Quantum Electromagnetic Zero-Point Energy of a Conducting Spherical Shell and the Casimir Model for a Charged Particle*

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The quantum electromagnetic zero-point energy of a conducting spherical shell of radius r has been computed to be $\Delta E(r)\cong 0.09\hbar c/2r$. The physical reasoning is analogous to that used by Casimir to obtain the force between two uncharged conducting parallel plates, a force confirmed experimentally by Sparnaay and van Silfhout. However, while parallel plates are attracted together because of the zero-point energy, a conducting sphere tends to be expanded. Thus although relevant for the understanding of the quantum-mechanical zero-point energy, the result invalidates Casimir's intriguing model for a charged particle as a charged conducting shell with Poincaré stresses provided by the zero-point energy and a unique ratio for $e^2/\hbar c$ independent of the radius.

I. INTRODUCTION

THE attractive force between two conducting parallel plates was calculated by Casimir[1] using the notions of quantum electromagnetic zero-point energy for the normal modes in the region between the plates, and experimental work by Sparnaay[2] and van Silfhout[3] has confirmed the result. The present paper reports an analogous calculation for the forces on an uncharged conducting spherical shell.

The motivation for the calculation arises from Casimir's intriguing model[4] for a charged particle. The model cuts across the lines of classical and quantum electrodynamics, beginning with the Abraham-Lorentz model of classical electron theory and then adding Poincaré stresses from the zero-point energy of quantum electrodynamics in a manner which makes the vanishing of the stresses independent of the radius of the configuration. The model suggests a unique value $C=e^2/\hbar c$ necessary for the vanishing of the self-stresses, and this value is determined unambiguously from the geometrical considerations used to evaluate the zero-point energy of a conducting spherical shell. However, the calculated magnitude of the zero-point energy turns out to be of the opposite sign from that proposed in the model, and any connection between the values for the zero-point energy and the fine-structure constant α is not immediately apparent.

In the Sec. II, we outline briefly the Casimir model for a charged particle. In Sec. III, we carry out the required calculation of the zero-point energy of a spherical conducting shell and conclude that the result, although relevant for understanding quantum-mechanical zero-point energy invalidates the proposed model.

II. CASIMIR MODEL FOR A CHARGED PARTICLE

In the early part of this century, there was considerable interest in models for charged particles within classical electromagnetic theory, and today the Abraham-Lorentz model still finds its way into textbooks[5] on electromagnetism. However, this classical electron theory was beset by numerous difficulties, and since quantum mechanics and notably quantum electrodynamics have succeeded essentially by sidetracking or ignoring problems of electron structure, the question has not been one of great interest recently.

One of the difficulties with the classical electron theories was the need for the *ad hoc* stresses postulated by Poincaré in order to make stable the particle's finite charge configuration despite the repulsion of the distinct parts of the distribution. Since all the effects of classical electromagnetism seemed to have been already incorporated, the Poincaré stresses were presumed to be "mechanical," nonelectromagnetic.

In 1953, Casimir, surely encouraged by his successful calculations of the attraction of two conducting parallel plates due to quantum electromagnetic zero-point energy, suggested that Poincaré's stresses could be viewed as a quantum electromagnetic effect due to zero-point energy. The idea is strikingly simple, and has the further virtue that now the electron model is filled out entirely by electromagnetic effects.

The Abraham-Lorentz model is extended by Casimir as follows. In its rest frame, a charged particle is regarded as a conducting spherical shell carrying a homogeneous surface charge of total magnitude e. Taking the radius as a in the intermediate stages of the model,

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[1] H. B. G. Casimir, Koninkl. Ned. Akad. Wetenschap, Proc. **51**, 793 (1948).

[2] M. J. Sparnaay, Physics **24**, 751 (1958).

[3] A. van Silfhout, *Dispersion Forces Between Macroscopic Objects* (Drukkerij Holland, N. V., Amsterdam, 1966). Lifshitz, Derjaugin, and collaborators have carried out extensive theoretical and experimental work on dispersion forces; see B. V. Derjaugin, I. I. Abrikossova, and E. M. Lifshitz, Quart. Rev. (London) **10**, 295 (1956); also, the references in van Silfhout's thesis.

[4] H. B. G. Casimir, Physica **19**, 846 (1956). The model has been termed suggestively a "mousetrap" to catch $e^2/\hbar c$.

[5] See, for example, J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1963). Recent work on classical electron theory includes that by F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965); S. Coleman, Rand Corporation Report No. RM-2820-PR, 1961 (unpublished).

we find that the electrostatic energy of the configuration is, in Gaussian units,

$$E_e = e^2/2a\,, \tag{1}$$

with a corresponding tension $e^2/8\pi a^4$ tending to expand the sphere. On the other hand, the presence of the conducting boundary alters the zero-point energy of the universe. Arguing by analogy with the parallel-plate calculations, Casimir suggested that the zero-point energy might tend to collapse the sphere, giving an energy

$$E_z = -C(\hbar c/2a)\,, \tag{2}$$

where C is a constant, and hence a corresponding tension $-C\hbar c/8\pi a^4$. This latter tension would supply the Poincaré stress, making the configuration stable independent of the value of the radius a provided that

$$e^2/8\pi a^4 - C\hbar c/8\pi a^4 = 0\,; \tag{3}$$

i.e., provided that the total charge on the sphere is such that

$$e^2/\hbar c = C\,. \tag{4}$$

We notice that the condition (4) is independent of the radius a of the configuration, and we may even allow $a \to 0$ so as to again avoid further questions of electron structure. This suggests that if by some chance—perhaps best compared to that for the Bohr model of the atom—the model did represent an approximation to nature, it might be possible to calculate the value of the fine-structure constant α as the Casimir constant appearing in the zero-point energy of a conducting spherical shell. The constant C is a pure number following uniquely and unambiguously from the electromagnetic normal modes of a sphere. The idea recalls a suggestion, which the author has heard attributed to Feynman, that the fine-structure constant might sometime be calculated in terms of Bessel functions.

Indeed the situation seems to be even more encouraging. Thus if we take the parallel-plate result for the energy of two conducting plates of area A and separation d,

$$\Delta E = -\pi^2 \hbar c A/720 d^3\,, \tag{5}$$

and very roughly approximate a sphere of radius a as two parallel plates of area πa^2 a distance a apart, then substituting into Eq. (5),

$$\Delta E_{\text{sphere}} \sim -0.09\hbar c/2a\,, \tag{6}$$

giving a value for Casimir's constant C only about 10 times as large as that of the fine-structure constant. This might be regarded as relatively good agreement for this rough approximation.

Considering the apparent beauty of the model, it seems most melancholy to report that the results of the full calculation of the zero-point energy of a conducting spherical shell which follows in the remainder of this paper shows that actually the contribution of the zero-point energy in Eq. (6) is positive; that is to say, it is of the opposite sign from that suggested. Thus instead of balancing the electrostatic repulsion, the quantum zero-point force also expands the sphere. Our calculation invalidates the Casimir model in the form given here.

III. CALCULATION OF QUANTUM ELECTROMAGNETIC ZERO-POINT ENERGY OF A CONDUCTING SPHERICAL SHELL

A. Physical Problem of Zero-Point Energy

In classical electromagnetic theory, a conducting shell may contain radiation describable in terms of a linear superposition of a discrete set of normal modes

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \operatorname{Re}\sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{f}_{\mathbf{k}}(\mathbf{x}) \exp[-i\omega_{\mathbf{k}}t]\,, \\ \mathbf{B}(\mathbf{x},t) &= \operatorname{Re}\sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{g}_{\mathbf{k}}(\mathbf{x}) \exp[-i\omega_{\mathbf{k}}t]\,. \end{aligned} \tag{7}$$

This same system is described for the quantized electromagnetic field in terms of collection of independent harmonic oscillators having energy eigenstates

$$E_{\mathbf{k}} = (n+\tfrac{1}{2})\hbar\omega_{\mathbf{k}}\,, \quad n=0,\,1,\,2,\,\cdots\,, \tag{8}$$

where the $\omega_{\mathbf{k}} = ck$ are precisely the frequencies of the classical normal modes. The allowed energies do not start at $E_{\mathbf{k}}=0$ but rather have a lowest possible energy $\frac{1}{2}\hbar\omega_{\mathbf{k}}$. Thus naively, even in its lowest possible energy state, the quantum system has a ground-state or zero-point energy

$$E = \sum_{\mathbf{k}} \tfrac{1}{2}\hbar\omega_{\mathbf{k}}\,. \tag{9}$$

Since there are an infinite number of normal modes of increasingly high frequency, this energy E is infinite. However, rather than inquire about this infinity, we will reformulate the problem in a manner analogous to that suggested by Casimir[6] for two conducting parallel plates. We will not calculate the zero-point energy E for a sphere but rather the difference ΔE in zero-point energy between two configurations.

The system to be considered consists of a large conducting sphere of radius R enclosing the quantization "universe," and a small concentric sphere of variable radius. The zero-point energy of the inner sphere $\Delta E(a,R)$ will be the charge in the zero-point energy of the total system when the radius of the inner sphere is changed from a radius a to some radius R/η, $\eta > 1$ which is a fixed fraction of the radius of the universe. If we label the zero-point energies E_{I}, E_{II}, E_{III}, and E_{IV} as in Fig. 1, then

$$\Delta E(a,R) = (E_{\text{I}} + E_{\text{II}}) - (E_{\text{III}} + E_{\text{IV}})\,. \tag{10}$$

After making this subtraction, we will allow the radius R to increase indefinitely ($R \to \infty$) and obtain the zero-

[6] H. B. G. Casimir, Philips Res. Rept. **6**, 162 (1951).

