In the above calculation we have also not used the mixing between η^0 and X^0. As has been pointed out in Ref. 4, its inclusion reduces the rate by a factor of 0.68 or enhances it to double its value depending on the sign of the mixing angle, and it is also easy to predict a similar branching ratio for the decays. We find that $R'\sim 0.20$ for the $M1$ transition case, to be compared to the rate ~ 0.42 when $E1$ is predominant (see Ref. 6). In any case the experimental limit is $\lesssim 0.9\%$ or 0.6% which is quite large. Hence the $M1$ transition can also contribute appreciably. In Fig. 1 we have plotted the photon spectrum in the ω-dominance model.

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Calculation of the Sixth-Order Contribution from the Fourth-Order Vacuum Polarization to the Difference of the Anomalous Magnetic Moments of Muon and Electron*

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We present the details of a calculation of the sixth-order contribution to $\frac{1}{2}(g_\mu-g_e)$ from the proper fourth-order vacuum polarization. As a byproduct of this calculation, we have also obtained the finite part of the fourth-order contribution to the charge-renormalization constant Z_3.

I. INTRODUCTION

ONE of the classical successes of quantum electrodynamics has been the prediction of radiative corrections to the Dirac value of the gyromagnetic ratio of the electron and of the muon. To first order in the fine structure constant α, these corrections[1] are predicted to be the same for the electron and the muon:

$$\tfrac{1}{2}(g_e-2)^{(2)}=\tfrac{1}{2}(g_\mu-2)^{(2)}=\alpha/2\pi. \tag{1}$$

This is, however, no longer true at higher orders. Already at fourth order in the electric charge constant e ($e^2/4\pi=\alpha$), the Feynman diagram shown in Fig. 1(a) gives a sizable contribution to $\frac{1}{2}(g_\mu-2)$, while the corresponding diagram obtained by interchanging the muon and electron lines [see Fig. 1(b)] gives a very small contribution to $\frac{1}{2}(g_e-2)$. All other diagrams involve only one kind of lepton, and therefore their contributions do not depend on the masses.

The total contribution to the electron g factor in fourth order is given by

$$\tfrac{1}{2}(g_e-2)^{(4)}=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{197}{144}+\tfrac{1}{12}\pi^2+\tfrac{3}{4}\zeta(3)-\tfrac{1}{2}\pi^2\ln 2+\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2+O\left[\left(\frac{m_e}{m_\mu}\right)^4\ln\frac{m_\mu}{m_e}\right]\right\}=-0.3284784(\alpha/\pi)^2, \tag{2}$$

where $\zeta(3)$ is the Riemann zeta function of argument 3, defined in the Appendix. The terms independent of the ratio m_e/m_μ were calculated by Karplus and Kroll,[2] Sommerfield,[3] and Petermann[4] using standard quantum electrodynamics, and by Terent'ev[5] using dispersion techniques. We have calculated the term $(1/45)\times(m_e/m_\mu)^2$, which comes from the diagram shown in Fig. 1(b).[6]

The corresponding contribution to the muon g factor in fourth order is

$$\tfrac{1}{2}(g_\mu-2)^{(4)}=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{97}{144}+\tfrac{1}{12}\pi^2+\tfrac{3}{4}\zeta(3)-\tfrac{1}{2}\pi^2\ln 2+\tfrac{1}{3}\ln\frac{m_\mu}{m_e}+\tfrac{1}{4}\pi^2\frac{m_e}{m_\mu}-4\left(\frac{m_e}{m_\mu}\right)^2\ln\frac{m_\mu}{m_e}+3\left(\frac{m_e}{m_\mu}\right)^2+O\left[\left(\frac{m_e}{m_\mu}\right)^3\right]\right\}=(+0.765779\pm 7\times 10^{-6})(\alpha/\pi)^2. \tag{3}$$

This includes the contribution from the diagram shown in Fig. 1(a), which was first estimated by Suura and

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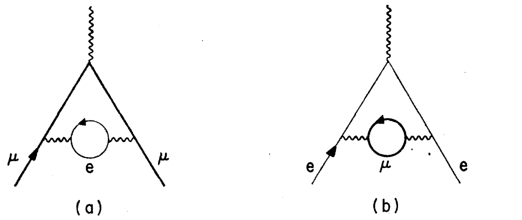

FIG. 1. Feynman diagrams contributing to the difference of the muon and electron g factors in fourth order.

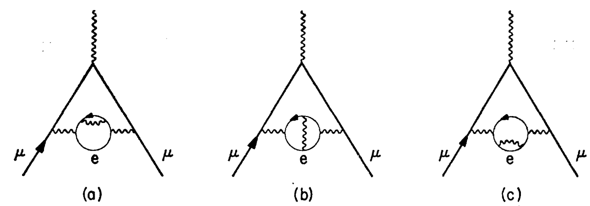

FIG. 2. Feynman diagrams representing the proper fourth-order vacuum polarization contribution to the difference of the muon and electron g factors in sixth order.

