Correlation between Transcendental and Polynomial Lagrangians*

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A calculational procedure is defined to permit the computation of radiative corrections for nonderivative transcendental interactions. The method, a generalization of the Fradkin iterative procedure, is guaranteed to maintain the correct analyticity structure of at least the lowest-order corrections, and suggests a correlation between transcendental and conventional polynomial interactions, along with simple conditions for equivalence. A class of chiral models, treated in this sense and tested in an approximate way which partially violates the chiral symmetry, appears to be dynamically empty under this equivalence.

I. INTRODUCTION

INTEREST has recently been expressed in certain highly nonlinear, or transcendental interaction, Lagrangians, in connection with the low-energy predictions of chiral symmetry.[1] The estimates obtained involve only tree graphs, while the question of radiative (closed-path Feynman integral) corrections is completely open. The purpose of this paper is to define a calculational procedure in which the radiative corrections derived from a nonderivative transcendental Lagrangian may be put into a one-to-one correspondence with the ordinary radiative corrections of a related polynomial interaction with modified coupling constant. This correlation suggests simple conditions for equivalence between certain transcendental and polynomial interactions, which may be of some value in studying the dynamical content of chiral symmetry. However, because of the restriction to nonderivative interactions, these results are not directly applicable to the chiral models.

The method of calculation will be a variant of Fradkin's elegant iterative procedure,[2] in which all the n-point Green's functions are expanded in powers of the complete transcendental interaction. The modification of Fradkin's procedure is one which ensures the correct analyticity, and hence unitarity, of the lowest-order terms of the expansion for an Hermitian interaction. The computation is most simply done in configuration space, and we accordingly choose a convenient form of the regularization which must be used to give meaning to the manipulations involved in summing over products of otherwise singular boson propagators: The zero-mass causal propagator

$$D_c(x)=\frac{i}{4\pi^2}\frac{1}{x^2+i\epsilon},\quad x^2=\mathbf{x}^2-x_0{}^2,$$

is replaced by[3] $i/(x^2+L+i\epsilon)$, where L is a small, real, positive (length)^2, and we omit the factor $4\pi^2$. The restriction to zero boson mass is for illustrative convenience only, while any form of regularization can be used for which both $-i\Delta_c{}^{\mathrm{Reg}}(x)$ and $-i[\Delta_c{}^{\mathrm{Reg}}(0)-\Delta_c{}^{\mathrm{Reg}}(x)]$ are real and positive for $x^2=\lambda>0$, and for which $-i\Delta_c{}^{\mathrm{Reg}}(0)$ diverges as the regularization is removed.

As long as regularization is retained, we have in lowest orders a theory with proper cut structure, but one whose numerical values depend upon the regularization parameters. This analysis does not specify what happens in that final limit in which the regularization is removed; but it does suggest that there are classes of interactions for which there is no need to calculate any radiative corrections at all, since these theories will, at best, be expected to give only trivial phase and mass renormalizations. This expectation is suggested by observing the results of an interchange of limits, corresponding to the removal of regularization before final calculations are performed; and it must be emphasized that this procedure in part defines the correlation scheme. However, within such classes of trivial interactions, this can be made quite plausible by simple examples where one finds results in agreement with those obtained by keeping the regularization until the very last step.

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[1] The original observation was made by S. Weinberg, Phys. Rev. Letters 18, 188 (1967). A recent summary has been given by P. Chang and F. Gürsey, Phys. Rev. 164, 1752 (1967).

[2] E. S. Fradkin, Nucl. Phys. 49, 624 (1963); 76, 588 (1966).

[3] For nonzero boson mass μ, this corresponds to writing

$$\Delta_c{}^{\mathrm{Reg}}(x)=\int_0^\infty d\eta^2\rho(\eta^2)\Delta_c(x;\eta^2),$$

with $\rho(\eta^2)=\delta(\eta^2-\mu^2)-\frac{1}{2}\theta(\eta-\mu)[L/(\eta^2-\mu^2)]^{1/2}J_1((L(\eta^2-\mu^2)^{1/2})$. The transform $\tilde{\Delta}_c{}^{\mathrm{Reg}}(k)$ thus has the usual cut structure and is given by the appropriate Bessel functions, $(i/4\pi^2)\Delta_c{}^*(k;L)$. This resembles the regularization used, in a related context, by G. Efimov, Yadern. Fiz. 2, 180 (1965) [English transl.: Soviet J. Nucl. Phys. 2, 126 (1966)]. In the usual way of regularizing,

$$\Delta_c{}^{\mathrm{Reg}}(x)=\sum_{i=1}^{r}C_i\Delta_c(z;\mu_i{}^2),$$

with $r\geq 3$, so that the C_i satisfy

$$\textstyle\sum_i C_i=\sum_i C_i\mu_i{}^2=0$$

in order to remove all singular dependence at the light cone. For example, if $r=3$, $\mu=\mu_1=0$, $\mu_3/\mu_2=\xi>1$, then as $x^2\rightarrow 0+$, $-i\Delta_c{}^{\mathrm{Reg}}(0)=\mu_2{}^2[\xi^2/(\xi^2-1)]\ln\xi$, which is positive and diverges as $\mu_2\rightarrow\infty$.