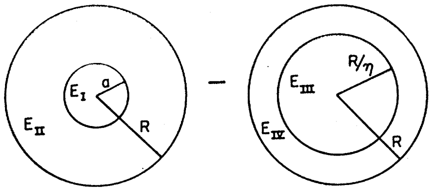

FIG. 1. Spherical configurations for finite quantization universe of radius R.

point energy of a spherical shell

$$\Delta E(a)=\lim_{R\to\infty}\Delta E(a,R)\,. \tag{11}$$

It will turn out that the energies $\Delta E(a,R)$ and $\Delta E(a)$ are finite, although the energies E_{I}, E_{II}, E_{III}, and E_{IV} are infinite.

The value for $\Delta E(a,R)$ for large R is found[7] to depend upon only the radius a of the physical sphere, and to be independent of the radius R of the quantization universe. Also this energy is independent of the value chosen for η provided that $R/\eta\gg a$ and $R\gg R/\eta$. We note, however, that if η is chosen very close to 1, so that $R\sim R/\eta$, then there would be an additional energy owing to the attraction of the inner comparison sphere to the outer sphere of the universe. This will not occur if $\eta>1$ is held fixed while R increases. The independence of the value of $\Delta E(a,R)$ from the values of R and η used in the intermediate calculations allows us to speak of the limit $\Delta E(a)$ in Eq. (11) as the zero-point energy of the spherical shell of radius a.

In the above, we have brushed over a significant mathematical problem. The subtractions involved in (10) involve the subtraction of infinite quantities, more specifically, of divergent series. Mathematical techniques for smoothing divergent series have been developed[8] to a very sophisticated degree, and the only question arising is as to the physical quantity appropriate for use as a cutoff. Here we take a cue from the behavior of physically realizable conductors in contrast to the mathematical idealization which gave the infinite series. A real conductor will conduct well at low frequencies or long wavelengths, but will become a poorer conductor as the frequencies increase or the wavelengths decrease. We thus require that any cutoff introduced in the mathematics should depend only on the wavelength and should cut off the short wavelengths. Once we have made this requirement, there are a wide variety of specific mathematical procedures which may be employed; *all* of those which meet certain well-defined mathematical criteria will give the same result.[9]

B. Electromagnetic Normal Modes in Conducting Concentric Spherical Shells

The calculation of the series for the zero-point energies E_{I} and E_{III} requires a knowledge of the TE and TM normal modes in a conducting spherical shell, and for E_{II} and E_{IV} a knowledge of the normal modes in the annular region between two conducting concentric spherical shells. The analysis leading to the normal modes appears in many textbooks[10] and will not be repeated here. A summary of the normal modes appears in Table I.

The author has prepared[11] an extensive analysis of these normal modes in an article which will be referred to as A.

Substituting the normal modes into the expression of the previous section,

$$\begin{aligned}\Delta E(a)=\lim_{R\to\infty}\lim_{\lambda\to 0}\tfrac{1}{2}\hbar c\Bigg[\sum_{l=1}^{\infty}(2l+1)\Bigg(&\sum_{s=1}^{\infty}\big[k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a))+\bar{k}_{ls}(a)\mathfrak{F}(\lambda\bar{k}_{ls}(a))\big]\\&+\sum_{S=1}^{\infty}\big[K_{lS}(a,R)\mathfrak{F}(\lambda K_{lS}(a,R))+\bar{K}_{lS}(a,R)\mathfrak{F}(\lambda\bar{K}_{lS}(a,R))\big]-\sum_{s=1}^{\infty}\bigg[k_{ls}\Big(\frac{R}{\eta}\Big)\mathfrak{F}\Big(\lambda k_{ls}\Big(\frac{R}{\eta}\Big)\Big)+\bar{k}_{ls}\Big(\frac{R}{\eta}\Big)\mathfrak{F}\Big(\lambda\bar{k}_{ls}\Big(\frac{R}{\eta}\Big)\Big)\bigg]\\&-\sum_{S=1}^{\infty}\bigg[K_{lS}\Big(\frac{R}{\eta},R\Big)\mathfrak{F}\Big(\lambda K_{lS}\Big(\frac{R}{\eta},R\Big)\Big)+\bar{K}_{lS}\Big(\frac{R}{\eta},R\Big)\mathfrak{F}\Big(\lambda\bar{K}_{lS}\Big(\frac{R}{\eta},R\Big)\Big)\bigg]\Bigg)\Bigg]\,,\end{aligned}\tag{12}$$

where $\mathfrak{F}$ is a suitable cutoff function vanishing for large argument. We will have occasion to use the notation $\Delta E(a,R)$ for the expression in (12) before the limit $R\to\infty$ and the notation $\Delta E(a,R,\lambda)$ for the expression before both the R and λ limits.

[7] See Sec. III E, in particular the comments following Eq. (66).
[8] See, for example, G. H. Hardy, *Divergent Series* (Oxford University Press, London, 1965).
[9] See Ref. 8, or the summary of such questions in K. Knopp, *Theory and Application of Infinite Series* (Blackie and Sons Ltd., London, 1964), Chap. XIII.
[10] J. C. Slater and N. H. Frank [*Electromagnetism* (McGraw-Hill Book Co., New York, 1947)] present a detailed and readable account.
[11] T. H. Boyer (to be published), hereafter referred to as A.

C. Preliminary Considerstion of Finiteness or Divergence of Zero-Point Energy

The first step in the evaluation of $\Delta E(a)$ is to prove that a cutoff function $\mathfrak{F}$ depending only on the wave number can actually give a finite value $\Delta E(a,R,\lambda)$ to the infinite series in l and s before allowing the limits $\lambda\to 0$ and $R\to\infty$. The convergence of each of the series in (12), e.g.,

$$\sum_{l=1}^{\infty}(2l+1)\sum_{s=1}^{\infty}k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a))\,,$$

follows from the following list of extremely rough limits[12] appearing in the analysis of A,

$$k_{ls}(a)=j_{l+\frac{1}{2},s}/a\geqslant[l+\pi(s-1)]/a\,, \tag{13}$$

$$K_{lS}(a,R)=x_{l+\frac{1}{2},R/a,S}/a\geqslant[l+\pi(S-1)]/R\,, \tag{14}$$

$$\bar{k}_{ls}(a)=\bar{j}_{l+\frac{1}{2},s}'/a\geqslant[l+\pi(s-1)]/a\,, \tag{15}$$

$$\bar{K}_{lS}(a,R)=\bar{x}_{l+\frac{1}{2},R/a,S}'/a\geqslant[l+\pi(S-2)]/R\,. \tag{16}$$

The notation of A appears as the center term in the inequalities. As an example, we will carry out the proof for

$$\sum_{l=1}^{\infty}(2l+1)\sum_{s=1}^{\infty}k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a))$$

when $\mathfrak{F}(x)=\exp(-x)$. Thus

$$0\leqslant\sum_{l=1}^{\infty}(2l+1)\sum_{s=1}^{\infty}k_{ls}(a)\exp[-k_{ls}(a)\lambda]\leqslant\sum_{l=1}^{\infty}(2l+1)\sum_{s=1}^{\infty}\frac{l+\pi(s-1)}{a}\exp\left(-\frac{l+\pi(s-1)}{a}\lambda\right)$$
$$=\left[\sum_{l=1}^{\infty}(2l+1)\frac{l}{a}\exp\left(-\frac{l}{a}\lambda\right)\right]\left[\sum_{s=1}^{\infty}\exp\left(-\frac{\pi(s-1)}{a}\lambda\right)\right]+\left[\sum_{l=1}^{\infty}(2l+1)\exp\left(-\frac{l}{a}\lambda\right)\right]$$
$$\times\left[\sum_{s=1}^{\infty}\frac{\pi(s-1)}{a}\exp\left(-\frac{\pi(s-1)}{a}\lambda\right)\right]. \tag{17}$$

For $\lambda>0$, each of the series in brackets converges by the ratio test. The original series is thus a bounded series of positive terms, and hence is convergent.

It will be of some interest later to have an upper bound to the rate of divergence as $\lambda\to 0$ of the series appearing in (12). We can obtain this by using the bounds of (13)–(16) and then summing the bounding series such as those which appear in (17). By using the device of term-by-term differentiation allowed by the uniform convergence of the series for $\lambda\geqslant\epsilon>0$, we have for example

$$\sum_{l=1}^{\infty}(2l)\exp\left(-\frac{l}{a}\lambda\right)=-2a\frac{\partial}{\partial\lambda}\sum_{l=1}^{\infty}\exp\left(-\frac{l}{a}\lambda\right)$$
$$=-2a\frac{\partial}{\partial\lambda}\left(\frac{\exp(-\lambda/a)}{1-\exp(-\lambda/a)}\right) \tag{18}$$

and analogous expressions for all the remaining series appearing in (17). Summing explicitly, we see that each of the series in (12) diverges at most as $O(\lambda^{-4})$.

