Rare Decay Mode $\eta \to \pi^+\pi^-\pi^0\gamma$ and the Relation between Vector-Dominance and Current-Algebra Calculations

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The decay rate for $\eta \to \pi^+\pi^-\pi^0\gamma$ has been calculated when the photon is emitted in the $M1$ state. Since for such a transition the three pions are in a $3^P=1^-$ and $I=0$ state, the ω-meson-dominant calculation is expected to give a reasonable estimate of the rate of the $\eta \to \pi^+\pi^-\pi^0\gamma$ decay mode. Further, comparison of the vector-meson-dominant result with that obtained by applying current commutation relations predicts that these two results are equivalent, provided the Kawarabayashi-Suzuki relation $g_{\omega 3\pi}=3\mu^2F_\pi^{-2}g_{\omega\pi\gamma}$ be valid when $g_{\omega 3\pi}$ and $g_{\pi\omega\gamma}$ are the coupling constants in $\omega \to 3\pi$ and $\omega \to \pi\gamma$ decays. The ω-meson-dominant calculation together with the Kawarabayashi-Suzuki relation predicts $\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \to \pi^0 2\gamma)=0.2\%$, in reasonable agreement with experiment.

1. INTRODUCTION

It is well known that there has been a renewed interest in estimating correctly the various η-meson decays. This essentially stems from the fact that $\eta \to 3\pi$ branching ratios have of late been of interest following a number of calculations based on current algebra. Further, because of the probable C-violation effects in η decays, various rare modes of C-conserving η decays—namely, the $\eta \to \gamma e^+e^-$, $\gamma\mu^+\mu^-$, e^+e^-, $\mu^+\mu^-$, etc., modes—have been theoretically calculated, and experimental searches for these decay modes are also in progress. In a similar fashion there has recently been some experimental indication of the probable existence[1] of the rare decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$.

The present experimental result[2] states that the branching ratio $R=\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma)/\Gamma(\pi^+\pi^-\pi^0) \leq 0.9\%$ and $R'=\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \to \pi^0\gamma\gamma) < 0.6\%$. Normally, one expects, purely from the phase-space factor and the magnitude of the fine-structure constant, that the branching ratio $R \simeq 1$. However, one then notices that the two pions in the $3\pi\gamma$ mode must be in a state of relative angular momentum $l=1$, leading to a centrifugal barrier effect which then suppresses the decay rate. An upper limit of $R' \simeq 0.23\%$ was predicted by Singer,[3] using a ρ-dominant model for the decay mechanism. If C conservation holds and the photon in $3\pi\gamma$ decay is emitted through an $M1$ transition, then the 3π state has $T=0$ and $J^P=1^-$, whereas for the $E1$ transition $J^P=1^+$; hence π^0 will be in an s state relative to $\pi^+\pi^-$, and thus the decay then essentially follows from the ρ-dominant model. An interesting calculation based on current algebra has been performed by Sarker.[4] He has considered the case when $E1$ is dominant. This is essentially equivalent to the ρ-dominant calculation. The recent calculation of Intemann and Lapidus[5] based on current algebra also confirms[6] this contention. The purpose of the present paper is to estimate the $M1$ transition amplitude for $\eta \to \pi^+\pi^-\pi^0\gamma$ decay using both the ω-dominance mechanism (which has the right quantum number as the pions are in the $J^P=1^-$ and $I=0$ state) and the algebra-of-currents[7] approach coupled with the hypothesis of partially conserved axial-vector current (PCAC).[8]

In Sec. 2 we calculate the matrix elements for $\eta \to 3\pi\gamma$ and $\eta \to \pi^0 2\gamma$ in the ω-dominance model. In the subsequent section we employ the techniques of the algebra of currents and show further the relationship of this calculation to the vector-dominance calculation for the $\eta \to 3\pi\gamma$ decay process. It is interesting to note that this can be established through the Kawarabayashi-Suzuki[9] relation [see Eq. (3.4) below] between $g_{\omega 3\pi}$ and $g_{\omega\pi\gamma}$ couplings defined in Eqs. (2.1) and (2.2). Using the Kawarabayashi-Suzuki relation [see Eq. (3.4) below] and vector-dominance matrix elements, we find R' $\sim 0.20\%$. Further, to distinguish the various vector-dominance calculations, we calculate the energy spectrum of the photon in $\eta \to 3\pi\gamma$.

