and then consider the limiting case in which C denotes branch cuts along $(-\infty, -x_-)$ and $(x_+, +\infty)$. If we construct the conformal map $z(x)$ in which the physical region $D=(-1, +1)$ is mapped onto itself and C is mapped onto a unifocal ellipse, then an expansion in polynomials in $z(x)$ will converge to $f(x)$ uniformly throughout any region that is bounded away from C. Now consider in the x plane the potential function $V(z)$ introduced in the previous paragraph. On C as well as on D, the potential $v(x)=V[z(x)]$ must be a constant. In fact, if we have unit charges on C and D, the complex potential $\phi(x)=v(x)+iw(x)$, where w is the stream function, is directly related to the mapping function. Noting that $v=0$ on D and $v=V_C$ (a constant) on C, and also that w increased by 2π in a circuit of D, we see that the function

$$z(x)=\sinh\phi(x)=i\cos w\sinh v-\sin w\cosh v \qquad (1)$$

maps the equipotential C onto a unifocal ellipse and D onto itself.

II. OPTIMIZED MAPPING

Since the rate of convergence is determined by the parameter $R=e^{+VC}$, we wish to choose C so that V_C will be as large as possible. We shall use in our discussion a basic theorem on electrostatics; for the sake of completeness we include a sketch of its proof. It is convenient here to redefine the potential so that $v=0$ on C and $v=V_C$ on D.

Consider a second curve C', such that C lies entirely within the closed region bounded by C' and $f(x)$ is also analytic in the interior. Let $v'(x)=v(x)+\delta v(x)$, where $v'=0$ on C', $v'=V_{C'}$ on D, and we again consider unit charges. On C we have $\delta v=v'\geq 0$. At any point inside of C we can write

$$\delta v(x')=-\int_C \delta v(x)\frac{\partial G(x,x')}{\partial n\,|dx|}, \qquad (2)$$

where $G(x,x')$ is the Green's function for the boundary conditions $G=0$ on C and $G=\text{const}$ on D, with no net charge on D. The charge density induced on C by a unit positive charge at x' is $\partial G(x,x')/\partial n$. Since this charge density is everywhere nonpositive, we see that at every point inside of C we have $\delta v(x)\geq 0$. Therefore $V_{C'}\geq V_C$ and we have proved that the best asymptotic rate of convergence on the physical region is obtained by choosing C to enclose as large an area as possible.

There is a second criterion for an optimum mapping that is especially important for applications such as that in Ref. 1. Suppose that, in addition to the singularities on and outside of C, $f(x)$ has a simple pole of residue r_p at the point x_p that lies inside of C. Then an expansion in polynomials in z will not converge throughout the interior of C, but only in a smaller region that is limited by the equipotential curve $v(x)=V_p$ that passes through x_p. We write

$$f(x)=\bar{f}(x)+r_p/(x-x_p) \qquad (3)$$

and expand the remainder $\bar{f}$ in polynomials that do converge throughout the interior of C. Our second criterion is that the mapping should make the pole term as visible as possible, that is, that the coefficients of the expansion should depend as sensitively as possible on the pole term being included with exactly the right residue. For large n the ratio of the nth coefficient in the expansion for f to the nth coefficient in the expansion for $\bar{f}$ is determined, roughly speaking, by the ratio

$$e^{-n(VC-V_p)}/e^{-nVC}=e^{nV_p},$$

so that we wish to make V_p as large as possible. According to Eq. (2), we have $V_p'>V_p$, where V_p' is the potential at x_p when the boundary is C'. Therefore our second criterion also leads to the requirement that C should enclose as large an area as possible.

A third criterion arises when we try to extrapolate from experimental data in the physical region to some distant point x_p at which the scattering amplitude is analytic. For example, it is known that the discontinuities across branch cuts can be expressed in terms

of the amplitudes for physical processes.[4] However, the values of the variables usually do not lie in the physical regions for these processes, so that an extrapolation must be resorted to. If the point x_p lies inside C, then our polynomial expansion will converge there, and the rate of convergence is, roughly speaking, given by e^{-nV_p}.[3] By the same arguments as above, we see that, to get the most rapid convergence, V_p should be as large as possible, and therefore C should enclose as large an area as possible. Note that for this third criterion (that the extrapolating expansion should converge as rapidly as possible) it does not matter at all where the point x_p lies, as long as it is a point where the function is analytic. The same expansion is an optimum for all points.