Wichman[7] and by Petermann.[8] Recently it has been calculated exactly by Elend.[9]

The complete calculations of the sixth-order radiative corrections to the electron and muon g factors have not been carried out yet. Interest in this type of calculations has been largely motivated by the increasing experimental accuracy in the measurements of the anomalous magnetic moments of the electron and the muon. At present, $\frac{1}{2}(g_e-2)$ is known to an accuracy of 30 ppm[10]:

$$\tfrac{1}{2}(g_e-2)_{\text{expt}}=(1\,159\,557\pm30)\times10^{-9}, \tag{4}$$

and there are experiments in progress[11] which are likely to improve this accuracy to a few ppm. For the negative muon, the most precise experimental result which has been reported is[12]

$$\tfrac{1}{2}(g_\mu-2)_{\text{expt}}=(116\,645\pm33)\times10^{-8}. \tag{5}$$

It is clear from these results that calculations of the α^3 radiative corrections, as well as other types of corrections (strong-interaction corrections due to the hadronic contributions to the photon propagator,[13] and weak interaction corrections[14]) which might be of the same order of magnitude as the α^3 radiative corrections, are needed.

A significant step in the estimate of the sixth-order radiative corrections is the calculation of the difference between the muon and electron g factors. In fact, the terms containing $[\ln(m_\mu/m_e)]^2$, and most of the terms containing $\ln(m_\mu/m_e)$, contributing to this difference have already been calculated by several authors.[15,16] However, most of the remaining terms have not been calculated as yet.

In this paper we present the details of a calculation of the sixth-order contribution from the proper fourth-order vacuum polarization graphs to $\frac{1}{2}(g_\mu-g_e)$. The relevant Feynman diagrams are shown in Fig. 2. The method of our calculation is discussed in Sec. II, and the explicit evaluation of the integrals involved is carried out in Sec. III. We should remark that all our results are gauge-invariant.

We have also evaluated the contribution to $\frac{1}{2}(g_\mu-g_e)$ from the Feynman diagram shown in Fig. 3, and our result agrees with a previous calculation by Kinoshita.[15]

The results of these calculations are presented in Sec. IV.

We have included an Appendix which contains a list of the special integrals used in the calculation.

II. METHOD OF CALCULATION

We start from the general expression for the renormalized photon propagator

$$D_{\mu\nu}(p)=-i\frac{g_{\mu\nu}}{p^2}+i\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{\Pi(p^2)}{p^2}, \tag{6}$$

with

$$\frac{\Pi(p^2)}{p^2}=\frac{1}{\pi}\int_0^\infty\frac{dt}{t}\,\frac{\operatorname{Im}\Pi(t)}{t-p^2}. \tag{7}$$

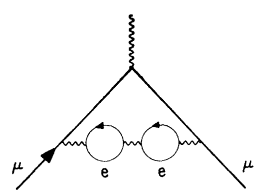

FIG. 3. Double bubble contribution to the difference of the muon and electron g factors in sixth order.

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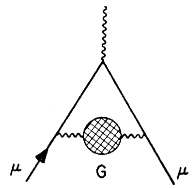

FIG. 4. Diagram representing a set of vacuum polarization graphs contributing to the anomalous magnetic moment of the muon.

The spectral function $\mathrm{Im}\Pi(p^2)$ is given by[17]

$$\theta(p)\,\mathrm{Im}\Pi(p^2)=-\frac{1}{6p^2}\sum_n(2\pi)^4\delta^{(4)}(p-p_n)\times\langle 0|J^\mu(0)|n\rangle\langle n|J_\mu(0)|0\rangle, \quad (8)$$

where $\sum_n$ also means summation over the phase space available to particles in each possible state n.

The contribution to the anomalous magnetic moment of the muon from a given set of vacuum polarization corrections of the type shown in Fig. 4 can be simply obtained in the following way: The photon propagator $-ig_{\mu\nu}/p^2$ in the second-order graph (the graph corresponding to the Schwinger term) is replaced by

$$(-ig_{\mu\nu})\left(-\frac{\Pi^{(G)}(p^2)}{p^2}\right), \quad (9)$$

where

$$\Pi^{(G)}(p^2)$$

is the contribution to $\Pi(p^2)$ from the set of graphs G. From Eq. (7) it can be seen that this expression is a superposition of propagators, with mass squared t, weighted by the function

$$\frac{1}{\pi}\frac{\mathrm{Im}\,\Pi^{(G)}(t)}{t}. \quad (10)$$

It is therefore clear that the resulting contribution to $\frac{1}{2}(g_\mu-2)$ has the following structure:

$$\tfrac{1}{2}(g_\mu-2)^{(G)}=\frac{1}{\pi}\int_0^\infty\frac{dt}{t}\,\mathrm{Im}\,\Pi^{(G)}(t)K_\mu^{(2)}(t), \quad (11)$$

where

$$K_\mu^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)\int_0^1 dz\frac{z^2(1-z)}{z^2+(t/m_\mu^2)(1-z)} \quad (12)$$

is the second-order contribution to $\frac{1}{2}(g_\mu-2)$ from the exchange of a photon with squared mass t.