II. CORRELATIONS

We consider the simplest interaction Lagrangian $\mathcal{L}'=-\bar{\psi}V(\phi)\psi$, where $V(\phi)$ denotes a local function of a scalar boson field $\phi(x)$, of form

$$V(z)=g_0 zU(z^2),$$

with

$$U(z^2)=\int_0^\infty d\alpha^2\chi(\alpha^2)e^{-\alpha^2z^2}$$

and $\chi(\alpha^2)$ to be specified below. Corresponding to this $\mathcal{L}'$, consider first n vertices in any diagram of order $g_0{}^n$, in which all the boson lines are internal to the diagram, i.e., virtual. Each point x_i lies on a fermion line, the same or a different line depending on the process, and between each point are exchanged all possible numbers of virtual bosons; by symmetry, for this interaction, n must be even. In addition, each point x_i may emit and reabsorb its own cloud of virtual particles. The latter contributions are frequently considered too singular to be sensible and, typically, are omitted from such peratization calculations; but when regularization is used, these terms are finite and should be included for completeness.[4] They turn out to be the crucial mechanism by which at least the lowest-order Fradkin iterations are guaranteed to be unitary, and provide sufficient damping to produce our results in the final regularization limit, $L\to 0$.

If we define $\lambda_{ij}=(x_i-x_j)^2$, and initially assume that all points are spacelike with respect to each other ($\lambda_{ij}>0$, $i\neq j$), then the function corresponding to all virtual exchanges among the n points, and no external lines ending on them, is given by the simple functional expression[5]

$$F(1[0],2[0],\cdots,n[0])$$
$$=\exp\left[-\tfrac{1}{2}i\int\int\frac{\delta}{\delta\phi(u)}\Delta_c{}^{\mathrm{Reg}}(u-v)\frac{\delta}{\delta\phi(v)}\right] V(\alpha_0\phi(x_1))\cdots V(\alpha_0\phi(x_n))\big|_{\phi=0},\quad (1)$$

where the number inside each square bracket denotes the number of external lines at the corresponding point. It is convenient to introduce the representation

$$V(\alpha_0 z)=\int_{-\infty}^{+\infty}d\omega\,\tilde{V}(\omega)e^{i\omega z},$$

with

$$\tilde{V}(\omega)=-\frac{ig_0\omega}{4\sqrt{\pi}}\int_0^\infty\frac{d\alpha^2}{\alpha^3}\chi(\alpha^2)e^{-\omega^2/4\alpha^2},$$

in which case (1) becomes[6]

$$F(1[0],2[0],\cdots,n[0])=\left(\frac{-ig_0}{4\sqrt{\pi}}\right)^n\int_0^\infty\frac{d\alpha_1{}^2}{\alpha_1{}^3}\chi(\alpha_1{}^2)\cdots$$
$$\times\int_0^\infty\frac{d\alpha_n{}^2}{\alpha_n{}^3}\chi(\alpha_n{}^2)\int_{-\infty}^{+\infty}d\omega_1\cdots\int_{-\infty}^{+\infty}d\omega_n\omega_1\cdots\omega_n$$
$$\times\exp\left[-\frac{1}{4}\left(\frac{\omega_1{}^2}{\alpha_1{}^2}+\cdots+\frac{\omega_n{}^2}{\alpha_n{}^2}\right)-\tfrac{1}{2}\sum_{ij}\frac{\omega_i\omega_j}{L+\lambda_{ij}}\right].\quad (2)$$