Another approach to the problem of extrapolation shows the unity of the second and third criteria. If the scattering amplitude $f(x)$ is approximated by polynomials $P_n[z(x)]$ that satisfy the constraint $P_n[z(x_p)]=\gamma_p$, then on D the convergence parameter of the polynomials is e^{V_C} if and only if $\gamma_p=f(x_p)$, but it is $e^{V_C-V_p}$ if $\gamma_p\neq f(x_p)$.[5] The relative convergence parameter, which determines the sensitivity to γ_p, is e^{V_p}.

Instead of the mapping $z(x)$, for a given boundary curve C, we could consider other mappings $z'(x)$ in which the image of D is not a straight line but the images of C and D are still equipotentials. For example, C and D could be mapped into concentric circles. However, at corresponding points we must have $v(x)=V(z)=V'(z')$; this is the principle underlying the use of conformal mapping to solve electrostatics problems. Therefore the convergence parameters $R=e^{V_C}$ and $R_p=e^{V_p}$ would be the same, whether one expanded in polynomials in z or in z'. For a given C it is evident that there are many conformal maps that would lead to convergence properties that are essentially equivalent. We choose the elliptic mapping on the basis of convenience.

Note that further improvement would be possible in principle if the contribution of parts of the branch cuts were included explicitly, by analogy with Eq. (3), or if C were to enclose parts of other Riemann sheets.

III. ELLIPTICAL TRANSFORMATION

We first map the x plane onto an auxiliary w plane in which the cuts run along $(-\infty, -W)$ and (W,∞):

$$w=(x-x_0)/(1-xx_0). \tag{4}$$

With the abbreviation $X_\pm=(x_\pm^2-1)^{1/2}$, we have

$$\begin{aligned} x_0&=(x_--x_+)/(x_+x_-+X_+X_--1),\\ W&=(x_+X_-+x_-X_+)/(X_++X_-). \end{aligned} \tag{5}$$

The mapping from the symmetrized plane to a unifocal ellipse is given by

$$\begin{aligned} z&=\sin\Phi(w,k_0),\\ \Phi(w,k)&=\pi F(\sin^{-1}w,k)/2K(k), \end{aligned} \tag{6}$$

where $F(\phi,k)$ and $K(k)=F(\frac{1}{2}\pi,k)$ are, respectively, the incomplete and complete elliptic integrals of the first kind. Their modulus is $k_0=1/W$. It is easy to check that $x=\pm1$ maps into $z=\pm1$.

A standard method of computing elliptic integrals is by a repeated use of the Gauss transformation[6]

$$(1+k_{n+1})F(\sin^{-1}v_{n+1},k_{n+1})=F(\sin^{-1}v_n,k_n); \tag{7a}$$

therefore

$$\Phi(v_{n+1},k_{n+1})=\Phi(v_n,k_n), \tag{7b}$$

where

$$\begin{aligned} k_{n+1}&=(1-k_n')/(1+k_n')=k_n^2(1+k_n')^{-2},\\ v_{n+1}&=\frac{v_n(1+k_n')}{1+(1-k_n^2v_n^2)^{1/2}},\\ k_n'&=(1-k_n^2)^{1/2}. \end{aligned} \tag{8}$$

In particular, let $k_0=1/W$ and $v_0=w$; then in the calculation of z from Eq. (6) we may use any of the sequence of values (v_n,k_n) generated by Eq. (8). Now note that

$$\lim_{n\to\infty}k_n=0.$$

Since $F(\phi,0)=\phi$, we find that

$$z=\lim_{n\to\infty}\sin\Phi(v_n,k_n)=\lim_{n\to\infty}v_n. \tag{9}$$

The Gauss transformation is thus especially convenient for construction of our mapping function. Not only is the convergence extremely rapid, since $k_{n+1}\approx\frac{1}{4}k_n^2$, but we even avoid calculation of the arcsine and sine.[7]

Note, in particular, that the transformation $w=v_0\to v_1$ maps the cut plane into a circle of radius $k_1^{-1/2}$. The mapping $v_1\to z$, which flattens this circle into the ellipse, is the transformation derived in most textbooks.[8]

The rate of convergence of an expansion on $-1\leq z\leq1$ is determined by

$$R=e^{V_C}=a+(a^2-1)^{1/2}, \tag{10}$$

where a is the semimajor axis of the ellipse. This is given by the formulas

$$a=\cosh\frac{\pi K(k_1')}{4K(k_1)}=\cosh\frac{\pi K(k_0')}{2K(k_0)}, \tag{11}$$

[4] R. E. Cutkosky, Phys. Rev. Letters 4, 624 (1960); J. Math. Phys. 1, 429 (1960).