In the calculation of the zero-point energy ΔE of two conducting parallel plates, Fierz[13] showed that with a cutoff $\mathfrak{F}(\lambda k)=\exp(-\lambda k)$ the energy $E(\lambda)=\sum\frac{1}{2}\hbar\omega\mathfrak{F}((\lambda/c)\omega)$ in a rectangular volume with sides $L\times L\times d$ could be given as a power series

$$E(\lambda)=\frac{\partial}{\partial\lambda}\left(\frac{b_1}{\lambda^3}+\frac{b_2}{\lambda^2}+\frac{b_3}{\lambda}+b_4+b_5\lambda+\cdots\right). \tag{19}$$

The final result for ΔE was finite because all the terms involving inverse powers of λ cancelled in the subtractions analogous to those of Eq. (10). It appears as though in the case of present interest, the energy of the individual regions again diverges as powers of λ^{-1},

$$E_{\rm I}(a,\lambda)\sim\frac{\partial}{\partial\lambda}\left(\frac{c_1(a)}{\lambda^3}+\frac{c_2(a)}{\lambda^2}+\frac{c_3(a)}{\lambda}\right),$$
$$E_{\rm II}(a,\lambda)\sim\frac{\partial}{\partial\lambda}\left(\frac{C_1(a,R)}{\lambda^3}+\frac{C_2(a,R)}{\lambda^2}+\frac{C_3(a,R)}{\lambda}\right). \tag{20}$$

[12] The following comments may be helpful: (i) The first zero $j_{\nu,1}>\nu$ and the spacing between zeros $j_{\nu,s+1}-j_{\nu,s}\geqslant\pi$. (ii) See A, Fig. 4 or Fig. 7; $x_{\nu,K,S}\geqslant j_{\nu,S}/K$. (iii) The first zero satisfies $\bar{j}_{\nu,1}'>\nu$ and the spacing between zeros $\bar{j}_{\nu,s+1}'-\bar{j}_{\nu,s}'\geqslant\pi$. (iv) See A, Figs. 8 and 9; $\bar{x}_{\nu,K,1}'\geqslant\nu/K$ and $\bar{x}_{\nu,K,S}'\geqslant\bar{j}_{\nu,S-1}'$

[13] M. Fierz, Helv. Phys. Acta. **33**, 855 (1960); see also T. H. Boyer, thesis, Harvard University, 1968 (unpublished). Proper treatment of the $n=0$ modes in Fierz's work shows that $b_2=0$ in Eq. (19).

TABLE. I. Spherical electromagnetic normal modes.

Spherical region

TE: $k_{ls}(a)$: solutions of

$$j_l(ak_{ls}(a))=0,$$

TM: $\bar{k}_{ls}(a)$: solutions of

$$\left[\frac{d}{dr}[rj_l(r\bar{k}_{ls}(a))]\right]_{r=a}=0,$$

$$l=1,2,\cdots,\quad m=-l,-l+1,\cdots,l,\quad s=1,2,\cdots.$$

Annular region

TE: $K_{lS}(a,R)$: solutions of

$$\frac{j_l(aK_{lS}(a,R))}{n_l(aK_{lS}(a,R))}-\frac{j_l(RK_{lS}(a,R))}{n_l(RK_{lS}(a,R))}=0,$$

TM: $\bar{K}_{lS}(a,R)$: solutions of

$$\frac{\left[\frac{d}{dr}[rj_l(r\bar{K}_{lS}(a,R))]\right]_{r=a}}{\left[\frac{d}{dr}[rn_l(r\bar{K}_{lS}(a,R))]\right]_{r=a}}-\frac{\left[\frac{d}{dr}[rj_l(r\bar{K}_{lS}(a,R))]\right]_{r=R}}{\left[\frac{d}{dr}[rn_l(r\bar{K}_{lS}(a,R))]\right]_{r=R}}=0$$

$$l=1,2,\cdots,\quad m=-l,-l+1,\cdots,l,\quad S=1,2,\cdots.$$

(It is by no means clear from the theory of Dirichlet series[14] that $E_{\rm I}$ and $E_{\rm II}$ here have Laurent expansions about the point $\lambda=0$, and the author's work indeed suggests the contrary.) If the result $\Delta E(a,R)$ in Eq. (10) is to be finite for $\lambda\rightarrow 0$, then we require that

$$[c_i(a)+C_i(a,R)]-[c_i(R/\eta)+C_i(R/\eta,R)]=0,\quad i=1,2,3.\quad (21)$$

At an early stage in this work, the author tried to fit $E_{\rm I}(a,\lambda)$ using orthogonal polynomials in λ, and found the results of Table II for the coefficients $c_i(a)$. The first coefficient $c_i(a)$ is just that expected from considering the number of normal modes in a large volume, independent of the shape of the volume, such as is done in the derivation of the Rayleigh-Jeans or Planck formula for black-body radiation. It is proportional to the volume enclosed and hence the required cancellation will occur in Eq. (21). The second coefficient $c_2(a)$ seems to vanish, so that here Eq. (21) is irrelevant. The coefficient $c_3(a)$ will depend linearly on a, $c_3(a)=-(2/3\pi)a$, and if $C(a,R)=-(2/\pi)(R-a)$, then we will indeed have full cancellation. The coefficients were evaluated for the spherical case alone, since only for this situation had we computed a sufficiently large collection of normal modes.

The arguments regarding the coefficients c_i, C_i are particularly relevant for the calculation of temperature radiation in a conducting sphere; here they are intended merely as suggestive of what is involved. The actual calculation for $\Delta E(a)$ which follows makes no use of Eq. (21).

[14] See V. Berstein, *Progrès Récent de la Théories des Séries de Dirichlet* (Gauthier-Villars, Paris, 1933).

TABLE II. Coefficients for divergent terms in $E_I(a)$.

Coefficient	TE modes	TM modes	TE+ME modes
$c_1(a)$	$(4/3\pi)a^3$	$(4/3\pi)a^3$	$(8/3\pi)a^3$
$c_2(a)$	$-\frac{1}{2}a^2$	$\frac{1}{2}a^2$	0
$c_3(a)$	$(2/3\pi)a$	$-(4/3\pi)a$	$-(2/3\pi)a$

D. Mathematical Relations Involving Spherical Normal Modes

In order to proceed to the evaluation of $\Delta E(a)$ from Eq. (12), we must first develop a number of mathematical relations connecting the frequencies and the index labels of the normal modes.

The spherical Bessel functions[15] may be written in terms of trigonometric functions and polynomials in x^{-1},

$$\begin{aligned}xj_l(x)&=\sin(x-\tfrac{1}{2}l\pi)A_l(x)+\cos(x-\tfrac{1}{2}l\pi)B_l(x),\\ xn_l(x)&=-\cos(x-\tfrac{1}{2}l\pi)A_l(x)+\sin(x-\tfrac{1}{2}l\pi)B_l(x),\end{aligned}\quad (22)$$

where

$$A_l(x)=\sum_{r=0}^{r=[\frac{1}{2}l]}(-1)^r\frac{A_{l,r}}{(4x^2)^r},$$

$$B_l(x)=\frac{1}{2x}\sum_{r=0}^{r=[\frac{1}{2}l-\frac{1}{2}]}(-1)^r\frac{B_{l,r}}{(4x^2)^r},\quad (23)$$

and

$$A_{l,r}=\frac{(l+2r)!}{(2r)!(l-2r)!},\quad B_{l,r}=\frac{(l+2r+1)!}{(2r+1)!(l-2r-1)!}.$$

The derivatives of the functions are

$$\begin{aligned}\frac{d}{dx}[xj_l(x)]&=\cos(x-\tfrac{1}{2}l\pi)[A_l(x)+B_l'(x)]\\&\quad-\sin(x-\tfrac{1}{2}l\pi)[B_l(x)-A_l'(x)],\\ \frac{d}{dx}[xn_l(x)]&=\sin(x-\tfrac{1}{2}l\pi)[A_l(x)+B_l'(x)]\\&\quad+\cos(x-\tfrac{1}{2}l\pi)[B_l(x)-A_l'(x)].\end{aligned}\quad (24)$$

It follows from these expressions that if $k_{ls}(a)$ is the sth zero of $xaj_l(xa)$, then

$$ak_{ls}(a)-\tfrac{1}{2}l\pi=\pi s-\arctan\left[\frac{B_l(ak_{ls}(a))}{A_l(ak_{ls}(a))}\right],\quad (25)$$

or equivalently

$$\pi s=-\arctan\left[\frac{j_l(ak_{ls}(a))}{n_l(ak_{ls}(a))}\right].\quad (26)$$

Although the first form will be found more convenient for arithmetic computer computations, the second form

[15] For excellent short review of information on Bessel functions, see *Handbook of Mathematical Functions*, edited by M. Abramowitz and J. A. Stegun (Dover Publications, Inc., New York, 1965).

will be used in the mathematical analysis because the information available from classical function theory is linked to the Bessel functions and not to the coefficient functions $A_l(x)$ and $B_l(x)$. Similarly, for the TM modes $\bar{k}_{l\bar{s}}(a)$ which are zeros of

$$\left(\frac{d}{dx}(xj_l(x))\right)_{x=a\bar{k}_{l\bar{s}}(a)},$$

we have

$$a\bar{k}_{l\bar{s}}(a)-\tfrac{1}{2}(l-1)\pi=\pi(\bar{s}-1)-\arctan\left[\frac{B_l(a\bar{k}_{l\bar{s}}(a))-A_l'(a\bar{k}_{l\bar{s}}(a))}{A_l(a\bar{k}_{l\bar{s}}(a))+B_l'(a\bar{k}_{l\bar{s}}(a))}\right] \tag{27}$$

and

$$\pi(\bar{s}-1)=-\arctan\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]_{x=a\bar{k}_{l\bar{s}}(a)}. \tag{28}$$