The ω integrals of (2) are simple Gaussians and, with

$$D_{ij}=\frac{\delta_{ij}}{2\alpha_i{}^2}+\frac{1}{L+\lambda_{ij}},$$

yield

$$F(1[0],2[0],\cdots,n[0])=(-\tfrac{1}{2}ig_0)^n$$
$$\times\int_0^\infty\frac{d\alpha_1{}^2}{\alpha_1{}^3}\chi(\alpha_1{}^2)\cdots\int_0^\infty\frac{d\alpha_n{}^2}{\alpha_n{}^3}\chi(\alpha_n{}^2)\,[\det D]^{-1/2}$$
$$\times\sum_{\mathrm{perm}}(D^{-1})_{12}(D^{-1})_{34}\cdots(D^{-1})_{n-1,n},\quad (3)$$

where the permutation sum[7] in (3) is over all distinct products of the inverse matrix elements $(D^{-1})_{ij}$. For $n=2$, for example, this sum is just $(D^{-1})_{12}$, while for $n=4$ it is $(D^{-1})_{12}(D^{-1})_{34}+(D^{-1})_{13}(D^{-1})_{24}+(D^{-1})_{14}(D^{-1})_{23}$.

For fixed λ_{ij} we now remove the regularization by passing to the limit $L\to 0$, in which case it is easy to see that

$$\det D\to\prod_{i=1}^n\left(\frac{1}{L}+\frac{1}{2\alpha_i{}^2}\right)$$

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[4] One familiar usage is the generation of the change in the electron's wave-function renormalization constant under a gauge transformation of the third kind; see, e.g., B. Zumino, J. Math. Phys. **1**, 1 (1960).

[5] This follows directly from the formal functional solutions of J. Schwinger, Lecture Notes, 1954 (unpublished), and K. Symanzik, Z. Naturforsch. **9a**, 809 (1954).

[6] Because we retain the virtual clouds about each point, the sums over i and j run over all coordinates. For $n=2$, this sum may be written as

$$-\frac{1}{2}\frac{1}{L+\lambda}(\omega_1+\omega_2)^2-\frac{1}{2}\frac{\lambda}{L}\frac{1}{L+\lambda}(\omega_1{}^2+\omega_2{}^2),$$

which is negative for any positive $\lambda\equiv\lambda_{12}$. Hence the representation (2) is valid for all $\lambda>0$, and $i\tilde{F}(q)$ will be analytic in the cut q^2 plane, and real for positive q^2. The simplest nucleon self-energy correction,

$$\Sigma^{(2)}(\omega),$$

constructed from $\tilde{F}_2$, will then be analytic in the cut ω plane, and real for real $\omega^2<m^2$, where m denotes the nucleon mass. These properties are here guaranteed for any $L>0$, in contrast to the previous situations in which the virtual-point clouds were omitted, and where one finds clear violations of this cut structure; see, e. g., H. M. Fried, Nuovo Cimento **52**, 1333 (1967).

[7] This permutation sum may be displayed as the result of the operation

$$\frac{\partial}{\partial f_1}\cdots\frac{\partial}{\partial f_n}\exp(\tfrac{1}{2}\sum_{ij}f_i(D^{-1})_{ij}f_j)\big|_{f_l=0},$$

while the sum leading to (5) is given by

$$\left(\frac{\partial}{\partial f_1}\right)^2\left(\frac{\partial}{\partial f_2}\right)^2\frac{\partial}{\partial f_3}\cdots\frac{\partial}{\partial f_n}\exp(\tfrac{1}{2}\sum_{ij}f_i(D^{-1})_{ij}f_i)\big|_{f_l=0}.$$

while

$$(D^{-1})_{ij}\to\frac{(-)^{i+j}}{\lambda_{ij}}\left(\frac{1}{L}+\frac{1}{2\alpha_i^2}\right)^{-1}\left(\frac{1}{L}+\frac{1}{2\alpha_j^2}\right)^{-1},\quad i\neq j$$

so that

$$F(1[0],2[0],\cdots,n[0])\to g_1^n\sum_{\text{perm}}\frac{1}{\lambda_{12}}\frac{1}{\lambda_{34}}\cdots\frac{1}{\lambda_{n-1,n}},\quad(4)$$

where

$$g_1=\lim_{L\to0}g_0\int_0^\infty d\alpha^2\chi(\alpha^2)\left(1+\frac{2\alpha^2}{L}\right)^{-3/2}.$$

Equation (4) is just that function of n vertices and no external boson lines which appears in the perturbation expansion of the interaction $\mathcal{L}'=-g_1\bar{\psi}\phi\psi$, for a zero-mass boson. The same steps go through for the finite-mass case, with λ_{ij}^{-1} replaced in (4) by $-i\Delta_c(x_i-x_j)$, and for more general regularization, where L^{-1} in g_1 is replaced by the constant $-i\Delta_c^{\text{Reg}}(0)$, which diverges as the regularization is removed. Subsequent expressions calculated in terms of the $F(1[0],2[0],\cdots,n[0])$ of (4) will have the divergences characteristic of the g_1 perturbation expansion; and it may well be that the sensible method of calculating radiative corrections is to complete all calculations before removing the regularization.[8] Nevertheless, as a consequence of this modified Fradkin procedure (MFP), the transcendental interaction serves, in this limit, to generate an equivalent perturbative expansion with altered coefficients.

