Chiral Symmetry from Divergence Requirements

D. B. FAIRLIE AND K. YOSHIDA
Department of Mathematics, University of Durham, Durham, England
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The standard nonlinear pion and pion-nucleon Lagrangians which exhibit chiral invariance are derived by imposing conservation of the vector and the axial currents constructed from the fields. It is observed that the pion Lagrangian may be written in the form $-(V^2+A^2)$, and the standard pion-nucleon Lagrangian is modified, while retaining chiral invariance, in order to admit a similar representation by the addition of 2 four-fermion interactions. Contributions of these additional terms to low-energy s-wave nucleon-nucleon scattering are assessed, and are shown to contribute in the correct experimental proportions to the different channels. Finally, the results of dropping the assumption of bilinearity in the derivatives of the pion fields are discussed.

I. INTRODUCTION

THE structure of nonlinear Lagrangian theories which exhibit chiral invariance has recently been intensively investigated by several authors.[1–7] In particular, Brown[8] and Weinberg[9] have stressed that the requirement of chiral invariance forces an essentially unique choice for a pion Lagrangian, bilinear in the derivatives of the pion field. The present paper analyzes the construction of nonlinear Lagrangians satisfying the conventional divergence conditions on the vector and axial-vector currents, without assuming a broken chiral symmetry. In Sec. II, we recover the results of Brown,[8] of Weinberg,[9] and of Chang and Gürsey[3] on the pion Lagrangian by taking a nonlinear isospin-invariant Lagrangian, bilinear in the derivatives of the pion field, and construct currents upon which we impose conservation of the vector current (CVC) and partial conservation of the axial-vector current (PCAC). We find that the part of the Lagrangian that yields a conserved axial-vector current can be written, after choosing an appropriate normalization of the currents, in the manifestly chirally invariant form V^2+A^2, and that is identical when expressed in terms of the pion field with the customary form.[4,5] This result is similar to the observations of Nauenberg[10] and of Veltman,[11] that the divergence conditions in the presence of a vector field (which supplies the normalization) are all that is required to reproduce current-algebra calculations. What is remarkable is that in the V^2+A^2 form, the Lagrangian exhibits not merely invariance under chiral transformations and isotopic-spin rotations, but also the larger invariance under rotations in a six-dimensional space spanned by the isospin components of $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$.

In Sec. III, we construct the pion-nucleon Lagrangian from the divergence conditions on the currents. Again the results is essentially unique, i.e., Weinberg's Lagrangian, provided that the initial form is bilinear in the nucleon field. With the addition of 2 four-fermion interactions of the form

$$(\bar{N}\gamma_\mu\boldsymbol{\tau}N)^2+(G_A/G_V)^2(\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N)^2,$$

which do not break the chiral invariance, we find that once more the Lagrangian can be written in the suggestive form

$$\mathcal{L}=i(\bar{N}\gamma_\mu\partial_\mu N+m_N\bar{N}N)-\frac{g^2}{2m_\rho^2}\frac{\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}}{1+a^2\phi^2}\cdot\bar{N}\gamma_\mu\boldsymbol{\tau}N-\left(\frac{g}{m_\rho}\right)^2(V^2+A^2)+\text{PCAC term},\quad(1)$$

i.e., the sum of the free-nucleon Lagrangian, an effective $\bar{N}N\rho$ coupling, an $O(6)$ invariant term, and the symmetry-breaking term, which is a function of the pion field alone. g is the ρ-nucleon coupling constant. The chiral invariance of this Lagrangian, apart from the last term, is an explicit consequence of the transformations introduced by Schwinger[5] and by Weinberg.[9] We test our modification suggested by the V^2+A^2 interaction by computing s-wave N-N scattering lengths, and obtain good agreement for the ratio of the independent amplitudes, though the magnitude is too small.

In Sec. IV, we show that, if the restriction to quadratic dependence on the gradient of the pion field is relaxed, the imposition of the V^2+A^2 structure on the theory is enough to guarantee uniqueness.