The situation is quite analogous for the annular modes. The TE annular modes $K_{lS}(a,R)$ satisfy

$$\frac{j_l(aK_{lS}(a,R))}{n_l(aK_{lS}(a,R))}=\frac{j_l(RK_{lS}(a,R))}{n_l(RK_{lS}(a,R))}, \tag{29}$$

so that for integer S

$$aK_{lS}(a,R)-\tfrac{1}{2}l\pi+\arctan\left[\frac{B_l(aK_{lS}(a,R))}{A_l(aK_{lS}(a,R))}\right]=-\pi S+RK_{lS}(a,R)-\tfrac{1}{2}l\pi+\arctan\left[\frac{B_l(RK_{lS}(a,R))}{A_l(RK_{lS}(a,R))}\right] \tag{30}$$

and

$$\pi S=\arctan\left[\frac{j_l(aK_{lS}(a,R))}{n_l(aK_{lS}(a,R))}\right]-\arctan\left[\frac{j_l(RK_{lS}(a,R))}{n_l(RK_{lS}(a,R))}\right]. \tag{31}$$

The TM annular modes $\bar{K}_{lS}(a,R)$ are solutions of

$$\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]_{x=a\bar{K}_{l\bar{S}}(a,R)}=\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]_{x=R\bar{K}_{l\bar{S}}(a,R)} \tag{32}$$

giving for integer $\bar{S}$

$$a\bar{K}_{l\bar{S}}(a,R)-\tfrac{1}{2}(l-1)\pi+\arctan\left[\frac{B_l(a\bar{K}_{l\bar{S}}(a,R))-A_l'(a\bar{K}_{l\bar{S}}(a,R))}{A_l(a\bar{K}_{l\bar{S}}(a,R))+B_l'(a\bar{K}_{l\bar{S}}(a,R))}\right]$$
$$=-\pi(\bar{S}-1)+R\bar{K}_{l\bar{S}}(a,R)-\tfrac{1}{2}(l-1)\pi-\arctan\left[\frac{B_l(R\bar{K}_{l\bar{S}}(a,R))-A_l'(R\bar{K}_{l\bar{S}}(a,R))}{A_l(R\bar{K}_{l\bar{S}}(a,R))+B_l'(R\bar{K}_{l\bar{S}}(a,R))}\right] \tag{33}$$

and

$$\pi(\bar{S}-1)=\arctan\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]_{x=a\bar{K}_{l\bar{S}}(a,R)}-\arctan\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]_{x=R\bar{K}_{l\bar{S}}(a,R)}. \tag{34}$$

Equations (25)–(34) were derived for integer indices s, $\bar{s}$, S, $\bar{S}$. However, the expression can be regarded as giving these indices as analytic functions of the frequencies of the normal modes, or, in suitable domains, as giving implicitly the frequencies of the modes as a function of the indices. We will denote these functions in the obvious notation as $k_l(a,s)$, $s_l(a,k)$; $\bar{k}_l(a,\bar{s})$, $\bar{s}_l(a,\bar{k})$; $K_l(a,R,S)$, $S_l(a,R,K)$; $\bar{K}_l(a,R,\bar{S})$, $\bar{S}_l(a,R,\bar{K})$. From (26), (28), (31), (34), we note the relations among the functions

$$S_l(a,R,K)=s_l(R,K)-s_l(a,K),$$
$$\bar{S}_l(a,R,K)=\bar{s}_l(R,K)-\bar{s}_l(a,\bar{K})+1. \tag{35}$$

Graphs of the behavior of the frequencies as functions of the indices s, $\bar{s}$, S, $\bar{S}$ above the index $=1$ are shown for $l=5$ in Figs. 5 and 10 of A.

It will be of some importance in the later development of the calculation for $\Delta E(a)$ to obtain expressions for the derivatives of the frequencies with respect to the index and to consider the asymptotic expansions of these derivatives for large index l. In handling these expansions, it turns out to be more convenient to use the variable $\nu=l+\frac{1}{2}$ rather than l; in what follows, l and ν will be used simultaneously always with this connection.

The first derivatives of $s_l(a,k)$ and $\bar{s}_l(a,\bar{k})$ as functions of k and $\bar{k}$ may be found by differentiating (26) and (28) with regard to k and $\bar{k}$. Using the relation for the Wronskian of the spherical Bessel functions

$$j_l(x)n_l'(x)-n_l(x)j_l'(x)=1/x^2, \tag{36}$$

we find

$$\pi\frac{ds_l(a,k)}{dk}=af_l^{\mathrm{TE}}(ak)=\frac{a}{[akj_l(ak)]^2+[akn_l(ak)]^2},\quad(37)$$

$$\pi\frac{d\bar{s}_l(a,\bar{k})}{d\bar{k}}=af_l^{\mathrm{TM}}(a\bar{k})=\frac{a[1-l(l+1)/(a\bar{k})^2]}{\{[d(xj_l(x))/dx]^2+[d(xn_l(x))/dx]^2\}_{x=a\bar{k}}}.$$

The functions $f_l^{\mathrm{TM}}(x)$ and $f_l^{\mathrm{TE}}(x)$ go to zero rapidly as $x\to 0$, change most rapidly in the region $x\sim\nu+O(\nu^{1/3})$ and finally, approach the value 1 for $x\gg\nu$. The asymptotic forms in the various regions may be found from those for the cylindrical Bessel functions $J_\nu(x)$, $N_\nu(x)$ which are customarily given in the literature. The spherical Bessel functions $j_l(x)$, $n_l(x)$ are related to the cylinder functions as

$$j_l(x)=(\pi/2x)^{1/2}J_{l+\frac{1}{2}}(x),\quad n_l(x)=(\pi/2x)^{1/2}N_{l+\frac{1}{2}}(x).\quad(38)$$

For l fixed and $x\to 0$, we can use the expansions[16] of $j_l(x)$ and $n_l(x)$ in ascending powers of x, and conclude that as $x\to 0$

$$\begin{aligned}f_l^{\mathrm{TE}}(x)&\propto x^{2l+2}+O(x^{2l+4}),\\ f_l^{\mathrm{TM}}(x)&\propto -x^{2l+2}+O(x^{2l+4}).\end{aligned}\quad(39)$$

In the opposite limit of $x\to\infty$ for l fixed, we use the expansions of Eqs. (22)–(24), or more specifically the expansions (23) and (37) of A to find

$$f_l^{\mathrm{TE}}(x),f_l^{\mathrm{TM}}(x)\sim 1-l(l+1)/2x^2+O(x^{-4}).\quad(40)$$

The behavior of the functions for large index l can be obtained from the asymptotic expansions of Debye and Olver.[17] Thus for $x=\nu\,\mathrm{sech}\alpha$, α fixed, and $l+\frac{1}{2}=\nu\to\infty$,

$$\begin{aligned}f_l^{\mathrm{TE}}(x)&\sim\sinh\alpha\exp[-2\nu(\alpha-\tanh\alpha)][1+O(\nu^{-1})],\\ f_l^{\mathrm{TM}}(x)&\sim-\sinh\alpha\exp[-2\nu(\alpha-\tanh\alpha)]\\ &\quad\times[1+O(\nu^{-1})].\end{aligned}\quad(41)$$

In the transition region $x\sim\nu+\tau\nu^{1/3}$,

$$\begin{aligned}f_l^{\mathrm{TE}}(x)&\sim\frac{2^{1/3}}{\pi\nu^{1/3}}\\ &\quad\times[Ai^2(-2^{1/3}\tau)+Bi^2(-2^{1/3}\tau)]^{-1}+O(\nu^{-1}),\\ f_l^{\mathrm{TM}}(x)&\sim\frac{2^{2/3}}{\pi\nu^{1/3}}\\ &\quad\times[Ai'^2(-2^{1/3}\tau)+Bi'^2(-2^{1/3}\tau)]^{-1}+O(\nu^{-1}).\end{aligned}\quad(42)$$

Finally, for $x=\nu\sec\beta$,

$$f_l^{\mathrm{TE}}(x),\quad f_l^{\mathrm{TM}}(x)\sim\sin\beta\ [1+O(\nu^{-2})].\quad(43)$$

[16] See Eqs. (76) and (77) of A.

[17] See Eqs. (80), (81), and (40)–(43) of A.