Consider, next, the same set of n points with external boson lines attached to two points, say, x_1 and x_2, together with all possible virtual exchanges between and for all n points; this is simply given by (1) with $V(\phi(x_1))$ and $V(\phi(x_2))$ replaced by $V'(\phi(x_1))$ and $V'(\phi(x_2))$, respectively, where $V'(z)=dV/dz$. A completely analogous calculation goes through as before, with a somewhat more complicated permutation sum appearing[7] than that of (3); in the limit $L\to0$, however, it simplifies down to

$$F(1[1],2[1],3[0],\cdots,n[0])\sim(-\tfrac{1}{2}ig_0)^n\times\int_0^\infty\frac{d\alpha_1^2}{\alpha_1^3}\chi(\alpha_1^2)\cdots\int_0^\infty\frac{d\alpha_n^2}{\alpha_n^3}\chi(\alpha_n^2)\,[\det D]^{-1/2}\times i^2(D^{-1})_{11}(D^{-1})_{22}\sum_{\text{perm}}(D^{-1})_{34}\cdots(D^{-1})_{n-1,n},\quad(5)$$

where the meaning of the permutation sum is the same as in (3). This simplification occurs, as $L\to0$, because

$$(D^{-1})_{ii}\sim\left(\frac{1}{L}+\frac{1}{2\alpha_i^2}\right)^{-1},$$

and hence dominates any $(D^{-1})_{ij,i\neq j}$. Thus, in this limit we obtain

$$F(1[1],2[1],3[0],\cdots,n[0])\to g_1^n\times\sum_{\text{perm}}\frac{1}{\lambda_{34}}\cdots\frac{1}{\lambda_{n-1,n}},\quad(6)$$

which is just the form appearing in the corresponding perturbation expansions of the interaction $\mathcal{L}'=-g_1\bar{\psi}\phi\psi$. On the other hand, had both external boson lines ended on the same point, say, x_1, then $V(\phi(x_1))$ of (1) is replaced by $V''(\phi(x_1))$, where $V''(z)=d^2V(z)/dz^2$, and we obtain just (3) with an extra factor of $(D^{-1})_{11}$; in the limit $L\to0$ this yields

$$F(1[2],2[0],3[0],\cdots,n[0])\to g_2g_1^{n-1}\times\sum_{\text{perm}}\frac{1}{\lambda_{12}}\frac{1}{\lambda_{34}}\cdots\frac{1}{\lambda_{n-1,n}},\quad(7)$$

where

$$g_2=g_0\int_0^\infty d\alpha^2\chi(\alpha^2)\,2\alpha^2\left(1+\frac{2\alpha^2}{L}\right)^{-5/2}.$$

If the integrals defining g_1 and g_2 exist, we may expect $g_2\sim Lg_1$, and hence

$$F(1[2],2[0],\cdots,n[0])\sim(L/\lambda)F(1[1],2[1],3[0],\cdots,n[0]).$$

Quite generally, for the interaction considered here, processes with more than one external boson attached to the same point will vanish, in the limit $L\to0$, relative to processes with only one external boson line ending at any point, and the latter will be the obvious generalizations of (6),

$$F(1[1],2[1],\cdots,l[1],l+1[0],\cdots,n[0])=g_1^n\sum_{\text{perm}}\frac{1}{\lambda_{l+1,l+2}}\cdots\frac{1}{\lambda_{n-1,n}}.\quad(8)$$

[8] The difficult question, not attempted in this paper, is to prove that those interactions which vanish in the MFP limit, $g_1\to0$ as $L\to0$, will have radiative corrections which vanish as $L\to0$, when the regularization is removed at the last step of any arbitrarily complicated calculation. Since the terms of (4), defined for $\lambda_{ij}\neq0$, must generate the finite part of all radiative corrections, one might expect that this is true. It can be verified in simple examples, such as that of theory (b), where the g_0^2 order nucleon-nucleon scattering amplitude vanishes with L according to $\sim(L/2\alpha_0^2)^2g_0^2/$(momentum transfer) using the MFP, and to $\sim g_0^2L(L/2\alpha_0^2)^{3/2}$ when regularization is retained until after the appropriate Fourier transform is calculated. Similarly, the nucleon self-energy function of the same order vanishes as $\sim g_0^2(L/2\alpha_0^2)^2\times\ln L$ under the MFP, in comparison with $\sim g_0^2(L/2\alpha_0^2)^{3/2}$ obtained by holding the regularization until the last step. In these examples, one finds just zero, regardless of the sequence of operations. If this apparent equivalence could be proved for all radiative corrections, one would then be able, in principle, to circumvent unnecessary calculation. If it is not true for all radiative corrections, the MFP defined here is simply different from a theory defined in terms of the usual regularization procedure.