The explicit form of $K_\mu^{(2)}(t)$ is the following[18]:

for $0\leq t\leq 4m_\mu^2$, with $\tau=t/4m_\mu^2$

$$K_\mu^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)\left[\tfrac{1}{2}-4\tau-4\tau(1-2\tau)\ln(4\tau)-2(1-8\tau+8\tau^2)(\tau/(1-\tau))^{1/2}\arccos\sqrt{\tau}\right]; \quad (13a)$$

for $t\geq 4m_\mu^2$, with $x=\dfrac{1-(1-4m_\mu^2/t)^{1/2}}{1+(1-4m_\mu^2/t)^{1/2}}$

$$K_\mu^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)\left(\tfrac{1}{2}x^2(2-x^2)+(1+x)^2(1+x^2)\times\frac{\ln(1+x)-x+\frac{1}{2}x^2}{x^2}+\frac{1+x}{1-x}x^2\ln x\right). \quad (13b)$$

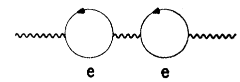

FIG. 5. The double bubble contribution to the fourth-order vacuum polarization.

The Feynman diagrams which contribute to the fourth-order vacuum polarization have been drawn in Figs. 5 and 6. Their contribution to $(1/\pi)\,\mathrm{Im}\Pi^{(4)}(t)$ has been calculated by Källén and Sabry,[19] and their result is the following:

$$\frac{1}{\pi}\mathrm{Im}\Pi^{(4)}(t)=\frac{1}{\pi}\mathrm{Im}\Pi^{*(4)}(t)-2\,\mathrm{Re}\Pi^{(2)}(t)\frac{1}{\pi}\mathrm{Im}\Pi^{(2)}(t). \quad (14)$$

Here $(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t)$ is the contribution from the set of proper graphs shown in Fig. 6, while

$$-2\,\mathrm{Re}\Pi^{(2)}(t)(1/\pi)\,\mathrm{Im}\Pi^{(2)}(t)$$

is the contribution from the double bubble graph shown in Fig. 5. Explicitly: with $\delta=(1-4m_e^2/t)^{1/2}$,

$$\frac{1}{\pi}\mathrm{Im}\Pi^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)(\tfrac{1}{2}-\tfrac{1}{6}\delta^2)\delta\theta(t-4m_e^2), \quad (15a)$$

$$\mathrm{Re}\Pi^{(2)}(t)=\left(\frac{\alpha}{\pi}\right)\left[\frac{8}{9}-\frac{1}{3}\delta^2+(\tfrac{1}{2}-\tfrac{1}{6}\delta^2)\delta\ln\frac{1-\delta}{1+\delta}\right], \quad \text{for } t\geq 4m_e^2; \quad (15b)$$

and

$$\frac{1}{\pi}\mathrm{Im}\Pi^{*(4)}(t)=\left(\frac{\alpha}{\pi}\right)^2\Bigg\{\delta\left[\tfrac{5}{8}-\tfrac{3}{8}\delta^2+(-\tfrac{1}{2}+\tfrac{1}{6}\delta^2)\times\ln(64\delta^4/(1-\delta^2)^3)\right]+\left[\frac{11}{16}+\frac{11}{24}\delta^2-\frac{7}{48}\delta^4+(\tfrac{1}{2}+\tfrac{1}{3}\delta^2-\tfrac{1}{6}\delta^4)\ln\frac{(1+\delta)^3}{8\delta^2}\right]\times\ln\frac{1+\delta}{1-\delta}-(\tfrac{1}{2}+\tfrac{1}{3}\delta^2-\tfrac{1}{6}\delta^4)\times\left[4\Phi\left(-\frac{1-\delta}{1+\delta}\right)+2\Phi\left(\frac{1-\delta}{1+\delta}\right)+\tfrac{1}{2}\pi^2\right]\Bigg\}\theta(t-4m_e^2), \quad (16)$$

where Φ is the Spence function defined in the Appendix.