The asymptotic forms for the derivatives of $f_l^{\mathrm{TE}}(x)$ and $f_l^{\mathrm{TM}}(x)$ can be found[18] for l fixed merely by differentiating (39) or (40) with respect to x. However, for the asymptotic forms in large ν, we can not use this procedure directly because a dependence upon ν enters both in the index $l=\nu-\frac{1}{2}$ in f_l^{TE}, f_l^{TM}, as well as in the expression for the argument. However, one may differentiate with respect to the parameter in x which does not involve ν. Thus, for example,

$$f_l^{\mathrm{TE}\prime}(\nu\,\mathrm{sech}\alpha)=-\frac{1}{\nu}\frac{d}{d\,\mathrm{sech}\alpha}f_l^{\mathrm{TE}}(\nu\,\mathrm{sech}\alpha)\quad(44)$$

and we may use the asymptotic form (41) for $f_l^{\mathrm{TE}}(\nu\,\mathrm{sech}\alpha)$ differentiating with regard to $\mathrm{sech}\alpha$ which is independent of ν. Similarly,

$$f_l^{\mathrm{TE}\prime}(\nu+\tau\nu^{1/3})=\frac{1}{\nu^{1/3}}\frac{d}{d\tau}f_l^{\mathrm{TE}}(\nu+\tau\nu^{1/3})\quad(45)$$

can be used together with the asymptotic form in (42). Thus from Eqs. (41)–(43) we learn that

$$\begin{aligned}f_l^{\mathrm{TE}(m)}(x),\ f_l^{\mathrm{TM}(m)}(x)&=O(\exp[-2\nu(\alpha-\tanh\alpha)]),\quad x=\nu\,\mathrm{sech}\alpha,\\ &=O(\nu^{-(m+1)/3}),\quad x=\nu=\tau\nu^{1/3},\\ &=O(\nu^{-m}),\quad x=\nu\sec\beta.\end{aligned}\quad(46)$$

The asymptotic behavior of the annular functions $S_l(a,R,K)$, $\bar{S}_l(a,R,\bar{K})$ and of their derivatives follows from their relations (35) to the spherical functions $s_l(a,k)$, $\bar{s}_l(a,\bar{k})$ which we have just considered.

E. Elimination of $R\to\infty$ Limit

From dimensional arguments alone, it is clear that the final zero-point energy $\Delta E(a)$ for the sphere must depend only upon the dimension a of the sphere and not on the radius R of the outer sphere which represents the limitation to a finite universe in the intermediate calculations, nor on the fixed ratio $\eta>1$ which represents the choice of a zero energy level in the intermediate calculations. Since the author has not been able to obtain a simple analytic expression for $\Delta E(a,R,\lambda)$, it will be convenient to remove the $R\to\infty$ limit in Eq. (12) before letting $\lambda\to 0$. We will explicitly justify any interchange of limits.

The procedure will be to rewrite the sums over the annular modes in Eq. (12) by using the Euler-Maclaurin summation formula with remainder. We explicitly group together the two sets $K_{lS}(a,R)$ and $K_{lS}(R/\eta,R)$ of TE annular modes, and similarly group the TM annular modes $\bar{K}_{lS}(a,R)$ and $\bar{K}_{lS}(R/\eta,R)$. Any regrouping of terms is possible since we have shown that all the sums are convergent. Thus for the TE modes

[18] See Theorem 3, of K. Knopp, *Theory and Application of Infinite Series* (Blackie and Son Ltd., London, 1964); see p. 542 for the justification of term-by-term differentiation of asymptotic series.

$$\sum_{S=1}^{\infty}[K_{lS}(a,R)\mathfrak{F}(\lambda K_{lS}(a,R))-K_{lS}(R/\eta,R)\mathfrak{F}(\lambda K_{lS}(R/\eta,R))]=\int_{S=1}^{\infty}dS[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))-K_l(R/\eta,R,S)$$
$$\times\mathfrak{F}(\lambda K_l(R/\eta,R,S))]+\tfrac{1}{2}[K_{l1}(a,R)\mathfrak{F}(\lambda K_{l1}(a,R))-K_{l1}(R/\eta,R)\mathfrak{F}(\lambda K_{l1}(R/\eta,R))]+\sum_{r=1}^{r=N}\frac{B_{2r}}{(2r)!}\frac{d^{2r-1}}{dS^{2r-1}}$$
$$\times[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))-K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))]-\int_{S=1}^{\infty}dS\,B_{2N}(S-[S])\frac{d^{2N}}{dS^{2N}}$$
$$\times[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))-K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))],\qquad(47)$$

where $B_{2N}(x)$ is the Bernoulli polynomial of order $2N$. All of the contributions which would appear from $S\to\infty$ limit vanish because of the cutoff function. The argument to follow shows that in Eq. (47) we may actually ignore all terms on the right-hand side beyond the initial integral. This holds because in the region where the correction terms between the sum and integral are important, the functions are approximately independent of a and R/η and hence cancel in the subtractions.

It is shown in A, Eqs. (79)–(85) that for large l the first zero of $K_l(R/\theta,R,S)$ for any $\theta>1$ is given by

$$K_{l1}(R/\theta,R)=k_{l1}(R)+\epsilon/R,\qquad(48)$$

where

$$\epsilon\sim(\nu^{1/3}/1.23\theta)\exp[-2\nu(\beta_\nu-\tanh\beta_\nu)],\qquad(49)$$

with

$$\operatorname{sech}\beta_\nu=\theta^{-1}(j_{\nu,1}/\nu)\sim\theta^{-1}[1+O(\nu^{-2/3})].\qquad(50)$$

Thus the zero depends only on the outer radius R, except for a correction term which decreases exponentially with increasing ν. But then in the first correction term in the Euler-Maclaurin formula, the principal contributions cancel as

$$\tfrac{1}{2}[(k_{l1}(R)+\epsilon/R)(\mathfrak{F}(\lambda k_{l1}(R))+\lambda(\epsilon/R)\mathfrak{F}'(\lambda k_{l1}(R))+\cdots)$$
$$-(k_{l1}(R)+\epsilon'/R)[\mathfrak{F}(\lambda k_{l1}(R))$$
$$+\lambda(\epsilon'/R)\mathfrak{F}'(\lambda k_{l1}(R))+\cdots]]$$
$$\sim O(\epsilon/R)+O(\epsilon'/R).\qquad(51)$$

The correction term decreases exponentially with increasing ν even if we let $\lambda\to0$, and, as $R\to\infty$, the contribution vanishes.

In order to make an analogous argument for each of the higher correction terms in (47), we reconsider the function $K_l(a,R,S)$ defined implicitly by Eq. (31). In the neighborhood of the first zero, the contribution from $\arctan[j_l(aK_l(a,R,S))/n_l(aK_l(a,R,S))]$ for large ν can be found, using the Debye asymptotic expansions for small argument. These show that the term decreases exponentially with increasing ν and the ratio of exponential decrease is governed by the ratio R/a, so that for $R\to\infty$ this term vanishes entirely. On the other hand, the term in $\arctan[j_l(RK_l(a,R,S))/n_l(RK_l(a,R,S))]$ near $S=1$ is finite and depends on a only through the exponential correction factor given above in (49) with $\theta\to R/a$. Thus in the neighborhood of $S=1$,

$$K_l(a,R,S)=k_l(R,S)+\delta_l(a,R,S),\qquad(52)$$

where $\delta_l(a,R,S)$ decreases exponentially with increasing ν and vanishes as $R\to\infty$. Differentiating with regard to S, we have the same result for any finite number of derivatives. But then we can turn back to any finite number of correction terms in the Euler-Maclaurin summation formula (47) and conclude that the principal contributions, which depend only upon the outer radius R, vanish, and the remaining correction terms decrease exponentially with increasing ν, vanishing for $R\to\infty$, and the conclusion is unaffected by the limit $\lambda\to0$.

We now proceed to the remainder term in (47). The function $B_{2N}(S-[S])$ is a periodic function of period one which agrees with the Bernoulli polynomial $B(S)$ for $0\leq S\leq1$. Since the function is bounded independently of ν as $|B_{2N}(S-[S])|\leq|B_{2N}|/(2N)!$, where B_{2N} is the $2N$ Bernoulli number, we wish to show that the remaining function in the integrand gives a finite integral decreasing rapidly with increasing ν. We rewrite the remainder integral as

$$\left|\int_{S=1}^{\infty}dS\,B_{2N}(S-[S])\frac{d^{2N}}{dS^{2N}}[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))-K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))]\right|$$
$$\leqslant\frac{|B_{2N}|}{(2N)!}\int_{S=1}^{S=S^*}dS\left|\frac{d^{2N}}{dS^{2N}}[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))-K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))]\right|$$
$$+\frac{|B_{2N}|}{(2N)!}\int_{S=S^*}^{\infty}dS\left|\frac{d^{2N}}{dS^{2N}}[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))]\right|$$
$$+\frac{|B_{2N}|}{(2N)!}\int_{S=S^*}^{\infty}dS\left|\frac{d^{2N}}{dS^{2N}}[K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))]\right|,\qquad(53)$$

where $S^*=S_l(R/\eta,R,\nu/b)$ with b fixed: $R/\eta<b<R$. When ν is large, the functions $K_l(a,R,S)$ and $K_l(R/\eta,R,S)$ can be expanded using the Debye asymptotic expansions for wave numbers $K<\nu/b$, and we can repeat the arguments given above to show that for the integral from $S=1$ to $S=S^*$, the principal contributions from $K_l(a,R,S)$ and $K_l(R/\eta,R,S)$ cancel leaving an integrand, and an integral, which decreases exponentially with increasing ν.