These statements depend upon the existence of the integrals

$$g_{l+1}/g_0=\int_0^\infty d\alpha^2\chi(\alpha^2)(2\alpha^2)^l\left(1+\frac{2\alpha^2}{L}\right)^{-\frac{1}{2}(3+2l)},$$

which are the only restrictions we need put on $\chi(\alpha^2)$. The ratio g_1/g_0 will always be given as a function of the ratio α_0^2/L, and it is useful to consider some special cases, as follows.

(a) $U(z^2)=e^{-z^2}$, $z\equiv\alpha_0\phi$, $\chi(\alpha^2)=\delta(\alpha^2-\alpha_0^2)$, and g_1/g_0 vanishes as $(L/2\alpha_0^2)^{3/2}$ in the MFP limit. If, incidentally, $\alpha_0\sim L^{1/2}$, we get a finite variant of the exponential form useful in removing gauge dependence of electromagnetic currents, but modified to a quadratic dependence on the boson field.[9] This provides one situation in which the MFP is clearly sensible, since we certainly would expect to obtain a polynomial interaction when $\alpha_0\rightarrow 0$.

(b) $U(z^2)=1/(1+z^2)$, $\chi(\alpha^2)=\alpha_0^{-2}e^{-\alpha^2/\alpha_0^2}$, and again g_1/g_0 vanishes, but this time more slowly, $g_1/g_0\sim(L/\alpha_0^2)$.

(c) $U(z^2)=z^2/(1+z^2)$, $\chi(\alpha^2)=-e^{-\alpha^2/\alpha_0^2}\partial/\partial\alpha^2$, and here g_1/g_0 is finite as $L\rightarrow 0$, $g_1/g_0\rightarrow 1$.

(d) $U(z^2)=z^4/(1+z^2)$, $\chi(\alpha^2)=\alpha_0^2e^{-\alpha^2/\alpha_0^2}(\partial/\partial\alpha^2)^2$, and g_1/g_0 diverges as $3\alpha_0^2/L$. This situation is analogous to that of an ordinary nonrenormalizable polynomial interaction, where the degree of divergence increases with the complexity of the process.

From these examples, it is clear that the ratio g_1/g_0 as $L\rightarrow 0$ is strongly dependent upon the form of $U(z^2)$, which observation forms the basis for the equivalence statements of Sec. III. It should also be noted that the identical MFP can be defined for interactions of form $\mathcal{L}'=-\bar{\psi}U(\alpha_0^2\phi^2)\psi$, $\mathcal{L}'=-W(\alpha_0^2\phi^2)$, with appropriate isotopic generalizations, but we defer consideration of these interactions until Sec. III.

III. EQUIVALENCE

Because of the circumstance that the parameters L and α_0^2 appear in the ratio α_0^2/L, it is possible to prove a statement of equivalence between special nonderivative transcendental and polynomial interactions. Suppose we are dealing with $\mathcal{L}'=-\bar{\psi}U(\alpha_0^2\phi^2)\psi$, where $U(z^2)$ has the property $U(\infty)=C$, a constant; then, under the MFP, $\mathcal{L}'$ is equivalent to an effective interaction $\mathcal{L}_{\rm eff}'=-C\bar{\psi}\psi$, which just corresponds to a fermion mass shift, $m_0\rightarrow m_0+C$. Similarly, if we begin with an interaction $\mathcal{L}'=-g_0\bar{\psi}\phi U(\alpha_0^2\phi^2)\psi$, where $U(\infty)=C$, a constant, then, under the MFP, $\mathcal{L}'$ is equivalent to an effective interaction $\mathcal{L}_{\rm eff}'=-(g_0C)\bar{\psi}\phi\psi$. Both statements are easily generalized to contain isotopics. It may be noted that they are the generalizations of example (c), and that they suggest a useful corollary which relates to the remaining examples (and can be obtained directly from the proofs to follow): Any $U(z^2)$ given as the ratio of two polynomials, or reducing to such a ratio for large values of their argument, can be put into equivalent polynomial form. If $U(z^2)=N_n(z^2)/D_m(z^2)$, then $U_{\rm eff}(z^2)=P_{n-m}(z^2)$, where N_n, D_m, P_l are polynomials of order n, m, and l, respectively. These statements are true because, under the MFP, the relevant parameter is α_0^2/L, and hence $\alpha_0^2\rightarrow\infty$ is equivalent to $L\rightarrow 0$. If, in the first case, $U(\alpha_0^2\phi^2)\rightarrow C$ as $\alpha_0^2\rightarrow\infty$, we obtain just δm_0; and we must recover the same result when $L\rightarrow 0$ in the MFP. The proofs for both cases may be sketched as follows.