2. MATRIX ELEMENT IN THE ω-DOMINANCE MODEL

In the ω-dominance model, we use the following direct coupling interaction Lagrangians

$$\mathcal{L}_I^1(\omega 3\pi)=(g_{\omega 3\pi}/\mu^3)\epsilon_{ijk}\epsilon_{\mu\nu\rho\sigma}q_\mu^{(i)}q_\nu^{(j)}q_\sigma^{(k)}\omega_\sigma\,, \tag{2.1}$$

$$\mathcal{L}_I^2(\omega\eta\gamma)=(g_{\omega\eta\gamma}/\mu)\epsilon_{\lambda\nu\rho\sigma}\epsilon_\lambda^{(\gamma)}\omega_\nu p_\rho^{(\eta)}k_\sigma^{(\gamma)}\,, \tag{2.2}$$

$$\mathcal{L}_I^3(\omega\pi\gamma)=(g_{\omega\pi\gamma}/\mu)\epsilon_{\lambda\nu\rho\sigma}\epsilon_\lambda^{(\gamma)}\omega_\gamma q_\rho^{(\pi)}k_\sigma^{(\gamma)}\,, \tag{2.3}$$

where $g_{\omega 3\pi}$, $g_{\omega\eta\gamma}$, and $g_{\omega\pi\gamma}$ are the dimensionless coupling constants for the respective interaction constants. The value of $g_{\omega 3\pi}$ is related to the observed decay width of $\omega \to 3\pi$ mode and $g_{\omega\pi\gamma}$ to the observed decay width of

[1] S. M. Flatté, Phys. Rev. Letters **18**, 976 (1967).
[2] L. R. Price and F. S. Crawford, Phys. Rev. Letters **18**, 1207 (1967).
[3] P. Singer, Phys. Rev. **154**, 1592 (1967).
[4] A. Q. Sarker, Phys. Rev. Letters **19**, 1261 (1967).
[5] G. W. Intemann and I. R. Lapidus, Phys. Rev. **165**, 1650 (1968).
[6] The current-algebra calculation of Sarker leads to a value of $R' \simeq 0.42\%$, whereas Intemann and Lapidus give a value of 0.23% only.
[7] M. Gell-Mann, Physica **1**, 63 (1964).
[8] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[9] K. R. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966).

$\omega\rightarrow\pi\gamma$. We need not evaluate $g_{\omega\eta\gamma}$ as it does not occur in R'.

The matrix element for $\eta\rightarrow 3\pi\gamma$ can be written down quite simply, using (2.1) and (2.2)

$$M_{(\eta\rightarrow 3\pi\gamma)}=\frac{g_{\omega\eta\gamma}g_{\omega 3\pi}}{\mu^4}\frac{2}{(p_\eta-k)^2-m_\omega^2}A',$$

$$A'=[q_a{}^\pi\cdot k^\gamma q_b{}^\pi\cdot p^\eta q_c{}^\pi\cdot\epsilon^\gamma-q_a{}^\pi\cdot k^\gamma p^\eta\cdot q_c{}^\pi\epsilon^\gamma\cdot q_b{}^\pi$$
$$-q_b{}^\pi\cdot k^\gamma q_a{}^\pi\cdot p^\eta q_c{}^\pi\cdot\epsilon^\gamma+q_b{}^\pi\cdot k^\gamma p^\eta\cdot q_c{}^\pi\epsilon^\gamma\cdot q_a{}^\pi$$
$$-q_c{}^\pi\cdot k^\gamma q_a{}^\pi\cdot p^\eta q_b{}^\pi\cdot\epsilon^\gamma+q_c{}^\pi\cdot k^\gamma p^\eta\cdot q_b{}^\pi q_a{}^\pi-\epsilon^\gamma], \quad (2.4)$$