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In order to evaluate the integral in Eq. (11), let us first analyze the behavior of the functions $K_\mu^{(2)}(t)$ and $\mathrm{Im}\Pi^{*(4)}(t)$ in the limits:

(a) For t small, the asymptotic expansion of $K_\mu^{(2)}(t)$, given in Eq. (13a), is $(\tau\to 0)$

$$K_\mu^{(2)}(t)=(\alpha/\pi)\left[\tfrac{1}{2}-\pi\sqrt{\tau}-4\tau\ln 4\tau-2\tau+O(\tau^{3/2})\right]. \quad (17)$$

(b) For t large, $(x\to 0)$, the asymptotic expansion of $K_\mu^{(2)}(t)$, given in Eq. (13b), is

$$K_\mu^{(2)}(t)=(\alpha/\pi)\left[\tfrac{1}{3}x+O(x^2\ln x)\right]. \quad (18)$$

(c) At $t=4m_e^2$, i.e., $\delta=0$, the function $\mathrm{Im}\Pi^{*(4)}(t)$ defined in Eq. (16) takes the value

$$(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t=4m_e^2)=(\alpha/\pi)^2\tfrac{1}{4}\pi^2. \quad (19)$$

(d) For t large, the asymptotic expansion of $(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t)$, with

$$\delta=(1-y)^{1/2} \quad\text{and}\quad y=4m_e^2/t, \quad (20)$$

is given by

$$(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t)=(\alpha/\pi)^2\left[\tfrac{1}{4}+\tfrac{3}{4}y+O(y^2\ln y)\right]. \quad (21)$$

Let us now discuss the calculation of the integral in Eq. (11). The contribution from the double bubble graph shown in Fig. 5 to $\frac{1}{2}(g_\mu-2)$ has already been given by Kinoshita.[15] It corresponds to the Feynman diagram shown in Fig. 3. We have also calculated this contribution, using our method, and we obtain the following result:

$$\left(\frac{\alpha}{\pi}\right)^3\left[\frac{2}{9}\left(\ln\frac{m_\mu}{m_e}\right)^2-\frac{25}{27}\ln\frac{m_\mu}{m_e}+\frac{\pi^2}{27}+\frac{317}{324}+O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]. \quad (22)$$

The contribution to $\frac{1}{2}(g_\mu-2)$ from the proper graphs shown in Fig. 6 has not been completely evaluated before. The integral to evaluate, corresponding to the Feynman diagrams shown in Fig. 2 is the following:

$$I\equiv\int_{4m_e^2}^{\infty}\frac{dt}{t}\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)K_\mu^{(2)}(t). \quad (23)$$

We separate it into three terms:

$$I=\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\int_{4m_e^2}^{\infty}\frac{dt}{t}K_\mu^{(2)}(t)+K_\mu^{(2)}(t=0)\int_{4m_e^2}^{\infty}\frac{dt}{t}\frac{1}{\pi}\times\left[\mathrm{Im}\Pi^{*(4)}(t)-\mathrm{Im}\Pi^{*(4)}(t=\infty)\right]+R, \quad (24)$$

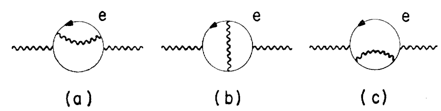

FIG. 6. The contribution to vacuum polarization from the proper fourth-order graphs.

with

$$R=\int_{4m_e^2}^{\infty}\frac{dt}{t}\left(\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\right)\times\left[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)\right]. \quad (25)$$

From Eqs. (17)–(21), it can be seen that all these integrals converge. We shall evaluate them neglecting terms of order $(\alpha/\pi)^3 m_e/m_\mu$ and smaller.

Let us first estimate the integral R defined in Eq. (25). We shall show that it is negligible. This is due to the particular behavior of the two factors in the integrand. The factor $(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t)-(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t=\infty)$ is only appreciable for $t\sim 4m_e^2$, but very small when t becomes of the order of $4m_\mu^2$. On the other hand, the factor $K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)$ is vanishing when $t\sim 4m_e^2$ and only becomes appreciable when $t\sim 4m_\mu^2$. In order to estimate R, we split the interval $4m_e^2$ to ∞ into two parts, obtaining

$$R=R_1+R_2, \quad (26)$$

with

$$R_1=\int_{4m_e^2}^{4m_\mu^2}\frac{dt}{t}\left(\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\right)\times\left[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)\right] \quad (27)$$

and

$$R_2=\int_{4m_\mu^2}^{\infty}\frac{dt}{t}\left(\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\right)\times\left[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)\right]. \quad (28)$$

Using Eq. (21), we find that for $t\geq 4m_\mu^2$,

$$\frac{1}{\pi}\left[\mathrm{Im}\Pi^{*(4)}(t)-\mathrm{Im}\Pi^{*(4)}(t=\infty)\right]\simeq\frac{3}{4}\left(\frac{\alpha}{\pi}\right)^2\frac{4m_e^2}{t};$$

thus,

$$R_2\simeq\frac{3}{4}\left(\frac{\alpha}{\pi}\right)^2\int_{4m_\mu^2}^{\infty}\frac{dt}{t}\frac{4m_e^2}{t}\left[K_\mu^{(2)}(t)-K_\mu^{(2)}(0)\right],$$

where $K_\mu^{(2)}(t)$ is given in Eq. (13b). It follows from Eq. (12) that $K_\mu^{(2)}(t)$ is monotonically decreasing for $t\geq 0$. Hence

$$|R_2|<(\alpha/\pi)^3\tfrac{3}{8}(m_e/m_\mu)^2$$

and therefore we neglect this contribution.