[5] J. L. Walsh (Ref. 3), p. 315.

[6] H. E. Fettis and J. C. Caslin, *Tables of Elliptic Integrals of the First, Second and Third Kind* (Office of Aerospace Research, U. S. Air Force, 1964).

[7] We shall be glad to supply interested readers with a FORTRAN program that generates this elliptic transformation.

[8] Z. Nehari, *Conformal Mapping* (McGraw-Hill Book Co., New York, 1952); H. Kober, *Dictionary of Conformal Mapping* (Dover Publications, Inc., New York, 1952).

and can also be calculated numerically from the sequence of Gauss transformations.

For later reference, we evalute the derivative

$$\frac{dz}{dw}=\prod_1^\infty \frac{dv_n}{dv_{n-1}}=\prod_1^\infty \frac{(1+k_nv_n^2)^2}{(1+k_n)(1-k_nv_n^2)}. \tag{12}$$

At $z=0$ this becomes

$$\frac{dw(0)}{dz}=\prod_1^\infty (1+k_n)=\frac{2}{\pi}K(k_0), \tag{13}$$

where we have used a standard formula for $K(k_0)$ that follows directly from the Gauss sequence, Eqs. (6) and (7). At $z=1$ we have

$$\frac{dw(1)}{dz}=\prod_1^\infty \frac{1-k_n}{1+k_n}=\prod_1^\infty k_{n-1}'=\prod_1^\infty \frac{k_n'^2}{(1+k_n)^2}.$$

Comparing the last two products above with each other and with Eq. (13), we find

$$dw(1)/dz=[2k_0'K(k_0)/\pi]^2. \tag{14}$$

The two degenerate forms of the ellipse are the circle and the parabola. In the context of our conformal mapping the circle is obtained in the limit in which the branch cuts are very far from the physical region. In the opposite limit, in which a branch point approaches very near to the end $x_e=\pm1$ of the physical region, the ellipse shrinks into the line segment $-1\leq z\leq+1$. However, if we examine the neighborhood of x_e in an expanded scale, we see that the ellipse must approach a parabola whose focus is at x_e. We can obtain the asymptotic form of the transformation in this neighborhood by mapping the plane directly onto the interior of a parabola.

On the real ξ axis let the physical region consist of the portion $\xi\geq0$ and let us choose the scale so that $\xi\leq-1$ is the branch cut. The transformation[8]

$$\zeta=\frac{1}{\pi^2}\left[\ln\frac{(\xi+1)^{1/2}+\xi^{1/2}}{(\xi+1)^{1/2}-\xi^{1/2}}\right]^2 \tag{15}$$

maps the physical region into itself, and the branch cut is mapped onto the parabola whose vertex is at $\zeta=-1$ and whose focus is at $\zeta=0$. To go back to our original variables x and z, we suppose that in the x plane the branch point is at $x=\pm(1+\delta)$. In the neighborhood of $x=\pm1$, the mapping $x\to w$ just amounts to a linear change of scale, and can be omitted. Let $\Delta=a-1$, where a is given by Eq. (11). We then have

$$\xi\delta=1\mp x,\quad \zeta\Delta=1\mp z. \tag{16}$$

For a simple way to estimate the accuracy of the parabolic approximation, Eqs. (15) and (16), to the elliptical mapping, let us compare the exact value of $dz(1)/dx$ as obtained from (14) with its approximation

$$dz(1)/dx\approx4\Delta/\pi^2\delta. \tag{17}$$

It is found that the approximation (17) is the leading term in an expansion in inverse powers of $\ln\delta$.