and similarly, using (2.2) and (2.3), we have

$$M_{(\eta\rightarrow\pi 2\gamma)}=\frac{g_{\omega\eta\gamma}g_{\omega\pi\gamma}}{\mu^2}\frac{4}{(p_\eta-k)^2-m_\omega^2}B',$$

$$B'=[k'^\gamma\cdot k^\gamma q^\pi\cdot p^\eta\epsilon'^\gamma\cdot\epsilon^\gamma-k'^\gamma\cdot k^\gamma\epsilon'^\gamma\cdot p^\eta\epsilon^\gamma\cdot q^\pi$$
$$-q^\pi\cdot k^\gamma k'^\gamma\cdot p^\eta\epsilon'^\gamma\cdot\epsilon^\gamma+q^\pi\cdot k^\gamma\epsilon'^\gamma\cdot p^\eta\epsilon^\gamma\cdot k'^\gamma$$
$$+\epsilon'^\gamma\cdot k^\gamma k'^\gamma\cdot p^\eta q^\pi\cdot\epsilon^\gamma-\epsilon'^\gamma\cdot k^\gamma p^\eta\cdot q^\pi k'^\gamma\cdot\epsilon'^\gamma]$$
$$+(k^\gamma\leftrightarrow k'^\gamma). \quad (2.5)$$

In writing the above expressions (2.4) and (2.5), we have suppressed the normalization factor for simplicity.

From (2.4) and (2.5), the branching ratio is then given in our ω-dominance model by

$$R'=\Gamma_{(\eta\rightarrow\pi^+\pi^-\pi^0\gamma)}/\Gamma_{(\eta\rightarrow\pi^0\gamma\gamma)}, \quad (2.6)$$

where

$$\Gamma_{(\eta\rightarrow 3\pi\gamma)}=\frac{1}{2M_\eta}\int\frac{d^3k^\gamma}{(2\pi)^3}\frac{d^3q_a{}^\pi}{(2\pi)^3}\frac{d^3q_b{}^\pi}{(2\pi)^3}\frac{d^3q_c{}^\pi}{(2\pi)^3}$$
$$\times\frac{1}{16k_0{}^\gamma q_{0a}{}^\pi q_{0b}{}^\pi q_{0c}{}^\pi}(2\pi)^4\delta^4(k^\gamma+q_a{}^\pi+q_b{}^\pi+q_c{}^\pi-p^\eta)$$
$$\times|M(\eta\rightarrow 3\pi\gamma)|^2 \quad (2.7)$$

and

$$\Gamma_{(\eta\rightarrow\pi^0\gamma\gamma)}=\frac{1}{2M_\eta}\int\frac{d^3q^\pi}{(2\pi)^3}\frac{d^3k^\gamma}{(2\pi)^3}\frac{d^3k'^\gamma}{(2\pi)^3}\frac{1}{8q_0{}^\pi k_0{}^\gamma k_0'^\gamma}$$
$$\times(2\pi)^4\delta^4(k^\gamma+k'^\gamma+q^\pi-p^\eta)|M_{(\eta\rightarrow\pi^0\gamma\gamma)}|^2. \quad (2.8)$$