Only the remaining two terms on the right-hand side still require consideration. Noting the relation (35), we see that

$$\frac{dK_l(a,R,S)}{dS}=\left(\frac{dS_l(a,R,K)}{dK}\right)^{-1}=\frac{\pi}{Rf_l^{\mathrm{TE}}(RK)-af_l^{\mathrm{TE}}(aK)}\,, \quad (54)$$

and, in applying increasingly high derivatives $d^nK_l(a,R,S)/dS^n$, we may use the large ν asymptotic forms of (41)–(46). We find for $m>1$,

$$\begin{aligned}\frac{d^mK_l(a,R,S)}{dS^m}&=O(\nu^{1-m})\,, \quad \nu/R\ll K_l(a,R,S)\ll\nu/a\,,\\ &=O(\nu^{-m/3})\,, \quad K_l(a,R,S)\sim(\nu+\tau\nu^{1/3})/a\,,\\ &=O(\nu^{1-m})\,, \quad \nu/a\ll K_l(a,R,S)\,.\end{aligned} \quad (55)$$

Thus if a sufficient number of terms are taken in the Euler-Maclaurin formula (47), we will have derivatives d^mK_l/dS^m which decrease with increasing ν at any desired power of $\nu^{-1/3}$. The derivatives also fall on the cutoff function $\mathfrak{F}(\lambda K_l(a,R,S))$ and each derivative brings out a factor of λ,

$$\frac{d}{dS}\mathfrak{F}(\lambda K_l(a,R,S))=\frac{dK_l(a,R,S)}{dS}\lambda\mathfrak{F}'(\lambda K_l(a,R,S))\sim\lambda O\left[\frac{1}{R}\mathfrak{F}(\lambda K_l(a,R,S))\right], \quad (56)$$

where we recall from (43) that for $K_l(a,R,S)=\nu\,(\sec\beta)/R>\nu/R$, $dK_l(a,R,S)/dS\sim O(1/R)$ in ν, and have assumed $\mathfrak{F}'(\lambda K_l(a,R,S))\sim O[\mathfrak{F}(\lambda K_l(a,R,S))]$, as is exactly the case for an exponential cutoff $\mathfrak{F}(\lambda L)=\exp(-\lambda K)$, $\mathfrak{F}'(\lambda K)=-\exp(-\lambda K)$. Now if N in Eq. (47) is chosen large enough, each term arising in the differentiation of

$$\left|\frac{d^{2N}}{dS^{2N}}[K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))]\right| \quad \text{for} \quad K_l(a,R,S)>K^*=\nu/b=K_l(a,R,S^*)$$

will decrease faster than $O(\nu^{-4})$ as $\nu\to\infty$ or else will have five or more factors of λ. We now take the absolute value of each one of these terms. We see that the integrand for each term is smaller than the summand in (17) where the series diverged at most as λ^{-4} as $\lambda\to0$, and hence in the limit $\lambda\to0$ those terms having five or more factors of λ vanish. On the other hand, for those terms which decrease as $O(\nu^{-4})$ we may bound them above by setting the cutoff function equal to 1. After integration from $S=S^*$ to ∞, these contribute at most as $O(\nu^{1/3})O(\nu^{-4})+O(\nu)O(\nu^{-4})\sim O(\nu^{-3})$ where the first contribution is from the transition region $K_l(a,R,S)\sim(\nu+\tau\nu^{1/3})/a$ and the second arises from the integration in the region where the Debye asymptotic expansions for large arguments may be used. Thus we note that if $x=\nu\sec\beta$, then

$$\int dx\,F_\nu(x)=\nu\int d(\sec\beta)F_\nu(\nu\sec\beta)$$

and we may use the asymptotic expansions in ν for $F_\nu(\nu\sec\beta)$. It follows from the asymptotic expansions of (43) that the integral with regard to the parameter $\sec\beta$ will be finite even for $S\to\infty$. The analysis may be repeated with R/η replacing a to arrive at the same conclusions for the very last term in (53).

The upshot of this painfully explicit analysis is that every term in the Euler-Maclaurin summation formula (47) beyond the first integral gives a contribution which on multiplication by $2l+1$ and summation over all l from $l=1$ to ∞ is a finite, continuous function as $\lambda\to0$ and which vanishes for $R\to\infty$.

From the form (46) of the asymptotic limits for the TM modes, we see that an exactly analogous argument may be made in this case also. Hence we may rewrite $\Delta E(a)$ as

$$\begin{aligned}\Delta E(a)=\lim_{R\to\infty}\lim_{\lambda\to0}\tfrac{1}{2}\hbar c\Bigg[\sum_{l=1}^{\infty}(2l+1)\Bigg(&\sum_{s=1}^{\infty}k_{ls}(a)\;\mathfrak{F}(\lambda k_{ls}(a))+\int_{s=1}^{\infty}dS\;K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S))+\sum_{\bar{s}=1}^{\infty}\bar{k}_{l\bar{s}}(a)\mathfrak{F}(\lambda\bar{k}_{l\bar{s}}(a))\\ &+\int_{S=1}^{\infty}d\bar{S}\;\bar{K}_l(a,R,\bar{S})\mathfrak{F}(\lambda\bar{K}_l(a,R,\bar{S}))-\sum_{s=1}^{\infty}k_{ls}(R/\eta)\mathfrak{F}(\lambda k_{ls}(R/\eta))-\int_{S=1}^{\infty}dS\;K_l(R/\eta,R,S)\mathfrak{F}(\lambda K_l(R/\eta,R,S))\\ &-\sum_{s=1}^{\infty}\bar{k}_{l\bar{s}}(R/\eta)\mathfrak{F}(\lambda\bar{k}_{l\bar{s}}(R/\eta))-\int_{S=1}^{\infty}d\bar{S}\;\bar{K}_l(R/\eta,R,\bar{S})\mathfrak{F}(\lambda\bar{K}_l(R/\eta,R,\bar{S}))\Bigg)\Bigg]\,.\end{aligned} \quad (57)$$

Since the functions relating the annular frequencies and indices are single-valued for positive frequencies above the lowest mode, the variables of integration may be changed from the indices S, $\bar{S}$ to the frequencies K, $\bar{K}$. Thus

$$\int_{S=1}^{\infty} dS\, K_l(a,R,S)\mathfrak{F}(\lambda K_l(a,R,S)) = \int_{K=K_{l1}(a,R)}^{\infty} dK \frac{dS_l(a,R,K)}{dK} K\mathfrak{F}(\lambda K),$$
$$\int_{\bar{S}=1}^{\infty} d\bar{S}\, \bar{K}_l(a,R,\bar{S})\mathfrak{F}(\lambda \bar{K}_l(a,R,\bar{S})) \quad \int_{\bar{K}=\bar{K}_{l1}(a,R)}^{\infty} d\bar{K} \frac{d\bar{S}_l(a,R,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda \bar{K}). \tag{58}$$

However, we can express $dS_l(a,R,K)/dK$ and $d\bar{S}_l(a,R,\bar{K})/d\bar{K}$ in terms of the functions s_l and $\bar{s}_l$ using Eq. (35). Rewriting all the annular modes in this fashion and rearranging terms,

$$\begin{aligned}\Delta E(a) = \lim_{R\to\infty}\lim_{\lambda\to 0} \tfrac{1}{2}\hbar c\Bigg[\sum_{l=1}^{\infty} (2l+1)\Bigg(\sum_{s=1}^{\infty} k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a)) - \int_{K=K_{l1}(a,R)}^{\infty} dK \frac{ds_l(a,K)}{dK} K\mathfrak{F}(\lambda K) \\ + \sum_{\bar{s}=1}^{\infty} \bar{k}_{l\bar{s}}(a)\mathfrak{F}(\lambda \bar{k}_{l\bar{s}}(a)) - \int_{\bar{K}=\bar{K}_{l1}(a,R)}^{\infty} d\bar{K} \frac{d\bar{s}_l(a,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) - \sum_{s=1}^{\infty} k_{ls}(R/\eta)\mathfrak{F}(\lambda k_{ls}(R/\eta)) \\ + \int_{K=K_{l1}(R/\eta,R)}^{\infty} dK \frac{ds_l(R/\eta,K)}{dK} K\mathfrak{F}(\lambda K) - \sum_{\bar{s}=1}^{\infty} \bar{k}_{l\bar{s}}(R/\eta)\mathfrak{F}(\lambda \bar{k}_{l\bar{s}}(R/\eta)) + \int_{\bar{K}=\bar{K}_{l1}(R/\eta,R)}^{\infty} d\bar{K} \frac{d\bar{s}_l(R/\eta,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) \\ + \int_{K=K_{l1}(a,R)}^{\infty} dK \frac{ds_l(R,K)}{dK} K\mathfrak{F}(\lambda K) - \int_{K=K_{l1}(R/\eta,R)}^{\infty} dK \frac{ds_l(R,K)}{dK} K\mathfrak{F}(\lambda K) + \int_{\bar{K}=\bar{K}_{l1}(a,R)}^{\infty} d\bar{K} \frac{d\bar{s}_l(R,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) \\ - \int_{\bar{K}=\bar{K}_{l1}(R/\eta,R)}^{\infty} d\bar{K} \frac{d\bar{s}_l(R,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) \Bigg) \Bigg].\end{aligned} \tag{59}$$

The last two lines cancel except for

$$-\int_{K=K_{l1}(R/\eta,R)}^{K=K_{l1}(a,R)} dK \frac{ds_l(R,K)}{dK} K\mathfrak{F}(\lambda K), \quad -\int_{\bar{K}=\bar{K}_{l1}(R/\eta,R)}^{\bar{K}=\bar{K}_{l1}(a,R)} d\bar{K} \frac{d\bar{s}_l(R,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}). \tag{60}$$

However, as noted in (48)–(51) the differences $K_{l1}(a,R)-K_{l1}(R/\eta,R)$ and $\bar{K}_{l1}(a,R)-\bar{K}_{l1}(R/\eta,R)$ decrease exponentially in ν, while from (42) we see that in this same region K, $\bar{K}\sim(\nu+\tau\nu^{1/3})/R$, $ds_l(R,K)/dK$, $d\bar{s}_l(R,\bar{K})/d\bar{K}$ $\sim O(\nu^{-1/3})$. Thus the contribution of these terms is finite when summed over $\sum_{l=1}^{\infty}(2l+1)$, is a continuous function of λ as $\lambda\to 0$, and vanishes as $R\to\infty$. They therefore make no contribution to $\Delta E(a)$.