IV. THRESHOLD PROPERTIES

In the elastic scattering of two particles of mass M_1 and M_2, let s, t, and u denote the Mandelstam variables and p^2 the barycentric momentum. Consider a scattering amplitude $f(s,x)$ that is free of kinematical singularities, and for which the coefficients in the usual partial-wave expansion

$$f(s,x)=\sum_0^\infty a_l(s)p_l(x), \tag{18}$$

where the $p_l(x)$ are normalized Jacobi polynomials, can be expanded as follows in powers of p^2 for some neighborhood of the elastic threshold $s=s_0$:

$$a_l(s)=p^{2l}\sum_0^\infty a_{ln}p^{2n}. \tag{19}$$

(We assume that there is no bound state at the threshold.) Denote by T and U the nearest t and u thresholds. In the x plane the singularities are given by

$$\begin{aligned} x_+-1&=T/2p^2,\\ x_--1&=U/2p^2-(M_1^2-M_2^2)^2/2sp^2\\ &=A/2p^2+O(1), \end{aligned} \tag{20}$$

where $A=U-(M_1-M_2)^2$. When $p^2\to0$, the convergence domain of the partial-wave expansion approaches a circle of radius $r_0/2p^2$, where $r_0=\min(T,A)$. Therefore the scattering lengths must contain a factor r_0^{-l} (in addition, perhaps, to other factors that depend on l more slowly, for large l, than $\exp\pm\epsilon l$ for $\epsilon>0$).

From (5) we find that

$$\begin{aligned} x_0&=p^2(A-T)/(A+T)+O(p^4),\\ W-1&=C/2p^2+O(1), \end{aligned}$$

where $C=2TA/(A+T)$. Furthermore, we have

$$k_0=W^{-1}=2p^2/C+O(p^4),\quad k_n=O(p^{2n+1}).$$

Therefore in the expansion

$$z=x\sum_0^\infty \beta_nx^n+z_0 \tag{21}$$

we have $\beta_n=O(p^{2n})$ and also $z_0=O(p^2)$. For $p^2\to0$ the ellipse in the z plane approaches a circle of radius C/p^2. The terms that we have represented by $O(p^{2m})$ can actually be expanded in powers of p^2.

Our modified expansion is

$$f(s,x)=\sum_0^\infty b_n(s)p_n(z), \tag{22}$$

where these $p_n(z)$ are any orthonormal polynomials over $(-1, 1)$. From (21) and (19) we see that at threshold

$$b_n(s)p^{-2n} \to b_{n0},$$

and, in fact, we may write

$$b_{n0}=\sum_{l=0}^{n} d_{nl}a_{l0}, \tag{23}$$

where the d_{nl} are some kinematic coefficients depending on M_i, T, and U. The exponential factor in the b_{n0} is r^{-n}, where $r=2C$ and

$$2\leq r/r_0\leq 4.$$

To summarize, we may say that the coefficients in our expansion (22) have near threshold essentially the same behavior as the usual partial-wave amplitudes. The improved convergence properties of our expansion are reflected in the fact that the sequence of sums of scattering lengths (23) consists of terms that are asymptotically smaller than the scattering lengths themselves. It would be extremely hard to measure directly enough of these scattering lengths with sufficient accuracy to test this feature of our theory. However, it may in some cases be possible to use this information effectively in the forward dispersion relations for the derivatives $\partial^n f(s,t)/\partial t^n$.

The imaginary part of $f(s,x)$ has as closest singularities $T_{\rm im}(s)$ and $U_{\rm im}(s)$, which are larger than T and U—much larger for s near to the elastic threshold s_0, if this is the lowest s threshold. These properties arise from the unitarity condition,[9] which is considered in more detail in Sec. VI. Therefore in the usual expansion the convergence for the imaginary part is more rapid than the convergence for the real part. We suggest that, in applications to partial-wave analysis of scattering data, the best procedure is to use separate transformations and expansions for $\mathrm{Re}[f(s,x)]$ and $\mathrm{Im}[f(s,x)]$. If s_0 is not the lowest threshold, we may study the properties of the coefficients for $\mathrm{Im}f$ by repeating the previous calculations with T and U replaced with $T_{\rm im}(s_0)$ and $U_{\rm im}(s_0)$. If s_0 is the lowest threshold, we may use the asymptotic expressions $T_{\rm im}\approx\xi/p^2$ and $U_{\rm im}\approx\eta/p^2$; the calculations following Eq. (17) can then be repeated with minor changes. In either case, one again finds more rapid convergence for the imaginary part than for the real part and a more rapid convergence than in the usual expansion.