3. CURRENT-ALGEBRA CALCULATION AND RELATIONSHIP TO VECTOR DOMINANCE

In order to apply the current-commutator relation for $M1$ photon decay in $\eta\rightarrow 3\pi\gamma$, we must disperse the three pions simultaneously because the quantum number of the three-pion system suggests that any two pions are in relative p wave leading to an $I=0$ state. It may be remarked that it is not necessary to disperse all the pions and the photon simultaneously (as was done in Ref. 5), and doing so does not lead to any new result except that of making the reduction phenomenon extremely involved. Following Chang,[10] we can write down the reduction of the three pions, and the matrix element $\langle 3\pi\gamma|T|\eta\rangle$ for the decay process $\langle\pi^a\pi^b\pi^c A^\rho|\eta\rangle$ contains the following set of matrix elements:

$$\int d^4xd^4yd^4z\,e^{iq_ax}e^{iq_by}e^{iq_cz}[(-iq_a{}^\mu)(-iq_b{}^\nu)(-iq_c{}^\lambda)$$
$$\times\langle A_\lambda|T\{A_a{}^\mu(x)A_b{}^\nu(y)A_c{}^\lambda(z)\}|\eta\rangle], \quad (3.1a)$$

$$\int d^4xd^4y\,e^{i(q_a+q_c)\cdot x}e^{iq_by}(-iq_a{}^\mu)(-iq_b{}^\nu)\epsilon_{cbd}$$
$$\times\langle A^\rho|T\{K_d{}^\mu(x),A_c{}^\nu(y)\}|\eta\rangle, \quad (3.1b)$$

$$\int d^4x\,e^{i(q_a+q_b+q_c)x}e^{iq_by}(-iq_a{}^\mu)\epsilon_{abd}\epsilon_{cde}$$
$$\times\langle\pi^\rho|A_e{}^\mu(x)|\eta\rangle, \quad (3.1c)$$

$$\int d^4xd^4y\,e^{iq_ax}e^{i(q_b+q_c)y}(-iq_a{}^\mu)$$
$$\times\langle\epsilon^\rho|T\{A_a{}^\mu(x),\sigma_{bc}(y)\}|\eta\rangle, \quad (3.1d)$$

$$\int d^4x\,e^{i(q_a+q_b+q_c)x}\delta_{bc}\langle\epsilon^\rho|\phi_{\pi_a}(x)|\eta\rangle. \quad (3.1e)$$

In evaluating terms in (3.1d) and (3.1e) we have utilized the current commutation relations (CCR) with the σ model as in Weinberg.[11] It may be noted that the terms in (3.1e) above represent the decay $\eta\rightarrow\pi+\gamma$, which is forbidden if charge-conjugation invariance is demanded, and similarly the terms in (3.1c) represent $A_1\rightarrow\eta+\gamma$ decay which is also forbidden by the same argument. We drop the terms in (3.1a), as they are highly momentum-dependent, i.e., cubic in the pion momenta, as has been shown by Rubinstein and Veneziano[12] and by Intemann and Lapidus.[5] The highly momentum-dependent form factor leads to a smaller branching ratio compared with the other terms. If further we drop the σ-dependent term in (3.1d) which represents $A_1\rightarrow\eta+\sigma+\gamma$ ($A_1{}^0\rightarrow\eta^0+\sigma^0+\gamma$), then we are left with terms in (3.1b) only. The neglect of the terms in (3.1d) may be justified as the decay $A_1\rightarrow\eta+\sigma+\gamma$ is not seen at all now. Thus we notice from the above discussion that the $\eta\rightarrow 3\pi\gamma$ decay process is related directly to the $\eta\rightarrow\pi 2\gamma$ process. Using the PCAC relation and rewriting the T product in (3.1b), we obtain the following expression for the $\eta\rightarrow 3\pi\gamma$ decay matrix element in the $q^2\rightarrow 0$ limit:

$$\langle\pi(q_a)\pi(q_b)\pi(q_c)\epsilon^\rho(k)|\eta(p_\eta)\rangle=3iF_\pi{}^{-2}(2\pi)^{-q/2}$$
$$\times(8q_aq_bq_c)^{-1/2}\int d^4xd^4y\,e^{i(q_b+q_a)x+iq_cy}(q_b-q_a)\epsilon_\mu\epsilon_{abd}$$
$$\times\langle\epsilon^\rho(k)|T[\phi_{\pi_c}(q_c),V_d{}^\mu(x)]|\eta(p_\eta)\rangle$$
$$+(\text{symmetric terms in } bc \text{ and } ca),$$