The same basic argument can be repeated to show that the lower limits of all the remaining integrals may be extended to $K=0$. Thus for $K\leq K_{l1}(a,R)$, $\bar{K}\leq\bar{K}_{l1}(a,R)$, we see from (41) that

$$\frac{ds_l(a,K)}{dK} = -\frac{a}{\pi} f_l^{\mathrm{TE}}(aK), \quad \frac{d\bar{s}_l(a,\bar{K})}{d\bar{K}} = -\frac{a}{\pi} f_l^{\mathrm{TM}}(a\bar{K})$$

are exponentially small in ν. Similar expressions hold for R/η replacing a. Thus exactly as in the previous paragraph the contributions are finite when summed over l, continuous for $\lambda\to 0$, and vanish as $R\to\infty$.

Combining the above results, we have

$$\begin{aligned}\Delta E(a) = \lim_{R\to\infty}\lim_{\lambda\to 0} \tfrac{1}{2}\hbar c\Bigg\{ \sum_{l=1}^{\infty} (2l+1)\Bigg[\Bigg(\sum_{s=1}^{\infty} k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a)) - \int_{K=0}^{\infty} dK \frac{ds_l(a,K)}{dK} K\mathfrak{F}(\lambda K) + \sum_{\bar{s}=1}^{\infty} \bar{k}_{l\bar{s}}(a)\mathfrak{F}(\lambda\bar{k}_{l\bar{s}}(a)) \\ - \int_{\bar{K}=0}^{\infty} d\bar{K} \frac{d\bar{s}_l(a,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) \Bigg) - \Bigg(\sum_{s=1}^{\infty} k_{ls}(R/\eta)\mathfrak{F}(\lambda k_{ls}(R/\eta)) - \int_{K=0}^{\infty} dK \frac{ds_l(R/\eta,K)}{dK} K\mathfrak{F}(\lambda K) \\ + \sum_{\bar{s}=1}^{\infty} \bar{k}_{l\bar{s}}(R/\eta)\mathfrak{F}(\lambda\bar{k}_{l\bar{s}}(R/\eta)) - \int_{\bar{K}=0}^{\infty} d\bar{K} \frac{d\bar{s}_l(R/\eta,\bar{K})}{d\bar{K}} \bar{K}\mathfrak{F}(\lambda\bar{K}) \Bigg)\Bigg]\Bigg\}.\end{aligned} \tag{61}$$

The expression for $\Delta E(a)$ can be transformed further by use of the Euler-Maclaurin summation formula in its simplest form,

$$\sum_{s=1}^{\infty} k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a))=\int_{s=1}^{\infty} ds\, k_l(a,s)\mathfrak{F}(\lambda k_l(a,s))+\tfrac{1}{2}[k_{l1}(a)\mathfrak{F}(\lambda k_{l1}(a))]+\int_{s=1}^{\infty} ds(s-[s]-\tfrac{1}{2})\frac{d}{ds}[k_l(a,s)\mathfrak{F}(\lambda k_l(a,s))]. \quad (62)$$

Now for $s>1$, $k_l(a,s)$ is a single-valued[19] function of s, and $s_l(a,k)$ is a single-valued function of k for all k, so that we can change the variable of integration from s to k. Furthermore,

$$\int_{k=0}^{k=k_{l1}(a)} dk\{s_l(a,k)-[s_l(a,k)]-\tfrac{1}{2}\}\frac{d}{dk}[k\mathfrak{F}(\lambda k)]=\int_{k=0}^{k=k_{l1}(a)} dk[s_l(a,k)-\tfrac{1}{2}]\frac{d}{dk}[k\mathfrak{F}(\lambda k)]$$
$$=\tfrac{1}{2}k_{l1}(a)\mathfrak{F}(\lambda k_{l1}(a))-\int_{k=0}^{k=k_{l1}(a)} dk\frac{ds_l(a,k)}{dk}k\mathfrak{F}(\lambda k). \quad (63)$$

Thus the sum over s can be converted into an integral

$$\sum_{s=1}^{\infty} k_{ls}(a)\mathfrak{F}(\lambda k_{ls}(a))=\int_{k=0}^{\infty} dk\frac{ds_l(a,k)}{dk}k\mathfrak{F}(\lambda k)+\int_{k=0}^{\infty} dk\{s_l(a,k)-[s_l(a,k)]-\tfrac{1}{2}\}\frac{d}{dk}[k\mathfrak{F}(\lambda k)]. \quad (64)$$

The same argument can be repeated for the contributions from the TM normal modes $\bar{k}_{l\bar{s}}(a)$ and for the terms where R/η appears instead of a.

It is clearly convenient to change the variables of integration from k to $x=ak$ and $x=(R/\eta)k$. If we also denote $s_l(1,k)=s_l(k)$ and $\bar{s}_l(1,k)=\bar{s}_l(k)$, then

$$\Delta E(a)=\lim_{R\to\infty}\lim_{\lambda\to 0}\tfrac{1}{2}\hbar c\Bigg[\sum_{l=1}^{\infty}(2l+1)\Bigg(\frac{1}{a}\int_{x=0}^{\infty} dx\{s_l(x)-[s_l(x)]+\bar{s}_l(x)-[\bar{s}_l(x)]-1\}\frac{d}{dx}[x\mathfrak{F}((\lambda/a)x)]$$
$$-\frac{\eta}{R}\int_{x=0}^{\infty} dx\{s_l(x)-[s_l(x)]+\bar{s}_l(x)-[\bar{s}_l(x)]-1\}\frac{d}{dx}[x\mathfrak{F}((\eta\lambda/R)x)]\Bigg)\Bigg], \quad (65)$$

where

$$s_l(x)=-\frac{1}{\pi}\arctan\frac{j_l(x)}{n_l(x)}=\frac{x}{\pi}-\tfrac{1}{2}l+\frac{1}{\pi}\arctan\frac{B_l(x)}{A_l(x)},$$
$$\bar{s}_l(x)=1-\frac{1}{\pi}\arctan\left[\frac{d}{dx}(xj_l(x))\Big/\frac{d}{dx}(xn_l(x))\right]=\frac{x}{\pi}-\tfrac{1}{2}(l-3)+\frac{1}{\pi}\arctan\frac{B_l(x)-A_l'(x)}{A_l(x)+B_l'(x)}. \quad (66)$$

The first term for $\Delta E(a)$ in (65) is independent of R, and in the limit $\lambda\to 0$ is given by const$/a$ or is infinite. The second term is exactly the same function where a is replaced by R/η. Thus if each term separately when summed over l makes a finite contribution for $\lambda\to 0$, then we may neglect the second term when taking the $R\to\infty$ limit.

F. Numerical Evaluation of $\Delta E(a)$

The form for $\Delta E(a)$ given in Eq. (65) is suitable for numerical evaluation, and the failure of the author to obtain an analytic solution has forced the use of this expedient. Since the expression in the first line of (65) seems to be finite, we have omitted the term depending on R.

It was found that the use of Riesz typical means[20] formed a convenient cutoff for the numerical work because this cutoff could be written as a polynomial in x and each monomial term could be integrated separately with the value of the cutoff parameter $M\sim 1/\lambda$ as a multiplicative factor. Hence numerical evaluation was faster than for an exponential cutoff, and also, in changing the parameter M, it was not necessary to start the integral over again, at $x=0$, but merely to integrate from the old to the new value of M, add the new integrals on to those previously obtained,

[19] The function $k_l(a,s)$ is single-valued in s for all real s, but $\bar{k}_l(a,\bar{s})$ is not single-valued in $\bar{s}$. The argument is phrased so as to include both cases.

[20] See Ref. 8, p. 86.

and then combine in polynomial form to give the new approximation for $\Delta E(a)$. The numerical evaluations compute

$$\Delta E(a)=\lim_{M\to\infty}\frac{\hbar c}{2a}\int_{x=0}^{x=M}dx\sum_{l=1}^{\infty}(2l+1)\{s_l(x)-[s_l(x)]+\bar{s}_l(x)-[\bar{s}_l(x)]-1\}\frac{d}{dx}\left[x\left(1-\frac{x}{M}\right)^{\kappa}\right], \tag{67}$$

where κ is any integer larger than some fixed κ_0. We note that the integrand involves an infinite series in l which is very rapidly convergent for any finite x. Thus using the Debye asymptotic expansions for large ν, $x=\nu\,\mathrm{sech}\alpha$,

$$\begin{aligned}s_l(x)&\sim\tfrac{1}{2}\exp[-2\nu(\alpha-\tanh\alpha)][1+O(1/\nu)],\\ \bar{s}_l(x)&\sim1-\tfrac{1}{2}\exp[-2\nu(\alpha-\tanh\alpha)][1+O(1/\nu)],\end{aligned} \tag{68}$$

so that in this region $x\ll\nu$, $[s_l(x)]=[\bar{s}_l(x)]=0$, and

$$s_l(x)-[s_l(x)]+\bar{s}_l(x)-[\bar{s}_l(x)]-1 \sim O(\exp[-2\nu(\alpha-\tanh\alpha)](1/\nu)). \tag{69}$$

The integrand in (67) is discontinuous at each of the TE and TM spherical normal modes and hence the integrations were carried out using Gaussian quadrature integration for each continuous section of the curve. Gaussian quadrature is particularly suitable here since the endpoints of the interval are not involved. All calculations were made in FORTAN IV double-precision arithmetic on the IBM 7094 computer.