(A) $\mathcal{L}'=-\bar{\psi}U(\alpha_0^2\phi^2)\psi$

$$F(1[0],2[0],\cdots,n[0])=\exp\left(-\tfrac{1}{2}i\frac{\delta}{\delta\phi}\Delta_c^{\rm Reg}\frac{\delta}{\delta\phi}\right)\times U(\alpha_0^2\phi^2(x_1))\cdots U(\alpha_0^2\phi^2(x_n))|_{\phi=0},\quad(9)$$

and one expects, from this form, $F\rightarrow C^n$ in the limit $\alpha_0\rightarrow\infty$, where $U(\alpha_0^2\phi^2)\rightarrow C$.

Inserting the representation

$$U=\int_{-\infty}^{+\infty}d\omega\; e^{i\omega\alpha_0\phi}\tilde{U}(\omega),$$

calculating the functional operations, reexpressing $\tilde{U}(\omega)$ in terms of $U(z^2)$, and performing the Gaussian integrals, we obtain the result

$$F(1[0],2[0],\cdots,n[0])=(2\pi\alpha_0^2)^{-n/2}\left[\det\left(\frac{1}{L+\lambda}\right)\right]^{-1/2}\times\int_{-\infty}^{+\infty}dz_1U(z_1^2)\cdots\int_{-\infty}^{+\infty}dz_nU(z_n^2)\times\exp\left\{-\frac{1}{2\alpha_0^2}\sum_{ij}z_i\left[\left(\frac{1}{L+\lambda}\right)^{-1}\right]_{ij}z_j\right\}.\quad(10)$$

Setting $z_i=(\alpha_0/\sqrt{L})\xi_i$, this may be written as

$$(2\pi)^{-n/2}\left[\det\left(\frac{L}{L+\lambda}\right)\right]^{-1/2}\times\int_{-\infty}^{+\infty}d\xi_1U\left(\frac{\alpha_0^2}{L}\xi_1^2\right)\cdots\int_{-\infty}^{+\infty}d\xi_nU\left(\frac{\alpha_0^2}{L}\xi_n^2\right)\times\exp\left\{-\tfrac{1}{2}\sum_{ij}\xi_i\left[\left(\frac{L}{L+\lambda}\right)^{-1}\right]_{ij}\xi_j\right\},\quad(11)$$

and we now pass to the limit $\alpha_0\rightarrow\infty$, $U(\infty)=C$, under

[9] As in the paper by F. Csikor and G. Pocsik, Nuovo Cimento 42, 1529 (1966).

the integrals which exist and give just

$$(2\pi)^{+n/2}\left[\det\left(\frac{L}{L+\lambda}\right)^{-1}\right]^{-1/2},$$

so that $F\to C^n$, as expected. Note that this is independent of L/λ_{ij}.

To show the equivalence of this limit with the $L\to 0$ limit of the MFP, we observe that

$$\left(\frac{L}{L+\lambda}\right)=\begin{bmatrix} 1 & L/(L+\lambda_{12}) & L/(L+\lambda_{13})\cdots \\ L/(L+\lambda_{21}) & 1 & L/(L+\lambda_{23})\cdots \\ \vdots & & 1 \end{bmatrix}\to\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1\end{bmatrix}$$

as $L\to 0$. Thus

$$\det\left(\frac{L}{L+\lambda}\right)\to 1 \quad\text{and}\quad \left[\left(\frac{L}{L+\lambda}\right)^{-1}\right]_{ij}\to\delta_{ij},$$

so that

$$\left(\frac{L}{L+\lambda}\right)\left(\frac{L}{L+\lambda}\right)^{-1}\equiv 1;$$

and (11) becomes

$$(2\pi)^{-n/2}\int_{-\infty}^{+\infty}d\xi_1 U\left(\frac{\alpha_0^2}{L}\xi_1^2\right)\cdots\int_{-\infty}^{+\infty}d\xi_n U\left(\frac{\alpha_0^2}{L}\xi_n^2\right)\exp(-\tfrac{1}{2}\sum_i\xi_i^2)$$

$$=\left[\left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{+\infty}d\xi\; U\left(\frac{\alpha_0^2}{L}\xi^2\right)e^{-\frac{1}{2}\xi^2}\right]^n=C'^n,\qquad(12)$$

and the limit $U(\infty)=C$ then yields $C'=C$. If external boson lines are attached to any point, we will always get extra factors of L/α_0^2 which cause these amplitudes to vanish; and hence we have demonstrated that $\mathcal{L}_{\rm eff}'=-C\bar{\psi}\psi$. Note that the constant C can be zero, e.g., for $U(z^2)=e^{-z^2}$; and therefore these interactions are completely damped out under the MFP.