V. HIGH-ENERGY PROPERTIES

We consider the leading terms when $p^2\to\infty$. For comparison, we first describe some familiar properties of the conventional expansion. Let $\mu_0^2=\min(U,T)$. From (20) we find that $\min(x_\pm-1)=\mu_0^2/2p^2$. The convergence parameter for the partial-wave expansion is, as obtained from (10),

$$R\approx 1+\mu_0/p\approx e^{1/l_0}, \tag{24}$$

where $l_0=p/\mu_0$. Thus for large l the coefficients decrease roughly as e^{-l/l_0}. This is in accord with the fact that μ_0^{-1} is the range of the forces. Note that at high energies $T_{\rm im}\to T$ and $U_{\rm im}\to U$. The real and imaginary parts of the amplitude have, therefore, the same rates of convergence when the energy is very high.

In discussing the elliptical transformation, we use the following asymptotic formula, which holds for $k\to 1$, for the complete elliptic integral:

$$K(k)\approx\ln(4/k'). \tag{25}$$

From (5) we have

$$W-1\approx\mu_e^2/2p^2, \tag{26}$$

where

$$\mu_e^2=(TU)^{1/2},$$

and hence

$$k_0\approx 1-\mu_e^2/2p^2, \quad k_0'\approx\mu_e/p. \tag{27}$$

Using Eq. (10) for the semimajor axis of the ellipse, we have

$$a-1\approx\tfrac{1}{32}\pi^4[\ln(4p/\mu_e)]^{-2}, \tag{28}$$

and therefore, writing for our elliptic expansion $R=e^{1/l_e}$, we have

$$l_e=(4/\pi^2)\ln(4p/\mu_e). \tag{29}$$

At first sight, the number of terms needed in our expansion increases only logarithmically with p, instead of linearly. However, this involves some tacit assumptions that we shall return to later.

One way to understand how the greatly enhanced rate of convergence of our modified expansion arises is to note that our transformation greatly magnifies the physical region in the neighborhood of $x=\pm 1$, which is where we expect the amplitudes to vary most rapidly with x. This plausible behavior of the amplitudes is, in fact, another reflection of the analyticity properties that we exploit in our expansion. We can calculate explicitly the magnification in the neighborhood of $x=+1$ by using the asymptotic limit obtained from Eq. (15):

$$1-z\approx\tfrac{1}{32}\pi^2\left(\ln\frac{4p}{\mu_e}\right)^{-2}\left(\ln\frac{(T-t)^{1/2}+(-t)^{1/2}}{(T-t)^{1/2}-(-t)^{1/2}}\right)^2. \tag{30}$$

This is valid for a fixed value of t in the limit $p\to\infty$. For $|t|\ll T$ we have

$$z-1\approx\tfrac{1}{8}\pi^2(t/T)\ln(4p/\mu_e). \tag{31}$$

In the neighborhood of $x=1$, therefore, $1-z$ is essentially $-t/T$, apart from a factor depending only logarithmically on the energy. Similar formulas hold for $1+z$, with t/T replaced with u/U.

The mapping shrinks the interior of the physical region. In particular, let us look at the point x_0 that is

[9] S. Mandelstam, Phys. Rev. 112, 1344 (1958); 115, 1752 (1959).

mapped into $z=0$; this is, asymptotically, the point

$$x_0=(U-T)/(U^{1/2}+T^{1/2})^2.$$

At this point, we find by use of (13) that

$$\left.\frac{dz}{dx}\right|_{x_0}=\tfrac{1}{8}\pi\frac{(T^{1/2}+U^{1/2})^2}{(TU)^{1/2}}\left(\ln\frac{4p}{\mu_e}\right)^{-1}. \tag{32}$$

Let us now return to the discussion of the rate of convergence. The precise meaning to be ascribed to the convergence parameter R is explained in the Introduction. That is, what we know is that the nth coefficient can be bounded by the expression $M(\epsilon)/(R-\epsilon)^n$ for any positive ϵ. This estimate is not sufficiently precise, for two reasons. First, we would like to have an estimate of the nonexponential factors in the coefficients, so that we could dispense with ϵ. Second, we would like to know how M depends on energy. For both these questions we need to know the nature of the singularities on C. It is evident that the behavior of $f(s,x)$ as $x\to\infty$ is critical; in our transformed variable z, $x=\infty$ corresponds to two finite points z_∞ on the edge of the domain of convergence. The nature of the singularities at $z=z_\infty$ is especially important at high energies, since these points then lie very close to the part of the physical region that corresponds to large-angle scattering.