II. NONLINEAR PION LAGRANGIAN

We consider a general nonlinear pion Lagrangian satisfying the requirements of isospin invariance and bilinear in the gradient of the pion field. It takes the form

$$\mathcal{L}=-\tfrac{1}{2}f(\phi^2)(\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}g(\phi^2)(\boldsymbol{\phi}\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}h(\phi^2),\quad(2)$$

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[5] J. Schwinger, Phys. Letters **24B**, 473 (1967).
[6] D. B. Fairlie and K. Yoshida, Ann. Phys. (N. Y.) **46**, 326 (1968).
[7] J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967).
[8] L. S. Brown, Phys. Rev. **163**, 1802 (1967).
[9] S. Weinberg, Phys. Rev. **166**, 1568 (1968).
[10] M. Nauenberg, Phys. Rev. **154**, 1455 (1967).
[11] M. Veltman, Phys. Rev. Letters **17**, 553 (1966).

where f, g, and h are arbitrary functions of ϕ^2. In the same fashion, vector and axial-vector currents carrying isospin can be constructed:

$$\mathbf{V}_\mu = -f_1(\phi^2)\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}\,,$$
$$\mathbf{A}_\mu = b_1(\phi^2)\partial_\mu\boldsymbol{\phi}+b_2(\phi^2)(\boldsymbol{\phi}\partial_\mu\boldsymbol{\phi})\boldsymbol{\phi}\,. \tag{3}$$

(The vector current could have been written down directly as $-f\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}$ from the isospin conservation by Noethers's theorem.) We relate the coefficients in the above expressions by the requirements that the vector current is conserved (CVC) and the axial current is partially conserved (PCAC) by inserting the equations of motion derived from (2) in the divergence equations

$$\partial_\mu\mathbf{V}_\mu=0\,,\quad \partial_\mu\mathbf{A}_\mu=F_\pi m_\pi{}^2\boldsymbol{\phi} \tag{4}$$

and equating the factors multiplying the invariants $(\partial_\mu\phi)^2$ and $\phi\partial_\mu\phi$ in (4). CVC demands $f_1=f$, as has been already anticipated. PCAC then imposes the following set of equations:

$$2f'g/f-g'=2[\ln(b_2/f)]'(f'-g)\,,$$
$$(b_1/f)'=-b_2/2f\,,$$
$$f'-g=2[\ln(b_1/f)]'(f+\phi^2f')\,, \tag{5}$$
$$h'b_2=(g-f')F_\pi m_\pi{}^2\,.$$

In the above equations, a prime denotes differentiation with respect to ϕ^2, F is the pion decay rate, and m_π is the pion mass. For any given $f(\phi^2)$ these equations have the unique solution

$$g(\phi^2)=\frac{2ff'+4a^2f^3+f'^2\phi^2}{f(1-4a^2\phi^2f)}\,,$$
$$b_1(\phi^2)=\tfrac{1}{2}cf^{1/2}(1-4a^2\phi^2f)^{1/2}\,,$$
$$b_2(\phi^2)=\tfrac{1}{2}c\frac{f'+4a^2f^2}{f^{1/2}(1-4a^{2\,2}f)^{1/2}}\,, \tag{6}$$
$$h(\phi^2)=\frac{F_\pi m_\pi{}^2}{2ca^2}\int f^{-1/2}d(1-4a^2f\phi^2)^{1/2}\,,$$

with two arbitrary constants a and c. Apart from the last equation in (5), the theory is invariant under a canonical transformation $\boldsymbol{\phi}\to\boldsymbol{\phi}K(\phi^2)$; choosing $K(\phi^2)$ to make g vanish, f is determined and we have the Schwinger Lagrangian[5]

$$-\frac{1}{2}\frac{(\partial_\mu\phi)^2}{(1+a^2\phi^2)^2}-\frac{1}{2}\frac{F_\pi m_n{}^2}{ca^2}\ln(1+a^2\phi^2)\,. \tag{7}$$