(B) $\mathcal{L}'=-ig_0\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\psi U(\alpha_0^2\boldsymbol{\pi}^2)\equiv f_0\boldsymbol{\Gamma}\cdot\mathbf{z}U(\mathbf{z}^2)$,

with

$$f_0=g_0/\alpha_0,\quad \boldsymbol{\Gamma}_i=-i\bar{\psi}(x_i)\gamma_5\boldsymbol{\tau}\psi(x_i),\quad \mathbf{z}=\alpha_0\gamma_5\boldsymbol{\pi}.$$

The equation analogous to (10) is, with n even,

$$F(1[0],2[0],\cdots,n[0])=(-if_0)^n(2\pi\alpha_0^2)^{-3n/2}\times\left[\det\left(\frac{1}{L+\lambda}\right)\right]^{-3/2}\left(\boldsymbol{\Gamma}_i\cdot\frac{\partial}{\partial\mathbf{f}_1}\right)\cdots\left(\boldsymbol{\Gamma}_n\cdot\frac{\partial}{\partial\mathbf{f}_n}\right)$$
$$\times\int d^3z_1U(\mathbf{z}_1^2)\cdots\int d^3z_nU(\mathbf{z}_n^2)\exp\left\{i\sum_i\mathbf{f}_i\cdot\mathbf{z}_i-\frac{1}{2\alpha_0^2}\sum_{ij}\mathbf{z}_i\cdot\mathbf{z}_j\left[\left(\frac{1}{L+\lambda}\right)^{-1}\right]_{ij}\bigg|_{\mathbf{f}\to 0}\right\},\qquad(13)$$

and we need only make the variable change $\mathbf{z}_i=(\alpha_0^2/L)^{1/2}\boldsymbol{\xi}_i$ and take the limit $U((\alpha_0^2/L)\boldsymbol{\xi}_i^2)\to C$ as $L\to 0$ to obtain

$$F(1[0],2[0],\cdots,n[0])\to(-if_0C)^n\left(\boldsymbol{\Gamma}_1\cdot\frac{\partial}{\partial\mathbf{f}_1}\right)\cdots\left(\boldsymbol{\Gamma}_n\frac{\partial}{\partial\mathbf{f}_n}\right)\exp[-\tfrac{1}{2}\alpha_0^2\sum_{ij}\mathbf{f}_i\cdot\mathbf{f}_j(\lambda_{ij})^{-1}|_{\mathbf{f}\to 0}$$
$$=(g_0C)^n\sum_{\rm perm}\frac{(\boldsymbol{\Gamma}_1\cdot\boldsymbol{\Gamma}_2)}{\lambda_{12}}\cdots\frac{(\boldsymbol{\Gamma}_{n-1}\cdot\boldsymbol{\Gamma}_n)}{\lambda_{n-1,n}},\qquad(14)$$

as expected. The generalization to the case of external boson lines to produce forms similar to (6), (7), and (8) is straightforward; one must, however, assume that $aU'(a)|_{a\to\infty}=0$, as is true for $U(a)=N_n(a)/D_n(a)$.

If, in either case, $U(\infty)$ is divergent, we produce the singular forms illustrated in example (d). The significance of the MFP may now be stated for the case of purely boson interactions, $\mathcal{L}'=-W(\alpha_0^2\phi^2)$. If W is a polynomial, the limit $L\to 0$ does not exist, corresponding to the presence of divergent virtual-point processes in every order (which sum to a divergent phase factor,[10] distinct from but added to that of the ordinary closed loops). There is also, in every order, finite dependence upon the λ_{ij}, which remains and defines the nontrivial theory (once the phase factor is recognized and removed). In the polynomial case there is no need to retain the virtual-point terms and they are conventionally dropped by omitting all factor pairings at the same point. On the other hand, for transcendental interactions, when the MFP limit does exist, the virtual-point processes are finite, in every order, but there is nothing left over and all the λ_{ij} dependence vanishes with L. For such interactions the MFP gives finite phases but no physics; or, for the interaction $\mathcal{L}'=-\bar{\psi}U(\alpha_0^2\phi^2)\psi$, $U(\infty)$ finite, just a fermion mass