Experiments suggest that at high energies the cross sections depend exponentially on some power of the momentum transfer.[10] It is plausible, therefore, that $f(s,x)$ might also have an exponential behavior at $x\to\infty$—perhaps at all energies, or perhaps just in the limit $s\to\infty$. This is also suggested by the idea of indefinitely rising Regge trajectories. If this is indeed the case, it would be very desirable to remove from $f(s,x)$ a factor f_0 that had the right behavior at $x\to\infty$, and was analytic and free from zeros in the cut plane. Then $f(s,x)$ could be represented at a product of $f_0(s,x)$ and a modulating factor $f_m(s,x)$ that was expanded in powers of z. We might then expect that the convergence of the expansion for f_m would be sufficiently rapid that we could really use l_e as an estimate of the number of terms required. Likewise, we suggest that in removing a pole term from $f(s,x)$, as in Eq. (3), the pole term should be multiplied by vertex functions that supply the right asymptotic behavior for large $|t|$ or $|u|$.

VI. UNITARITY

Unlike the partial-wave expansion, our expansion for $f(x)$ cannot satisfy elastic unitarity term by term. Instead, unitarity must be built into the sum as a whole. However, the positions of the singularities T, U, $T_{\rm im}$, and $U_{\rm im}$ are consistent with unitarity; $T_{\rm im}$ and $U_{\rm im}$ can even be generated from the unitarity relation.[9] As a result, if we express $\mathrm{Re}[f(x)]=f_1(x)$ and $\mathrm{Im}[f(x)]=f_2(x)$ by separate expansions, the asymptotic convergence rates of the two expansions are compatible with unitarity. This at least suggests that only small corrections need to be made for unitarity, if the expansions are truncated.

In practice, one wishes to approximate a scattering amplitude by a truncated expansion containing a finite number of parameters. These parameters are then to be determined by fitting to experimental data. We shall show here how small corrections can be made to a truncated expansion in such a way that the result will be explicitly unitary. For brevity, we consider only the scattering of spinless particles.

First, consider scattering below the inelastic threshold. Let $N+1$ be the number of parameters to be determined from the data. These may be considered as the phase shifts δ_l for the partial waves with $l=0,\cdots,N$. We write for the real and imaginary part

$$f_i(x)=\sum_0^N b_{in}p_n[z_i(x)]+\sum_0^\infty b_{in}'p_n[z_i(x)]+f_{Bi}(x), \tag{33}$$

where $z_1(x)$ and $z_2(x)$ are the transformed variables for the real and imaginary parts of f, and where $f_{Bi}(x)$ is the "Born approximation" [e.g., given by the pole terms in $f(x)$]. The b_{in}' are coefficients of small correction terms and are considered to be zero as a first approximation. The partial-wave amplitudes are $a_l=a_{1l}+ia_{2l}$, where

$$a_{il}=\tfrac{1}{2}\int_{-1}^{1}f_i(x)P_l(x)dx, \tag{34}$$

and where $P_l(x)$ is the Legendre polynomial. Now define

$$C_{iln}=\tfrac{1}{2}\int p_n[z_i(x)]P_l(x)dx \tag{35}$$

and denote the approximation of dropping the b_{in}' with a caret; this gives

$$\hat{a}_{il}=\sum_{n=0}^N C_{iln}b_{in}+a_{Bil}, \tag{36}$$

where the a_{Bil} are the contribution of the Born approximation. Solving (36) for the b_{in} gives the result

$$b_{in}=\sum_{l=0}^N D_{iNnl}(\hat{a}_{il}-a_{Bil}), \tag{37}$$

where

$$\sum_{l=0}^N D_{iNnl}C_{ilm}=\delta_{nm},\quad n,m\leq N.$$

The $\hat{a}_{il}$ for $l>N$ are expressed in terms of the $\hat{a}_{il}$ for $l\leq N$ by substituting (37) back into (36); with the notation

$$\Gamma_{iNlk}=\sum_{n=0}^N C_{iln}D_{iNnk}, \tag{38}$$

[10] J. Orear, Phys. Letters **13**, 190 (1964); A. D. Krisch, Phys. Rev. Letters **19**, 1149 (1967); T. T. Chou and C. N. Yang, *ibid.* **20**, 1213 (1968).

the result can be written as follows:

$$\hat{a}_{il}=\sum_{k=0}^{N}\Gamma_{iNlk}(\hat{a}_{ik}-a_{Bik})+a_{Bil}. \tag{39}$$