To evaluate R_1, defined in Eq. (27), we need a separation point between large and small values of t.

A natural choice is the geometrical mean of the two limits of integration. Thus we have

$$R_1=R_{11}+R_{12}, \tag{29}$$

with

$$R_{11}=\int_{4m_e^2}^{4m_em_\mu}\frac{dt}{t}(\cdots), \tag{30a}$$

$$R_{12}=\int_{4m_em_\mu}^{4m_\mu^2}\frac{dt}{t}(\cdots). \tag{30b}$$

For t in the interval $4m_e^2\leq t\leq 4m_em_\mu$, we can use the asymptotic expansion of $K^{(2)}(\tau)$ for τ small given in Eq. (17). Then

$$R_{11}=-\left(\frac{\alpha}{\pi}\right)\int_{4m_e^2}^{4m_em_\mu}\frac{dt}{t}\times\left[\tfrac{1}{2}\pi\left(\frac{t}{m_\mu^2}\right)^{1/2}+\frac{t}{m_\mu^2}\ln\frac{t}{m_\mu^2}+\tfrac{1}{2}\frac{t}{m_\mu^2}+\cdots\right]\times\left(\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\right).$$

The dominant part of this integral comes from the $(t/m_\mu^2)^{1/2}$ term. With $z=t/4m_e^2$ we have for the contribution from this term

$$R_{11}\simeq-\left(\frac{\alpha}{\pi}\right)\left(\frac{m_e}{m_\mu}\right)\int_1^{m_\mu/m_e}\frac{dz}{\sqrt{z}}\times\pi\left(\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(z)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(z=\infty)\right).$$

This term is clearly of order m_e/m_μ, the coefficient of which is given by the integral in the right-hand side. This integral converges in the limit $m_\mu/m_e\to\infty$ and could be evaluated analytically. We have evaluated it numerically to be equal to $6.9(\alpha/\pi)^2$. Thus we finally have

$$R_{11}\simeq-(\alpha/\pi)^3\,6.9m_e/m_\mu,$$

a term which we neglect.

For t in the interval $4m_em_\mu\leq t\leq 4m_\mu^2$, we can use the asymptotic expansion of $(1/\pi)\,\mathrm{Im}\Pi^{*(4)}(t)$, corresponding to small $y=4m_e^2/t$, given in Eq. (21). Notice that

$$y=4m_e^2/t=(m_e/m_\mu)^2/\tau.$$

Then R_{12} becomes

$$R_{12}\simeq\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_e}{m_\mu}\right)^2\frac{3}{4}\int_{m_e/m_\mu}^{1}\frac{d\tau}{\tau}\frac{1}{\tau}[K_\mu^{(2)}(\tau)-K_\mu^{(2)}(\tau=0)].$$

Again, we know that in the interval $m_e/m_\mu\leq\tau\leq 1$, we have $K_\mu^{(2)}(\tau)<\alpha/2\pi$. Therefore, we have at least

$$|R_{12}|<(\alpha/\pi)^3\,\tfrac{3}{8}m_e/m_\mu,$$

a contribution which we also neglect.

Next, let us consider the first term in Eq. (24). This is the only term left which still depends on the parameter m_e/m_μ. In order to extract this dependence, we split the interval $4m_e^2$ to ∞ into two parts:

$$L_1=\int_{4m_e^2}^{4m_\mu^2}\frac{dt}{t}K_\mu^{(2)}(t), \tag{31a}$$

$$L_2=\int_{4m_\mu^2}^{\infty}\frac{dt}{t}K_\mu^{(2)}(t). \tag{31b}$$

Of these, only L_1 depends on the mass ratio. We separate out this dependence by writing

$$L_1=2K_\mu^{(2)}(0)\ln\frac{m_\mu}{m_e}+\int_0^{4m_\mu^2}\frac{dt}{t}\times[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)]+S, \tag{32}$$

where

$$S=-\int_0^{4m_e^2}\frac{dt}{t}[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)] \tag{33}$$

can be shown to be of $O(m_e/m_\mu)$ by using Eq. (17).