Working to second order in the pion field, we find that this Lagrangian reduces to the free-pion Lagrangian if

$$c=F_\pi\,. \tag{8}$$

Also, the normalization of the pion field is taken so that $c=a^{-1}$. (Once we have established the chiral invariance of the theory, this normalization is seen to be necessary so that $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$ obey the Gell-Mann algebra, using canonical commutation relations for ϕ and the conjugate momentum.) This will be made more explicit later. The Weinberg Lagrangian[4] differs from (7) by having a reciprocal instead of a logarithm in the second term; it is obtained by first setting $f=1$ in the solution (6) and then making a canonical transformation to eliminate the term in $(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2$. Thus the pion field of Weinberg that satisfies strict PCAC with no three-pion terms is related to that of Schwinger[6] by

$$\boldsymbol{\phi}_W=\boldsymbol{\phi}_S/(1+a^2\phi_S{}^2)\,. \tag{9}$$

In Weinberg's theory[9] the symmetry-breaking term transforms as the isoscalar component of the $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\times SU(2)$; this is in analogy with the standard assumptions made in the symmetry breakdown of $SU(3)$.[12] He has also given the solution in the form of a power series in ϕ^2 for the symmetry-breaking term when it transforms as the isoscalar member of the $(\frac{1}{2}N,\frac{1}{2}N)$ representation; it is interesting to note that this solution is a finite polynomial of Nth degree in the variable $(1+a^2\phi^2)^{-1}$. The part of the Lagrangian that conserves both $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$ may be written in the form

$$-2a^2[V_\mu{}^2+(ac)^{-2}A_\mu{}^2]\,. \tag{10}$$

Thus we see that the Lagrangian is not only invariant under isospin, but also under chiral transformations which mix $\mathbf{V}_\mu$ and $(ac)^{-1}\mathbf{A}_\mu$ (provided that $a\neq 0$), and we can then choose the normalization of the axial current so that $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$ obey the Gell-Mann algebra by setting $a=c^{-1}$. If $a=0$, then (10) is canonically equivalent to the free-field kinetic-energy term and the currents then obey the commutation relations of the contracted group $SU(2)\times T^3$. In Eq. (10), however, for $a=c^{-1}$ we see that for the vector and axial-vector currents the Lagrangian is invariant, not only under chiral $SU(2)\times SU(2)$, but also under an even larger group of rotations in the six-dimensional space of the vectors $\mathbf{V}_\mu$ and $\mathbf{A}_\mu$. (Of course, the Poincaré invariance of the Lagrangian is taken for granted.) In order to discriminate among various possibilities for the symmetry-breaking term, an additional postulate is necessary: Brown[8] has shown that if the commutator of the axial charge with the pion field is an isoscalar, this assumption, together with PCAC, yields Weinberg's result.

III. PION-NUCLEON LAGRANGIAN

The theory presented in Sec. II may be readily extended to incorporate nucleon terms. Suppose that we write the Lagrangian, omitting the symmetry-

[12] A. E. S. Green and T. Sawada, Rev. Mod. Phys. 3, 95 (1967).

breaking term, as

$$\mathfrak{L}=i(\bar{N}\gamma_\mu\partial_\mu N+m\bar{N}N)-\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2/(1+a^2\phi^2)^2 -\alpha\phi^2(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})\cdot\mathbf{N}_\mu-\beta(\phi^2)\partial_\mu\boldsymbol{\phi}\cdot\mathbf{N}_\mu{}^5, \quad (11)$$

where α and β are arbitrary functions to be determined by the divergence conditions on the currents, and $\mathbf{N}_\mu$ and $\mathbf{N}_\mu{}^5$ are shorthand for $\bar{N}\gamma_\mu\frac{1}{2}\boldsymbol{\tau}N$ and $\bar{N}\gamma_\mu\gamma_5\frac{1}{2}\boldsymbol{\tau}N$, respectively. Since the Lagrangian satisfies isospin invariance, the vector current may be deduced by Noethers's theorem, or form the equations of motion:

$$\begin{aligned}\partial_\mu\mathbf{N}_\mu&=\alpha(\phi^2)(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})\times\mathbf{N}_\mu+\beta(\phi^2)\partial_\mu\boldsymbol{\phi}\times N_\mu{}^5,\\ \partial_\mu\partial_\mu\boldsymbol{\phi}/(1+a^2\phi^2)^2&=\partial\mu\{-\alpha\boldsymbol{\phi}\times\mathbf{N}_\mu+\beta\mathbf{N}_\mu{}^5\}\\ &\quad+\alpha(\phi^2)\partial_\mu\boldsymbol{\phi}\times\mathbf{N}_\mu+\text{term with}\\ &\qquad\text{isospin dependence }\boldsymbol{\phi}.\end{aligned} \quad (12)$$

The second equation suggests the replacement of $\partial_\mu\boldsymbol{\phi}/(1+a^2\phi^2)^2$ in (3) with $\partial_\mu\boldsymbol{\phi}/(1-a^2\phi^2)^2+\alpha\boldsymbol{\phi}\times\mathbf{N}_\mu-\beta N_\mu{}^5$ and the addition of a term containing $\mathbf{N}_\mu$ to make $\partial_\mu\mathbf{V}_\mu=0$. Thus,

$$\mathbf{V}_\mu=\mathbf{N}_\mu-\boldsymbol{\phi}\times[\partial_\mu\boldsymbol{\phi}/(1+a^2\phi^2)^2-\alpha\boldsymbol{\phi}\times\mathbf{N}_\mu+\beta\mathbf{N}_\mu{}^5]. \quad (13)$$

We determine the arbitrary functions α and β up to constants by the divergence condition on $\mathbf{A}_\mu$, here taken as $\partial_\mu\mathbf{A}_\mu=0$, since we are omitting the symmetry-breaking term. This gives

$$\alpha=\frac{2a^2}{1+a^2\phi^2},\quad \beta=\frac{2f}{m_\pi}\frac{1}{1+a^2\phi^2},$$

and we have

$$\begin{aligned}a\mathbf{A}_\mu&=\frac{1-a^2\phi^2}{1+a^2\phi^2}\left[\frac{\partial_\mu\boldsymbol{\phi}}{2(1+a^2\phi^2)}+\frac{f}{m_\pi}\mathbf{N}_\mu{}^5-a^2\boldsymbol{\phi}\times\mathbf{N}_\mu\right]\\ &\quad+\frac{2a^2\boldsymbol{\phi}}{1+a^2\phi^2}\boldsymbol{\phi}\cdot\left[\frac{\partial_\mu\boldsymbol{\phi}}{2(1+a^2\phi^2)}+\frac{f}{m_n}\mathbf{N}_\mu{}^5-a^2\boldsymbol{\phi}\times\mathbf{N}_\mu\right]\\ &\quad-a^2\boldsymbol{\phi}\times\mathbf{N}_\mu.\end{aligned} \quad (14)$$

The arbitrary constants are chosen so that the coupling constants in (11), when terms quadratic in the pion field alone are retained, are identified with the pseudovector and vector coupling constants[7]

$$f^2/4\pi=0.08,\quad g^2/4\pi=2.8,\quad \text{with}\quad a^2=g^2/2m^2\rho. \quad (15)$$

The expressions (11), (13), and (14) then agree with those obtained by Wess and Zumino[7] from chiral-invariance arguments, and the Lagrangian is identical with that employed by various authors. As remarked in the Introduction, the form of (10) suggests that we try to write (11) similarly. To accomplish this we are forced to introduce two extra four-point interactions that do not affect the structure of the vector and axial currents. These additional terms are just

$$-2a^2[N_\mu{}^2+(f/am\pi)^2(N_\mu{}^5)^2], \quad (16)$$

and the Lagrangian that includes both terms (11) and (15) may then be written as

$$\mathfrak{L}=i(\bar{N}\gamma_\mu\partial_\mu N+M\bar{N}N)-[2a^2/(1+a^2\phi^2)^2] \times(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})\cdot\mathbf{N}_\mu-2a^2(V^2+A_\mu{}^2). \quad (17)$$