The results of the integration for various values of the cutoff parameter M are given in Table III and seem to indicate

$$\Delta E(a)\cong+0.09\hbar c/2a. \tag{70}$$

The computer program was tested by omitting the sum over l and evaluating the expression $\Delta E_l(a)$ for fixed values of l where it can be proved that the expression does indeed converge. Although $\Delta E_l(a)$ for $l=0$ does not enter into the sum involved in $\Delta E(a)$, it can be computed exactly analytically as $\Delta E_{l=0}(a)=-(\pi/24)\times\hbar c/2a$. The numbers found using the cutoff with Riesz typical means appear in Table IV and seem to converge rapidly to the required limit. The fact that the $\kappa=2$ form gives the exact result each time is an accident due to the fact that each of the values of M used is a TE or TM normal mode.

TABLE III. Approximate values for $\Delta E(a)$ using Riesz means (in units of $\hbar c/2a$).

M	$\kappa=3$	$\kappa=4$	$\kappa=5$
9.095011	0.09580	0.09232	0.09125
12.485947	0.09481	0.09299	0.09245
15.514603	0.09372	0.09305	0.09280
17.838643	0.09221	0.09300	0.09292
20.121806	0.09278	0.09302	0.09297
23.591274	0.09325	0.09297	0.09300
26.791390	0.09282	0.09294	0.09299
29.642604	0.09202	0.09297	0.09298
32.334735	0.09344	0.09288	0.09296
35.1996423	0.09251	0.09288	0.09293
37.804940	0.09192	0.09285	0.09291
40.503839	0.09383	0.09284	0.09289

The calculation for $\Delta E(a)$ is incomplete unless we can determine whether the limit of Eq. (67) exists as a finite number and can estimate the rate of convergence so that we may be confident that the integral has indeed been carried sufficiently far that the value found will not differ significantly from the true limiting value. Unfortunately, we have not been able to derive a mathematically rigorous proof, and the apparently disappointing nature of the result has inclined us to be satisfied with the patent stability of the values seen in Table III, and some strong qualitative arguments.

Although from the point of view of computer calculations the use of Riesz typical means is far more convenient than an exponential cutoff, when we wish to make theoretical arguments the latter is more tractable. We emphasize that the limiting results are the same independent of the cutoff procedure.

If we take $\mathfrak{F}(\lambda x)=\exp(-x\lambda)$ in Eq. (65) and consider only a single term $\Delta E_l(a,\lambda)$ in the sum over l, it is easy to prove that $\Delta E_l(a)=\lim_{\lambda\to0}\Delta_l E(a,\lambda)$ is finite. This follows since we may separate off a finite integral from $x=0$ to, say, $x=2\nu$, and then apply the Euler-Maclaurin summation formula with remainder. The remainder term may be integrated by using the expansions for large x given by McMahon for the TE modes and by the author for the TM modes. [See Eqs. (23) and (38) of A.] The rate of convergence also follows from these expressions.

In Table V we list the values for $\Delta E_l(a)$ for $l=0$, 1, 2, 10, 11, 20, and 21. We note that as l increases through the values indicated, $\Delta E_l(a)$ seems to go very rapidly over to a value independent of l which is very nearly the negative of the value found for $\Delta E(a)$. Since $\Delta E_l(a)$ is a continuous function of l, it seems extremely likely, although we have not proved this, that this behavior will hold for arbitrarily large l.

For $x<\nu$, we may use the Debye asymptotic expansions to see that the integrand (69) appearing in Eq. (65) for $\Delta E_l(a)$ is exponentially small in ν until $x\sim\nu+O(\nu^{1/3})$. Thus $\Delta E_l(a,\lambda)$ falls off in λ roughly as $\exp(-\nu\lambda)$ and we may approximate

$$\Delta E_l(a,\lambda)\sim-q(\hbar c/2a)\exp(-\nu\lambda/a). \tag{71}$$

TABLE IV. Convergence of Riesz means for $\Delta E_l(a)$ (in units of $\hbar c/2a$) for $l=0$.

M	$\kappa=2$	$\kappa=3$	$\kappa=4$	$\kappa=5$
9.424777	−0.130899	−0.129445	−0.128718	−0.127292
18.849555	−0.130899	−0.130536	−0.130354	−0.129992
47.123889	−0.130899	−0.130841	−0.130812	−0.130754
84.823000	−0.130899	−0.130881	−0.130872	−0.130854
92.247779	−0.130899	−0.130885	−0.130877	−0.130863

TABLE V. Values computed for $\Delta E_l(a)$ (in units of $\hbar c/2a$).

l	$\Delta E_l(a)$
0	$-\pi/24\sim-0.130899$
1	-0.094541
2	-0.093969
10	-0.09375
11	-0.09375
20	-0.09373
21	-0.09373

But if we now assume that Eq. (71) is an equality, then

$$\Delta E(a)=\lim_{\lambda\to 0}\frac{\hbar c}{2a}\sum_{l=1}^{\infty}(2l+1)\int_{x=0}^{\infty}dx\{s_l(x)-[s_l(x)]+\bar{s}_l(x)-[\bar{s}_l(x)]-1\}\frac{d}{dx}\left[x\exp\left(-\frac{\lambda}{a}x\right)\right]$$

$$=\lim_{\lambda\to 0}\left(1+\lambda\frac{\partial}{\partial\lambda}\right)\sum_{l=1}^{\infty}\Delta E_l(a,\lambda)$$

$$=\lim_{\lambda\to 0}\left(1+\lambda\frac{\partial}{\partial\lambda}\right)\left(-q\frac{\hbar c}{2a}\frac{\exp(-\frac{3}{2}\lambda/a)}{1-\exp(-\lambda/a)}\right)$$

$$=+q\frac{\hbar c}{2a}, \quad (72)$$

which agrees within the stated approximation with the results given in Eq. (70) and Table III. Moreover, the discrepancies between the values of $\Delta E(a)$ and $-\lim_{l\to\infty}\Delta E_l(a)$ can also be accounted for satisfactorily by noting that if (71) holds for large l, then

$$\Delta E(a)=+q\frac{\hbar c}{2a}+\sum_{l=1}^{\infty}\left(\Delta E_l(a)+q\frac{\hbar c}{2a}\right). \quad (73)$$

Finally, the rate of approach to the limit in Eq. (72) is of the order of λ, which is again in rough agreement with the calculations using Riesz means with the correspondence $1/\lambda\sim M$.

G. Independent Numerical Check

Because of the surprise at the positive result for $\Delta E(a)$, we have made an effort to confirm the conclusion by a method as different as possible from that presented in the body of this paper. The availability of modern high-speed computers made possible a partial check by merely directly summing the numerical values for the modes as indicated in Eq. (12). With a collection of nearly 5000 spherical and annular modes at $\eta=2$, and using Riesz typical means $\kappa=3$ cut off at $1/\lambda\sim M=10.0$, we obtained $\Delta E(a=1, R=10, M=10)\sim+0.17\hbar c/2$ and $\Delta E_{l=1}(a=1, R=10, M=10)\sim-0.02\hbar c/2$. The order of magnitudes, the change in sign between the two quantities, and the fact that $\Delta E(a,R)>\Delta E(a)$ and $\Delta E_l(a,R)>\Delta E_l(a)$ are all consistent with the results $\Delta E(a)\sim+0.09\hbar c/a$, $\Delta E_{l=1}(a)\sim-0.0945\hbar c/2a$, obtained earlier for the limit $R\to\infty$. In the case of $\Delta E_{l=1}(a,R)$, it was possible to increase substantially the number of normal modes to obtain the limit as $M\to\infty$, $\Delta E_{l=1}(a=1, R=10)\sim-0.0218\hbar c/2$, in agreement with the value at $M=10$. Although this method of direct summation does allow this qualitative check, any attempt to check $\Delta E(a)$ further by increasing the values of M or of R significantly would at present run into prohibitive expenses in computer time.

H. Conclusions

Thus in summary, the hoped for connection between electromagnetic zero-point energy and charge quantization was not found; rather the Casimir model fails in the form described. The calculation gives a finite value for the zero-point energy of a conducting spherical shell, but the sign of the energy is opposite from that anticipated. Looked at from an entirely different point of view, the result shows the first example of the repulsive aspect of retarded dispersion forces which was conjectured by Verwey and Overbeek.[21] However, implications for the theory of dispersion forces will be considered elsewhere.

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[21] E. J. W. Verwey and J. Th. G. Overbeek, *Theory of Stability of Lyophobic Colloids* (Elsevier Publishing Co., Inc., Amsterdam, 1948), p. 104.