In conclusion, we therefore have

$$I=\left(\frac{\alpha}{\pi}\right)^3\left[\tfrac{1}{4}\ln\frac{m_\mu}{m_e}+\tfrac{1}{4}(I_1+I_2)+\tfrac{1}{2}I_3+O\left(\frac{m_e}{m_\mu}\right)\right], \tag{34}$$

where

$$\left(\frac{\alpha}{\pi}\right)I_1=\int_0^{4m_\mu^2}\frac{dt}{t}[K_\mu^{(2)}(t)-K_\mu^{(2)}(t=0)], \tag{35}$$

$$\left(\frac{\alpha}{\pi}\right)I_2=\int_{4m_\mu^2}^{\infty}\frac{dt}{t}K_\mu^{(2)}(t), \tag{36}$$

$$\left(\frac{\alpha}{\pi}\right)^2I_3=\int_{4m_e^2}^{\infty}\frac{dt}{t}\times\left[\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t)-\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t=\infty)\right]. \tag{37}$$

These integrals are calculated in Sec. III.

III. EVALUATION OF THE INTEGRALS I_1, I_2, AND I_3

Using the expression for $K_\mu^{(2)}(\tau)$ given in Eq. (13a), we obtain for I_1 in Eq. (35)

$$I_1=\int_0^1 d\tau\left\{-4-4(1-2\tau)\ln 4\tau-2(1-8\tau+8\tau^2)\frac{\arccos(\tau^{1/2})}{[\tau(1-\tau)]^{1/2}}\right\}. \tag{38}$$

The last term in the integrand gives a vanishing contribution to the integral (best seen by putting $\tau=\cos^2\varphi$) and the two first terms are trivial. We find

$$I_1=-2. \tag{39}$$

To integrate I_2 we use the variable

$$x=\frac{1-(1-4m_\mu^2/t)^{1/2}}{1+(1-4m_\mu^2/t)^{1/2}},\quad t=m_\mu^2\frac{(1+x)^2}{x}, \tag{40}$$

and obtain [see Eq. (13b)]

$$I_2=\int_0^1 dx\left(1-\frac{1}{1+x}+x\ln x-x\ln(1+x)+\frac{\ln(1+x)-x+\frac{1}{2}x^2}{x^3}\right). \tag{41}$$

Note that the last term is regular at $x=0$. All terms are now trivial and we obtain

$$I_2=\tfrac{3}{4}-\ln 2. \tag{42}$$

The integral I_3 is essentially the finite part of the fourth-order contribution to the charge renormalization constant Z_3, which is given by

$$Z_3^{(4)}=-\int_{4m_e^2}^{\Lambda^2}\frac{dt}{t}\frac{1}{\pi}\,\mathrm{Im}\Pi^{*(4)}(t), \tag{43}$$

where Λ is an ultraviolet cutoff. We find, up to terms vanishing as $\Lambda\to\infty$,

$$Z_3^{(4)}=-(\alpha/\pi)^2[\tfrac{1}{4}\ln(\Lambda^2/4m_e^2)+I_3]. \tag{44}$$

The divergent term has been calculated previously by Jost and Luttinger.[20] To our knowledge the finite part I_3 has, however, never been evaluated before.[20a]

Using Eq. (11), we find, in terms of the variable

$$\delta=(1-4m_e^2/t)^{1/2}, \tag{45}$$

$$I_3=\int_0^1 d\delta\frac{2\delta}{1-\delta^2}\Bigg\{-\tfrac{1}{4}+\delta[\tfrac{5}{8}-\tfrac{3}{8}\delta^2+(-\tfrac{1}{2}+\tfrac{1}{6}\delta^2)\ln(64\delta^4/(1-\delta^2)^3)]+\ln\frac{1+\delta}{1-\delta}\left[\frac{11}{16}+\frac{11}{24}\delta^2-\frac{7}{48}\delta^4+(\tfrac{1}{2}+\tfrac{1}{3}\delta^2-\tfrac{1}{6}\delta^4)\ln\frac{(1+\delta)^3}{8\delta^2}\right]-(\tfrac{1}{2}+\tfrac{1}{3}\delta^2-\tfrac{1}{6}\delta^4)\times\left[4\Phi\left(-\frac{1-\delta}{1+\delta}\right)+2\Phi\left(\frac{1-\delta}{1+\delta}\right)+\tfrac{1}{2}\pi^2\right]\Bigg\}. \tag{46}$$

[20] R. Jost and J. M. Luttinger, Helv. Phys. Acta **23**, 201 (1949).

[20a] *Footnote added in proof.* The finite part of $Z_3^{(4)}$ has recently been independently evaluated by C. R. Hagen and M. A. Samuel [Phys. Rev. Letters **20**, 1405 (1968)] in a different context.