Now the factor f/a_{m_π} is just $-G_A/G_V$.[7] Thus the additional contribution in (16) is of the form

$$-(g^2/m\rho^2)[N_\mu{}^2+(G_A/G_V)^2(N_\mu{}^5)^2].$$

We should like to use this theory to compute the N-N s-wave scattering lengths: We must take, in the spirit of the effective Lagrangian approach, not only the four-point interaction (17), but also the pole terms arising from the ρ and axial-vector couplings to the nucleon implied in (11).

To evaluate the scattering lengths we take the contribution from the contact interaction (16) together with the meson-exchange contribution to lowest order in the pion fields in the vector current of magnitude $-\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}$, which may be estimated by treating it as $(m_\rho/g)\rho_\mu$; and the term $-(1/2a)\partial_\mu\boldsymbol{\phi}$, the contribution of the pion exchange to the axial current, vanishes at threshold for N-N scattering. The effective singlet n-p scattering length and the singlet-to-triplet ratio are

$$-(g^2M_N/4\pi m_p{}^2)[1+\tfrac{3}{2}(G_A/G_V)^2]=-3.5\text{ F} \quad (18a)$$

and

$$\frac{\text{singlet }p\text{–}n}{\text{triplet }p\text{–}n}=-\frac{1}{3}\left[\frac{2+3(G_A/G_V)^2}{2-(G_A/G_V)^2}\right]=-3.75, \quad (18b)$$

with $G_A/G_V=1.2$. Comparison with the experimental values[12,13] of -23.69 F for the singlet scattering length and -3.99 for the ratio indicates that, unless the π-N Lagrangian (11) is augmented with explicit ρ and A, and possibly elementary deuteron contributions, this theory cannot account for the magnitude of this s-wave N-N scattering lengths.

It is noteworthy that this form (16) has already been considered as a N-N interaction by Nambu and Jona-Lasinio[14] from an entirely different viewpoint.

It may be easily verified that the modified Lagrangian (17) is still chirally invariant. This follows from the separate invariance of (11) and (17) under the chiral transformations[5,6]

$$N\to[1+ia^2\boldsymbol{\tau}\cdot(\boldsymbol{\phi}\times\delta\boldsymbol{\phi})]N, \quad (19)$$

and (16) has the appealing property that it separates into two terms: a strong-interaction form with an effective $\rho N\bar{N}$ coupling, together with the current interaction $a^2(V_\mu{}^2+A_\mu{}^2)$ to which must be added a symmetry-breaking term as in Eq. (7) if the PCAC condition rather than $\partial_\mu\mathbf{A}_\mu=0$ is imposed.

[13] E. Lomon and H. Feshbach, Rev. Mod. Phys. **39**, 611 (1967).
[14] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

IV. CURRENT-CURRENT LAGRANGIANS

In this section, we consider the pion Lagrangian once more but abandon the constraint that it depends only bilinearly on the derivative of the pion field, though we do not permit the appearance of higher derivatives. The easiest way to construct such arbitrary chiral-invariant Lagrangians is to work in the σ model, and then transform away the σ field by redefining the fields through[6]

$$\boldsymbol{\phi}\rightarrow\frac{a\boldsymbol{\phi}}{1+a^2\phi^2},\quad \sigma\rightarrow\frac{1}{2}\frac{1-a^2\phi^2}{1+a^2\phi^2}. \tag{20}$$

The currents may be derived in the σ model[15,16] which is manifestly covariant, by Noethers's theorem, and then converted by the use of Eq. (20) to give

$$\mathbf{V}_\mu=\tfrac{1}{2}\boldsymbol{\phi}\times(\partial\mathcal{L}/\partial\boldsymbol{\phi}_\mu), \tag{21a}$$

where $\partial\mathcal{L}/\partial\boldsymbol{\phi}_\mu$ denotes differentiation with respect to $\partial_\mu\boldsymbol{\phi}$, and

$$\mathbf{A}_\mu=\frac{1+a^2\phi^2}{4a^2}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\phi}_\mu}+\boldsymbol{\phi}\times\mathbf{V}_\mu. \tag{21b}$$

