Effect of a Degenerate Neutrino Sea on Electromagnetism*

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Modern cosmological theories imply the existence of a universal degenerate Fermi sea of neutrinos. The fact that the Fermi energy K_F varies from theory to theory could, in principle, be used to help decide which universe we live in. We show that a parity-violating term is introduced into Maxwell's equations as a result of the neutrino sea. In particular, we study whether a new, meaningful limit on the Fermi energy can be established by studying the propagation of light and the character of magnetic fields in such a neutrino sea. Unfortunately, the solutions to these equations show that the effect of the neutrino sea on electromagnetism is too small to be observed.

INTRODUCTION

It has been pointed out by Weinberg[1] that modern cosmological theories imply the existence of a universal degenerate Fermi sea of neutrinos. The fact that the magnitude of the Fermi energy K_F varies from theory to theory could, in principle, be used to help decide which cosmology best describes our universe.

Weinberg's order-of-magnitude estimates for various cosmologies are $K_F<10^{-30}$ MeV for the evolutionary cosmology, $K_F<e^{-10^{36}}$ MeV for the steady-state cosmology, and $K_F<10^{-8}$ MeV for the oscillating cosmology. Clearly, there is hope of detecting only the K_F value for the oscillating cosmology. Analysis of β-decay experiments[2–3a] indicate that if neutrinos are degenerate $K_F<1000$ eV, and if antineutrinos are degenerate $K_F<200$ eV.

In this paper, we investigate the propagation of light in a degenerate Fermi sea of neutrinos with the purpose of determining whether or not the anomalies caused by the sea can be used to establish meaningful limits on K_F. Section I contains two different derivations of a modification in Maxwell equations. The first is a phenomenological derivation and the second is a more detailed microscopic derivation. The two methods of derivation are shown to be mutually consistent. Section II contains a discussion of the solutions of the modified Maxwell equations. The possibility of observing the neutrino-sea-dependent terms is discussed. In order to ensure clarity, detailed calculations have been relegated to an Appendix.

The results point out what might have been suspected in the first place. In the cases examined, the extra term depends directly on the Fermi constant G. Because of its small size, the extra term is masked by the familiar electromagnetic effects. The experiment cannot be done because of the small size of the relevant parameters.

I. MODIFIED MAXWELL EQUATIONS

As stated in the Introduction, the first derivation of the neutrino-sea-dependent term in Maxwell's equations is rather heuristic. This derivation is motivated by the desire to add a parity-violating term to Maxwell's equations consistent with the usual requirements of Lorentz covariance and differential current conservation.

We assume that the neutrino sea is completely filled at an absolute temperature of $T=0°$ and thus characterized only by the Fermi energy K_F. We define a 4-vector K^μ so that it has components $K^0=K_F$, $\mathbf{K}=0$ in the rest frame of the neutrino sea; i.e., the frame in which k space is filled symmetrically about the origin. K^μ then characterizes the neutrino sea in an arbitrary frame.[4] We assume for simplicity that the extra term depends linearly on K^μ and the electromagnetic field tensor. We also exclude the possibility of derivative terms. The motivation to search for an extra term with parity opposite from the rest of the equation is based on the hope that the parity-violating effects can be more easily observed than those which do not violate parity.

For reference, we write Maxwell's equations:

$$\partial_\mu F^{\nu\mu}=4\pi J^\nu, \tag{1}$$

$$\epsilon_{\alpha\beta\delta\gamma}\partial^\beta F^{\delta\gamma}=0. \tag{2}$$

With the preceding remarks in mind, we see that the two possible candidates for extra terms are the 4-vectors

$$K^\alpha F_{\nu\alpha}, \quad K^\alpha F^{\delta\gamma}\epsilon_{\nu\alpha\delta\gamma}.$$

Thus the modified Maxwell equations which include parity-violating terms are[5]

$$\partial_\mu F^{\nu\mu}=4\pi J^\nu+C_1\epsilon^{\nu\alpha\delta\gamma}K_\alpha F_{\delta\gamma}, \tag{3}$$

$$\epsilon_{\alpha\beta\delta\gamma}\partial^\beta F^{\delta\gamma}=C_2 K^\delta F_{\alpha\delta}. \tag{4}$$

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[1] S. Weinberg, Nuovo Cimento 25, 15 (1962); Phys. Rev. 128, 1457 (1962).
[2] L. M. Langer and R. J. D. Moffat, Phys. Rev. 88, 689 (1952); D. R. Hamilton, W. D. Alford, and L. Gross, *ibid.* 92, 1521 (1953).
[3] S. C. Curran, J. Angus, and A. L. Cockcroft, Phil. Mag. 40, 53 (1949).
[3a] J. Bernstein, M. Ruderman, and G. Feinberg, Phys. Rev. 132, 1227 (1963).