Introducing the variable

$$x=(1-\delta)/(1+\delta), \tag{47}$$

we find

$$I_3=\int_0^1 dx\{R_1+R_2\ln x+R_3\ln((1+x)(1-x)^2)+R_4[-\ln x\ln((1+x)(1-x)^2)+4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2]\}, \tag{48}$$

where R_i are rational functions. We express them in terms of $1/x$, and

$$y=1/(1+x), \tag{49}$$

and we find

$$R_1=\tfrac{1}{2}(y+y^2-12y^3+12y^4), \tag{50}$$

$$R_2=\tfrac{1}{3}(6y-8y^2-27y^3+45y^4-14y^5), \tag{51}$$

$$R_3=\tfrac{2}{3}(-1/x+2y^2+8y^3-8y^4), \tag{52}$$

$$R_4=\tfrac{2}{3}(-1/x+2y-4y^3+12y^4-8y^5). \tag{53}$$

The integral over R_1 is trivial:

$$I_{31}=\int_0^1 dx\,R_1=\tfrac{1}{2}\ln 2-\tfrac{1}{4}. \tag{54}$$

The second integral only contains one nontrivial term (see the Appendix):

$$\int_0^1 dx\frac{\ln x}{1+x}=-\tfrac{1}{12}\pi^2. \tag{55}$$

In the remaining terms the logarithm disappears after a partial integration, the result being

$$I_{32}=\int_0^1 dx\,R_2\ln x=-\tfrac{1}{6}\pi^2+\frac{10}{3}\ln 2-\frac{55}{72}. \tag{56}$$

In the integral over R_3 only one term is nontrivial, leading to Spence-function values

$$\int_0^1 dx\frac{\ln[(1+x)(1-x)^2]}{x}=-\tfrac{1}{4}\pi^2. \tag{57}$$

After a partial integration, the remaining terms are trivial:

$$I_{33}=\int_0^1 dx\,R_3\ln[(1+x)(1-x)^2]=\tfrac{1}{6}\pi^2-\frac{10}{3}\ln 2+\frac{25}{27}. \tag{58}$$

The integral over R_4 has nontrivial parts coming from the first two terms in Eq. (53). Using

$$\frac{d}{dx}[4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2]=2\frac{\ln[(1+x)(1-x)^2]}{x}, \tag{59}$$

we find by a partial integration

$$\int_0^1 dx\left(-\frac{1}{x}+\frac{2}{1+x}\right)\{-\ln x\ln[(1+x)(1-x)^2]+4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2\}$$
$$=-\pi^2\ln 2+\int_0^1 dx\left[\left(\frac{3}{x}-\frac{2}{1+x}\right)\ln x-\frac{4}{x}\ln(1+x)\right]\times\ln[(1+x)(1-x)^2]. \quad (60)$$

All these terms lead to expressions involving $\zeta(3)$ (see the Appendix), and upon collection of terms we find

$$\frac{2}{3}\int_0^1 dx\left(-\frac{1}{x}+\frac{2}{1+x}\right)\{-\ln x\ln[(1+x)(1-x)^2]+4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2\}=\zeta(3). \quad (61)$$

The remaining terms in (53) can be reduced to Spence-function values and rational expressions. We find

$$\frac{2}{3}\int_0^1 dx[-4y^3+12y^4-8y^5]\times\{-\ln x\ln[(1+x)(1-x)^2]\}=\pi^2/24-\tfrac{2}{3}\ln 2+1/108 \quad (62)$$

and

$$\frac{2}{3}\int_0^1 dx[-4y^3+12y^4-8y^5][4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2]=-\pi^2/24+\tfrac{2}{3}\ln 2-7/54, \quad (63)$$

so that finally we get

$$I_{34}=\int_0^1 dx\,R_4\{-\ln x\ln[(1+x)(1-x)^2]+4\Phi(-x)+2\Phi(x)+\tfrac{1}{2}\pi^2\}$$
$$=\zeta(3)-13/108. \quad (64)$$

Collecting all terms, we find

$$I_3=\sum_{i=1}^{4} I_{3i}=\zeta(3)+\tfrac{1}{2}\ln 2-\frac{5}{24}. \quad (65)$$

IV. CONCLUSIONS

(a) The terms I_1, I_2, and I_3 in Eq. (34) are now known [see Eqs. (39), (42), and (65)]. Therefore, the contribution from the Feynman diagrams in Fig. 2 to $\frac{1}{2}(g_\mu-g_e)^{(6)}$, which we have called I, is the following:

$$I=(\alpha/\pi)^3[\tfrac{1}{4}\ln(m_\mu/m_e)+\tfrac{1}{2}\zeta(3)-5/12+O(m_e/m_\mu)]$$
$$\simeq 1.52(\alpha/\pi)^3.$$

(b) Our result for the contribution from the Feynman diagram shown in Fig. 3 to $\frac{1}{2}(g_\mu-g_e)^{(6)}$ agrees with that obtained by Kinoshita[15]:

$$\left(\frac{\alpha}{\pi}\right)^3\left[\frac{2}{9}\left(\ln\frac{m_\mu}{m_e}\right)^2-\frac{25}{27}\ln\frac{m_\mu}{m_e}+\frac{\pi^2}{27}+\frac{317}{324}+O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]$$
$$\simeq 2.72(\alpha/\pi)^3.$$