The condition $\partial_\mu\mathbf{A}_\mu=0$ imposes the restriction

$$\frac{1+a^2\phi^2}{4a^2}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\phi}}=\partial_\mu\boldsymbol{\phi}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\phi}_\mu}\boldsymbol{\phi}. \tag{22}$$

This is satisfied provided that $\mathcal{L}$ is a scalar function of the invariants constructed from the tensor

$$T^{ab}=\partial_\mu\phi^a\partial_\mu\phi^b/(1+a^2\phi^2), \tag{23}$$

where a and b are isospin labels. Now let us impose the condition that $\mathcal{L}$ has the form $2a^2(V^2+A^2)$. This gives

$$\tfrac{1}{2}(1+a^2\phi^2)^2(\partial\mathcal{L}/\partial\boldsymbol{\phi}_\mu)^2=\mathcal{L}. \tag{24}$$

The independent invariants that can be constructed from (23) are $T^{aa}T^{ab}T^{ba}$ and $T^{ab}T^{bc}T^{cb}$, and all others are reducible to functions of these three. Regarding T^{ab} as a 3×3 matrix, by the Cayley-Hamilton theorem[17] it satisfies the matrix equation

$$\mathbf{T}^3\times x\mathbf{T}^2+y\mathbf{T}-z\mathbf{I}=0, \tag{25}$$

where

$$x=T^{aa},\quad y=\tfrac{1}{2}[(T^{aa})^2-T^{ab}T^{ba}],$$
$$z=\tfrac{1}{3}[T^{ab}T^{bc}T^{ca}-\tfrac{3}{2}T^{ab}T^{ba}T^{cc}+\tfrac{1}{2}(T^{aa})^2], \tag{26}$$

and $\mathbf{I}$ denotes the unit matrix. By repeated use of the identity (25) all traces of powers of $\mathbf{T}$ higher than the third may be expressed in terms of traces of the first three.[18] It is convenient to take as an equivalent set of independent variables x, y, and z introduced in Eq. (26). Then $\mathcal{L}$ depends only upon x, y, and z, and Eq. (24) takes the form

$$x\left(\frac{\partial\mathcal{L}}{\partial x}\right)^2+(xy+3z)\left(\frac{\partial\mathcal{L}}{\partial y}\right)^2+yz\left(\frac{\partial\mathcal{L}}{\partial z}\right)^2+4y\frac{\partial\mathcal{L}}{\partial x}\frac{\partial\mathcal{L}}{\partial y}$$
$$+6z\frac{\partial\mathcal{L}}{\partial x}\frac{\partial\mathcal{L}}{\partial z}+4xz\frac{\partial\mathcal{L}}{\partial y}\frac{\partial\mathcal{L}}{\partial z}=\tfrac{1}{2}\mathcal{L}. \tag{27}$$

When the power-series solution of this equation is sought, the only permissible solution is

$$\mathcal{L}=\tfrac{1}{2}x \tag{28}$$

if the solution is chosen so that the Lagrangian may agree with the kinetic-energy term for a free pion to lowest order in ϕ^2. Thus the requirement of a V^2+A^2 interaction is equivalent to the assumption of terms of second degree in $\partial_\mu\boldsymbol{\phi}$ in the Lagrangian.

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[17] R. S. Rivlin, J. Rational Mech. Anal. **4**, 4681 (1955). For a "physicist's" reference, see S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[18] No invariants of the type $\phi^aT^{ab}\phi^b$, etc., will appear, because of the restriction (20) which annuls similar invariants in $\mathcal{L}(\boldsymbol{\phi},\sigma)$ as $\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}+\sigma\partial_\mu\sigma$ vanishes.