[10] This statement is strictly true for the simplest quadratic interaction, but must, in general, be amended to include the possibility of further renormalizations. Thus, for a π-π interaction of form $\mathcal{L}'=-\frac{1}{4}\lambda\pi^4$, $\langle\pi\rangle=0$, $\mathcal{L}'$ differs from the normal ordered interaction $-\frac{1}{4}\lambda:\pi^4:$ by the inclusion of the extra terms $-\frac{1}{4}\lambda\langle\pi^4\rangle-\frac{3}{2}\lambda\langle\pi^2\rangle:\pi^2:$, which correspond to a divergent phase and an infinite bare mass factor, respectively.

shift and nothing else. In order to have a nontrivial theory for such interactions, within the MFP, one must have an interaction which increases without limit for large values of its argument.

IV. APPLICATION

A tentative application of these ideas may be made for chiral theories of current interest, although we will be forced to employ an approximation. Thus, the Lagrangian of Chang and Gürsey,[1] excluding the symmetry-breaking term,

$$\mathcal{L}=-\bar{\psi}(\gamma_\mu\partial_\mu+mV(\alpha_0\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}))\psi-(1/16\alpha_0^2)\,\mathrm{tr}(\partial_\mu V\partial_\mu V^\dagger),$$

is too complicated for the present techniques because of the derivatives appearing in the pion term; and so we brutally replace the latter by $-\frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2$, which destroys the chiral invariance of the procedure, but makes a dynamical statement possible. We continue to assume, however, that $V(z)$ satisfies $V(-z)=V^\dagger(z)=V(z)^{-1}$, $z=\alpha_0\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}$, and choose a general form which satisfies this unitarity requirement,

$$V(z)=\frac{1+izF(z^2)}{1-izF(z^2)},$$

with F real. It is simplest to consider $\mathcal{L}'=-m\bar{\psi}V(z)\psi$ rather than to subtract off a mass term and use $(V(z)-1)$ for the interaction. Rationalizing, we have $V(z)=U_1(z^2)+izU_2(z^2)$, with

$$U_1=\frac{1-z^2F^2(z^2)}{1+z^2F^2(z^2)}\quad\text{and}\quad U_2=\frac{2F}{1+z^2F^2}.$$

If $\lim_{z^2\to\infty} z^2F^2(z^2)=\xi^2$, where ξ^2 denotes a nonoscillatory limit, $0\leq\xi^2\leq\infty$, then, according to the MFP, these functions may be replaced by $U_1^{\mathrm{eff}}=(1-\xi^2)/(1+\xi^2)=C$ and $U_2^{\mathrm{eff}}=0$. Depending upon the form chosen for F, C can take on any value between -1 and $+1$, and hence this interaction is equivalent to just a free, massive fermion field (of arbitrary parity convention). As expected, this result violates the chiral symmetry and cannot yet be taken seriously; but it raises suspicions as to the MFP dynamical content of the complete theory. There could be nontrivial content to the theory if the symmetry-breaking part of the interaction diverges, in the $z^2\to\infty$ limit, with sufficient strength; there would then remain the problem of disentangling the phases from the physics.

V. SUMMARY

It may be useful to summarize the observations of this paper. For a nonderivative transcendental interaction and with the aid of regularization, the quantities $F(1[i],2[j],3[k],\cdots)$ have been defined in a way which guarantees the correct cut structure of the lowest-order radiative corrections, and which exhibits strong damping as the regularization is removed. Such damping suggests a simple correlation between transcendental and ordinary polynomial interactions with modified coupling constants, in the MFP limit as the regularization is removed. For an interaction even in the boson field, $U(\alpha_0^2\phi^2)$, only a modified coupling constant which diverges, or equivalently a divergent value of $U(\infty)$, can correspond to a nontrivial theory under the MFP; for an interaction odd in the boson field, $V=g_0\phi U(\alpha_0^2\phi^2)$, a finite or divergent $U(\infty)$ can produce a nontrivial theory. The content of such nontrivial possibilities has not been considered here, although the special case $U\sim z^{2l}$, $z^2\to\infty$, $l>0$, yields an iterative expansion similar to that given by the perturbation expansion of the polynomial interaction $U=z^{2l}$.

An approximate and highly tentative application of the MFP to a class of chiral theories suggests that the latter contains no dynamics, in the sense of being able to generate nontrivial radiative corrections.

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