[10] Lay-Nam Chang, Phys. Rev. **162**, 1497 (1967).
[11] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966).
[12] H. R. Rubinstein and S. Veneziano, Phys. Rev. Letters **18**, 411 (1967); see also J. Pasupathy and R. E. Marshak, *ibid.* **17**, 888, (1966).

where in Eq. (1b) the right-hand side is essentially the matrix elements for the $\eta \to \pi^0 2\gamma$ decay. We can then write

$$\langle \epsilon^\rho | \eta \rangle = (3/6) F_\pi^{-2} (2\pi)^{-q/2} (8 q_a q_b q_c)^{-1/2} (q_b - q_a)_\mu \times \epsilon_{abd} M_{cd}{}^{\mu\rho} (p_\eta, (q_a + q_b)_k, q_c) + \text{symmetric terms},$$

where M denotes the $\eta \to \pi^0 2\gamma$ invariant matrix element. If we now use ω dominance for $\eta \to \pi^0 2\gamma$ decay, then we can write down the following expression for $M_{cd}{}^{\mu\rho}$:

$$M_{cd}{}^{\mu\rho} = \frac{g_{\omega\pi\gamma} g_{\eta\omega\gamma}}{\mu^2} \delta_{cd} \epsilon^{\mu\nu\sigma\epsilon} (q_a + q_b)^\nu q_c{}^\sigma \frac{(\delta^{\epsilon\phi} - p^\epsilon p^\phi / m_\omega{}^2)}{(p^2 - m_\omega{}^2)} \epsilon^{\phi\alpha\beta\rho} p_\eta{}^\alpha k^\beta + \text{symmetric terms}, \quad (3.2)$$

where

$$p = (p_\eta - k).$$

We thus obtain finally

$$\langle 3\pi\gamma | \eta \rangle = 3 F_\pi^{-2} (2\pi)^{-q/2} (8 q_a q_b q_c)^{-1/2} \epsilon_{abc} \frac{q_{\omega\pi\gamma} g_{\eta\omega\gamma}}{\mu^2} \times \left\{ \epsilon^{\mu\nu\sigma\epsilon} q_a{}^\mu q_b{}^\nu q_c{}^\sigma \frac{(\delta^{\epsilon\phi} - p^\epsilon p^\phi / m_\omega{}^2)}{(p^2 - m_\omega{}^2)} \epsilon^{\phi\alpha\beta\rho} p_\eta{}^\alpha k^\beta \epsilon^\rho \right\}. \quad (3.3)$$

In writing Eq. (3.2) we have used the Lagrangian Eq. (2.1).

In the ω-dominance model for $\eta \to 3\pi\gamma$, we have suppressed the normalization factor, using the Lagrangian, Eq. (2.1):

$$\langle 3\pi\gamma | \eta \rangle = \frac{g_{\omega 3\pi} g_{\eta\omega\gamma}}{\mu^4} \left[\epsilon^{\lambda\nu\rho\sigma} \epsilon_\gamma{}^\lambda p_\eta{}^\rho k_\gamma{}^\sigma \frac{\{\delta^{\epsilon\nu} - p^\epsilon p^\nu / m_\omega{}^2\}}{(p^\eta - k)^2 - m_\omega{}^2} \times \epsilon^{\mu\nu\sigma\epsilon} q_a{}^\mu q_b{}^\nu q_c{}^\sigma \right].$$

The vector-domiance and the current-algebra results are identical if

$$g_{\omega\to 3\pi} = 3(\mu^2 / F_\pi{}^2) g_{\omega\pi\gamma}, \quad (3.4)$$

which is the Karawabayashi-Suzuki relation. This relation was obtained from a direct application of the current commutation techniques to the $\omega \to 3\pi$ decay process in Ref. 9.