For $l\leq N$ we may define

$$\hat{a}_{il}\equiv a_{il}, \tag{40}$$

where the complex amplitudes $a_l=i(1-e^{2i\delta l})$ are unitary by construction. However, the amplitudes $\hat{a}_l$ determined from (39) for $l>N$ are not, in general, unitary. Let

$$\eta_l{}^2=\hat{a}_{1l}{}^2+(\hat{a}_{2l}-1)^2, \tag{41}$$

and let us write

$$a_l=\hat{a}_l+a_l'. \tag{42}$$

We shall choose the a_l' to have the smallest possible magnitudes that will make the a_l unitary. This condition is easy to see geometrically if we represent $\hat{a}_l$ in the complex plane. The unitary amplitudes lie on a circle of unit radius centered at $(0,i)$. The point on the circle closest to $\hat{a}_l$ is given by

$$\begin{aligned} a_{1l}&=\hat{a}_{1l}/\eta_l,\\ a_{2l}-1&=(\hat{a}_{2l}-1)/\eta_l. \end{aligned} \tag{43}$$

The b_{in}' in Eq. (33) are another representation of the a_l', but they are not needed directly. If any η_l differ from unity by more than a few percent, this is a sign that a larger N should be used.

Above the inelastic region, unitarity gives only an inequality; we can use the same method as in the elastic region, but the corrections are, in general, smaller. Since the expansion for f_2 converges faster than the expansion for f_1, we may use different upper limits N_1 and N_2 in the sums (33); for a similar degree of accuracy in the two sums, we may take

$$N_2\ln R_2\approx N_1\ln R_1, \tag{44}$$

where R_1 and R_2 are the two convergence parameters. There are then N_1+N_2+2 parameters that are to be fitted to the given data. In the elastic case, there is no point in using a smaller number of terms in the second expansion, because it does not involve any new parameters. For $l\leq N_2$ we write $a_l=i(1-\eta_l e^{2i\delta l})$, and for $N_2<l\leq N_1$, a_{2l} is given by (39) while a_{1l} is a parameter. The correction procedure for $l>N_1$ is as follows: If η_l as given by (41) is less than unity, it is considered to represent a real inelasticity, and no correction is applied. Otherwise, formula (43) is applied.

In meson-baryon scattering, the nearest singularities T_{im} and U_{im} correspond to inelastic processes, provided that the energy is raised above the inelastic threshold by a rather small amount. In this case, one expects that the η_l as calculated above for large l will all be slightly less than 1, so that no unitarity correction would have to be applied to the high partial waves, but, rather, that these would all turn out automatically to be slightly inelastic. In nucleon-nucleon scattering, on the other hand, the nearest singularities always correspond to elastic scattering. As a result, the contribution of the nearest part of the branch cut can be expressed explicitly in terms of the meson-nucleon coupling constant and included as a term $f_{02}(x)$ in Eq. (33).

To some extent, the rapidly convergent property of our expansion is lost when we reexpress the sum in terms of the ordinary partial-wave expansion; the amplitude $f(x)$ is a sum of many partial waves, and the ones with large l may be quite small but add up to a non-negligible total contribution. The advantages of working with the ordinary partial waves and keeping the elliptical expansion in the background are quite numerous. The constraint on the b_{in} for $n\leq N$ that is imposed by the condition that the partial waves for $l\leq N$ be unitary is a very complicated nonlinear one, and would be hard to satisy directly. Secondly, the complicated mathematical calculations that are required in order to effect the transformation and to impose unitarity are summed up in the coefficients Γ_{iNlk}, which can be calculated once (for a given energy) and used repeatedly. Furthermore, the energy dependence of the scattering amplitude is not easy to study in terms of the transformed expansion; the coefficients b_{in} have very complicated analytic properties as functions of the energy, since each of them brings in singularities that lie on the second sheet of the scattering amplitude. In studying the energy dependence, therefore, one would certainly wish to go back to the partial-wave amplitudes a_l or to the scattering amplitude at a fixed value of the momentum transfer.

Nevertheless, the elliptical transformation has a very important effect on the partial-wave expansion. In the first place, the waves with $l>N$, which are generated by the first few terms, may contribute very significantly to the amplitude. In addition, the tests for the validity of the truncation of the series with N_i terms should be made by reconstructing the b_{in} by means of the coefficients D_{iNnl}.