(c) We have also calculated the contribution to $\frac{1}{2}(g_e-2)^{(4)}$ from the Feynman diagram shown in Fig. 1(b). The result is

$$\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2+O\left[\left(\frac{m_e}{m_\mu}\right)^4\ln\frac{m_\mu}{m_e}\right]\right\}=5.2\times 10^{-7}(\alpha/\pi)^2.$$

(d) As a byproduct of our calculation of I_3 [see Eqs. (37) and (44)] we have obtained the finite part of the fourth-order contribution to the charge renormalization constant Z_3. Thus, up to terms vanishing as $\Lambda\to\infty$, we have

$$Z_3^{(4)}=-(\alpha/\pi)^2(\tfrac{1}{2}\ln(\Lambda/m_e)+\zeta(3)-5/24).$$

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APPENDIX

In this Appendix we list the special integrals which have been used in the calculation discussed in the text.

The function $\Phi(x)$ (which we call the Spence function) is defined for complex z by the integral[21]

$$\Phi(z)=\int_1^z dt\frac{\ln(1+t)}{t} \quad \textbf{(A1)}$$

along any path that does not cross the real axis between $-\infty$ and -1. It has a cut from $-\infty$ to -1 along the real axis, and on the real axis it is conventionally defined by

$$\Phi(x)=\lim_{\epsilon\to 0}\mathrm{Re}\,\Phi(x+i\epsilon)=\int_1^x dt\frac{\ln|1+t|}{t}. \quad \text{(A2)}$$

In the preceding sections we have only used the Spence function for values between -1 and 1. For a detailed discussion, we refer to Källén and Sabry.[19] Various integrals can be reduced to special values of the Spence function:

$$\Phi(0)=-\tfrac{1}{12}\pi^2, \quad \text{(A3)}$$

$$\Phi(-1)=-\tfrac{1}{4}\pi^2. \quad \text{(A4)}$$

[21] This function is closely related to the dilogarithm [Spence's integral for $n=2$ as defined, e.g., in M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1965), Eq. (27.7.1)].

Below we list the integrals used in the text:

$$\int_0^1 dx \frac{\ln(1+x)}{x} = -\Phi(0), \tag{A5}$$

$$\int_0^1 dx \frac{\ln x}{1+x} = \Phi(0), \tag{A6}$$

$$\int_0^1 dx \frac{\ln(1-x)}{x} = \Phi(-1) - \Phi(0), \tag{A7}$$

$$\int_0^1 dx \frac{\ln x}{1-x} = \Phi(-1) - \Phi(0). \tag{A8}$$

For $x > 1$, Riemann's ζ function

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{A9}$$

has the integral representation[22]

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^1 dt \frac{[-\ln(1-t)]^{x-1}}{t}. \tag{A10}$$

Especially, we have

$$\zeta(3) = \frac{1}{2} \int_0^1 dt \frac{[\ln(1-t)]^2}{t} = 1.202056903\cdots. \tag{A11}$$

[22] This follows from Eq. (23.2.7) in Abramowitz and Stegun (Ref. 21).

A host of integrals can be expressed in terms of $\zeta(3)$. We list below the ones needed in the text:

$$\int_0^1 dx \frac{\ln x \ln(1+x)}{x} = -\tfrac{3}{4}\zeta(3), \tag{A12}$$

$$\int_0^1 dx \frac{\ln x \ln(1-x)}{x} = \zeta(3), \tag{A13}$$

$$\int_0^1 dx \frac{\ln x \ln(1+x)}{1+x} = -\tfrac{1}{8}\zeta(3), \tag{A14}$$

$$\int_0^1 dx \frac{\ln x \ln(1-x)}{1+x} = -\tfrac{1}{4}\pi^2 \ln 2 + \frac{13}{8}\zeta(3), \tag{A15}$$

$$\int_0^1 dx \frac{[\ln(1+x)]^2}{x} = \tfrac{1}{4}\zeta(3), \tag{A16}$$

$$\int_0^1 dx \frac{\ln(1+x) \ln(1-x)}{x} = -\tfrac{5}{8}\zeta(3). \tag{A17}$$

The derivation of Eq. (61) can be somewhat simplified if the following integrals are used:

$$\int_0^1 dx \frac{\Phi(x) - \Phi(0)}{x} = \tfrac{3}{4}\zeta(3), \tag{A18}$$

$$\int_0^1 dx \frac{\Phi(-x) - \Phi(0)}{x} = -\zeta(3), \tag{A19}$$

$$\int_0^1 dx \frac{\Phi(x)}{1+x} = -\tfrac{1}{4}\zeta(3), \tag{A20}$$

$$\int_0^1 dx \frac{\Phi(-x)}{1+x} = \tfrac{5}{8}\zeta(3) - \tfrac{1}{4}\pi^2 \ln 2. \tag{A21}$$