Having established the connection between the vector dominance and CCR results, we now compute the branching ratio for the $\eta \to 3\pi\gamma$ and $\eta \to \pi^0 2\gamma$ decay processes. Using the vector-meson dominant result, Eqs. (2.7) and (2.8), for $\eta \to 3\pi\gamma$ and $\eta \to \pi^0 2\gamma$ and the Kawarabayashi-Suzuki relation (3.4), we find[12]

$$R' = \Gamma(\eta \to 3\pi\gamma) / \Gamma(\eta \to \pi\gamma\gamma) \simeq 0.2\%.$$

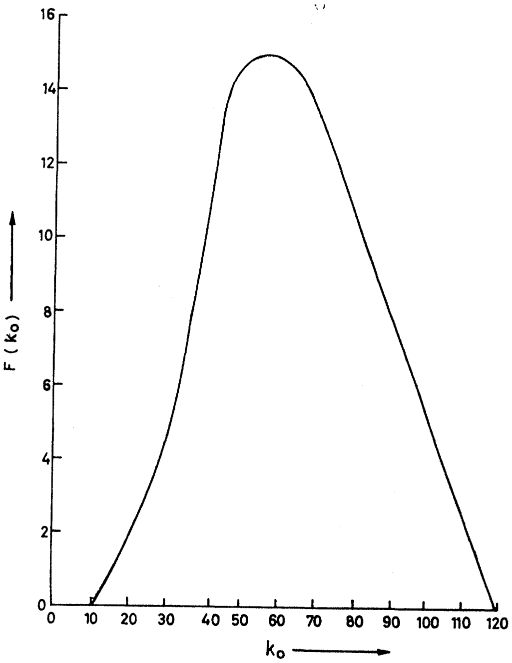

FIG. 1. The calculated photon energy spectrum in the decay $\eta \to \pi^+ \pi^- \pi^0 \gamma$ in the η rest frame, using the ω-dominance model. (k_0 is given in MeV.)

4. CONCLUSION

It remains now to discuss the inner bremsstrahlung contribution to the $\eta \to 3\pi\gamma$ decay. As pointed out in Ref. 5, the direct emission term is 5 times as large as the inner bremsstrahlung contribution, so we have not considered the bremsstrahlung contribution for $M1$ photon emission. It was also pointed out in Ref. 5 that Singer's ρ-dominance calculation agrees with[13] that obtained by current algebra, a result demonstrating the equivalence of the two calculations for $\eta \to 3\pi\gamma$ decay when a photon is emitted in the $E1$ state. The present calculation shows similarly that for $M1$ photon emission both methods again give identical[14] results if we assume the Kawarabayashi-Suzuki (K-S) relation [Eq. (3.4)] and further that the neglect of σ terms is justified.[15] Here we would like to remark that Sakurai[16] has pointed out the equivalence of current algebra and ρ-pole-dominance calculations, through the use of the K-S relation $g_\rho{}^2 = 2 m_\rho{}^2 F_\pi{}^2$. From the very small observed rate $\eta \to 3\pi\gamma$, it is unlikely that the σ meson contributes significantly. However, this point needs further investigation.

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[13] We do not give details of the phase-space calculation as it is given in great detail in Ref. 5.

[14] See, however, Ref. 6.

[15] We must emphasize here that the ω-dominance $\eta \to 3\pi\gamma$ amplitude agrees with the current-algebra result if we calculate $\eta \to \pi^0 2\gamma$ decay via the ω meson again which seems to be quite reasonable [see Alles *et al.*, Nuovo Cimento **45**, 272 (1966)].

[16] J. J. Sakurai, Phys. Rev. **156**, 1508 (1967).

In the above calculation we have also not used the mixing between η^0 and X^0. As has been pointed out in Ref. 4, its inclusion reduces the rate by a factor of 0.68 or enhances it to double its value depending on the sign of the mixing angle, and it is also easy to predict a similar branching ratio for the decays. We find that $R'\sim0.20$ for the $M1$ transition case, to be compared to the rate ~0.42 when $E1$ is predominant (see Ref. 6). In any case the experimental limit is $\lesssim0.9\%$ or 0.6% which is quite large. Hence the $M1$ transition can also contribute appreciably. In Fig. 1 we have plotted the photon spectrum in the ω-dominance model.