VII. SUMMARY

In an earlier paper,[1] we used the elliptic transformation for the extraction of coupling constants from fixed-energy differential cross-section data and demonstrated its utility for this purpose. We also showed how error estimates could be based on the convergence theory. Another application along the same lines would be the determination of the real part of the forward scattering amplitude $\text{Re}f(0°)$ from the Coulomb interference term in the differential cross section. In this case, one is interested in the residue of the pole at $x=1$ (from Coulomb scattering there is also a known double pole at $x=1$ that can be explicitly subtracted). In experiments on the Coulomb interference term, one typically obtains very precise data in a limited range $-1\ll x_2\leq x\leq x_1<1$. In such a case, the interval (x_2,x_1) should be

interpreted as the "physical region" and mapped into $(-1, 1)$, in order that the greatest sensitivity to $\mathrm{Re}f(0°)$ may be obtained.

We expect that the application of the theory described in this paper to the partial-wave analysis of scattering data will be especially important in practice. In this case, what the transformation achieves is the imposition of certain relations among the higher partial waves that occur when they are generated by forces of the Yukawa type, which extend over a finite distance. Similar relations are obtained when the scattering amplitudes are constructed by solving the Schrödinger equation with a phenomenological potential that contains adjustable parameters, but the transformation method is both simpler and less model-dependent. With the transformation method it is not quite automatic that the scattering amplitude satisfies the unitarity condition, but it is easy to make the necessary small adjustments. In doing this, the effective potential is, in effect, parametrized by the amplitudes for the first few partial waves, and the superior convergence of the transformed series is taken into account through the existence of a set of coefficients $\Gamma_{ll'}$ that relate partial waves for $l>N$ to those with $l'\leq N$.

If the scattering amplitude happens to be dominated by a single resonance of high spin, the amplitude that is well represented by a single term of the partial-wave expansion would require many terms of our expansion. In such a case, we propose that the resonance be introduced explicitly, and our expansion used for the nonresonant background.

The fact that the parabolic transformation (15) can be interpreted as the limiting form of the elliptic transformation suggests that it can also be used directly as the basis for rapidly convergent expansions. For example, an expansion in Laguerre polynomials converges inside a parabolic domain of analyticity. If there is a physical region and a branch cut, extending, respectively, to $+\infty$ and to $-\infty$, we suggest the mapping of the cut plane into a parabolic region through use of Eq. (15). Unfortunately, a general theory of the expansion coefficients is not available. However, for a given sequence of polynomials $p_n(\zeta)$ that are orthonormal over $(0,\infty)$, we may base convergence estimates on the sequence of values $p_n(-1)$. That is, since $\zeta=-1$ is just at the edge of the domain of convergence, the coefficients a_n may be expected to decrease roughly as $1/p_n(-1)$ for large n.

The parabolic transformation would apply to the electromagnetic form factors, since these have a cut plane of analyticity and a physical region extending to infinity. A second example is given by the forward dispersion relations for processes such as p-p and K^+-p scattering, in which there is an "unphysical region." Let s_0 denote the threshold for the direct channel, s_2 the physical threshold for the crossed channel, and s_1 the unphysical threshold $(s_0>s_1>s_2)$. Let

$$f(s)=F(s)-\frac{1}{2\pi i}\int\frac{\mathrm{disc}[F(s')]ds'}{s'-s}\quad\text{(physical regions)} \tag{45}$$

(ignoring subtractions), where $F(s)$ is the forward scattering amplitude. If we take

$$\xi=\frac{s-s_0}{s-s_2}\frac{s_1-s_2}{s_0-s_1}, \tag{46}$$

the unphysical region is mapped onto the parabola by (15), and $f(s)$ has the right analyticity properties to allow an expansion in polynomials in $\zeta(s)$.

The convergence rate of a polynomial fit to data can always be increased further if the nearest branch cuts are treated explicitly and only the most distant parts of the cuts are mapped onto the ellipse (or parabola). However, it is usually necessary to tolerate a certain amount of model dependence in doing this, because the effect of a branch cut cannot be represented exactly by only a few parameters. A small number of parameters referring to the cuts can be determined directly from the data by employing the "subtraction principle" summed up in Eq. (3). In addition, we may in some cases use experimental data for other processes to determine the discontinuities across the branch cuts.