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Calculation of the Sixth-Order Contribution from the Fourth-Order Vacuum Polarization to the Difference of the Anomalous Magnetic Moments of Muon and Electron*

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We present the details of a calculation of the sixth-order contribution to $\frac{1}{2}(g_\mu-g_e)$ from the proper fourth-order vacuum polarization. As a byproduct of this calculation, we have also obtained the finite part of the fourth-order contribution to the charge-renormalization constant Z_3.

I. INTRODUCTION

ONE of the classical successes of quantum electrodynamics has been the prediction of radiative corrections to the Dirac value of the gyromagnetic ratio of the electron and of the muon. To first order in the fine structure constant α, these corrections[1] are predicted to be the same for the electron and the muon:

$$\tfrac{1}{2}(g_e-2)^{(2)}=\tfrac{1}{2}(g_\mu-2)^{(2)}=\alpha/2\pi. \quad (1)$$

This is, however, no longer true at higher orders. Already at fourth order in the electric charge constant e ($e^2/4\pi=\alpha$), the Feynman diagram shown in Fig. 1(a) gives a sizable contribution to $\frac{1}{2}(g_\mu-2)$, while the corresponding diagram obtained by interchanging the muon and electron lines [see Fig. 1(b)] gives a very small contribution to $\frac{1}{2}(g_e-2)$. All other diagrams involve only one kind of lepton, and therefore their contributions do not depend on the masses.

The total contribution to the electron g factor in fourth order is given by

$$\tfrac{1}{2}(g_e-2)^{(4)}=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{197}{144}+\tfrac{1}{12}\pi^2+\tfrac{3}{4}\zeta(3)-\tfrac{1}{2}\pi^2\ln2+\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2+O\left[\left(\frac{m_e}{m_\mu}\right)^4\ln\frac{m_\mu}{m_e}\right]\right\}=-0.3284784(\alpha/\pi)^2, \quad (2)$$

where $\zeta(3)$ is the Riemann zeta function of argument 3, defined in the Appendix. The terms independent of the ratio m_e/m_μ were calculated by Karplus and Kroll,[2] Sommerfield,[3] and Petermann[4] using standard quantum electrodynamics, and by Terent'ev[5] using dispersion techniques. We have calculated the term $(1/45)\times(m_e/m_\mu)^2$, which comes from the diagram shown in Fig. 1(b).[6]

The corresponding contribution to the muon g factor in fourth order is

$$\tfrac{1}{2}(g_\mu-2)^{(4)}=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{97}{144}+\tfrac{1}{12}\pi^2+\tfrac{3}{4}\zeta(3)-\tfrac{1}{2}\pi^2\ln2+\tfrac{1}{3}\ln\frac{m_\mu}{m_e}+\tfrac{1}{4}\pi^2\frac{m_e}{m_\mu}-4\left(\frac{m_e}{m_\mu}\right)^2\ln\frac{m_\mu}{m_e}+3\left(\frac{m_e}{m_\mu}\right)^2+O\left[\left(\frac{m_e}{m_\mu}\right)^3\right]\right\}=(+0.765779\pm7\times10^{-6})(\alpha/\pi)^2. \quad (3)$$

This includes the contribution from the diagram shown in Fig. 1(a), which was first estimated by Suura and

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[2] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950). Their calculation, however, contained an error which was corrected by Sommerfield (Ref. 3) and Petermann (Ref. 4).
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[5] M. V. Terent'ev, Zh. Eksperim. i Teor. Fiz. **43**, 619 (1962) [English transl.: Soviet Phys.—JETP **16**, 444 (1963)].
[6] To our knowledge, this term has not been taken into account before.