[4] Although we have specified a particular frame in which the neutrino sea is at rest, the dynamical equations are still Lorentz-covariant.

[5] The modified equations with terms of the same parity included would be

$$\partial_\mu F^{\nu\mu}=4\pi J^\nu+C_1\epsilon^{\nu\alpha\delta\gamma}K_\alpha F_{\delta\gamma}+D_1K_\delta F^{\nu\delta}$$

and

$$\epsilon_{\alpha\beta\delta\gamma}\partial^\beta F^{\delta\gamma}=C_2K^\delta F_{\alpha\delta}+D_2\epsilon_{\alpha\beta\delta\gamma}K^\beta F^{\delta\gamma}.$$

The coefficients C_1 and C_2 are assumed to be constant, which corresponds to assuming a constant spatial density of neutrinos in the universe.[6] We can prove that $C_2=0$ as follows. Contract Eq. (4) with ∂^α. The left-hand side of the equation vanishes identically, and we have $C_2K^\delta\partial^\alpha F_{\alpha\delta}=0$. Substitution of $\partial^\alpha F_{\alpha\delta}$ from Eq. (1) into this expression yields

$$4\pi C_2K^\delta J_\delta=0.$$

Since J_δ is arbitrary, this implies $C_2\equiv 0$.

The modified equations now assume the form

$$\partial_\mu F^{\nu\mu}=4\pi J^\nu+C_1\epsilon^{\nu\alpha\delta\gamma}K_\alpha F_{\delta\gamma}\,, \tag{5}$$

$$\epsilon_{\alpha\beta\delta\gamma}\partial^\beta F^{\delta\gamma}=0. \tag{6}$$

These equations are consistent with current conservation, which can be seen by contracting Eq. (5) with ∂^ν. The constant C_1 will be determined in the microscopic derivation.

By writing this out in the more familiar vector notation,

$$\nabla\cdot\mathbf{E}=4\pi\,,\qquad \nabla\cdot\mathbf{B}=0\,,$$
$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t\,,\quad \nabla\times\mathbf{B}=4\pi\mathbf{J}+2C_1\mathbf{B}+\partial\mathbf{E}/\partial t\,,$$

we see that the neutrino sea introduces a parity-violating term in the induction law.

The microscopic derivation is based on elementary-particle formalism. The object of the calculation is to determine the first-order weak correction to the photon propagator produced by the neutrino sea. Feynman graphs for this process are shown in Fig. 1. The photon propagator is altered because of the interaction of the virtual e^+-e^- pair with the neutrino sea. The formalism is developed and the detailed calculation carried out in the Appendix. The method of calculation is first to determine the modification of the electron propagator, and then to use the modified electron propagator in the usual calculation of the photon polarization tensor. The result of the calculation is that the polarization tensor is modified by the finite term

$$i\pi^{\mu\nu}(q)=e^2\left(\frac{GK_F{}^3}{\sqrt{2}}\right)\frac{1}{9\pi^4}\epsilon^{0\mu\nu\alpha}q_\alpha+O\left(\frac{q^2}{M_e{}^2}\right),$$

where we have evaluated K^μ in the rest frame of the sea.

In order to compare this with the first derivation and

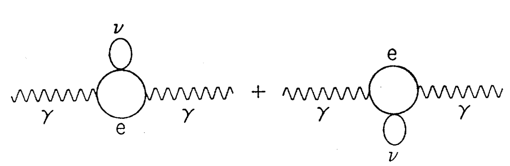

FIG. 1. Feynman diagrams for the first-order neutrino-sea contribution to the photon propagator.

[6] If C_1 or K^α is a function of x, then we no longer have differential current conservation.

to determine the constant C_1, we use the fact that the exact photon propagator satisfies

$$D_{\mu\nu}(q)=D_{\mu\nu}{}^0(q)+D_{\mu\alpha}{}^0(q)\pi^{\alpha\lambda}D_{\lambda\nu}(q)$$

or

$$(D^{-1})_{\mu\nu}=((D^0)^{-1})_{\mu\nu}-\pi_{\mu\nu}$$

The determinant of $(D^{-1})_{\mu\nu}=((D^0)^{-1})_{\mu\nu}-\pi_{\mu\nu}$ then gives the dispersion relations for the electromagnetic field.

If we now find the dispersion relations for the modified Maxwell equations, they must be the same as the dispersion relations derived from the microscopic picture. This is carried out in the Appendix, and we see that the dispersion relations are identical if we set $C_1=(G/\sqrt{2})\times(K_F{}^2e^2/18\pi^4)$. This determines the constant C_1 and shows that the two derivations lead to consistent results.

II. DISCUSSION

The modified Maxwell equations are

$$\nabla\cdot\mathbf{E}=4\pi\rho\,,\qquad \nabla\cdot\mathbf{B}=0\,,$$
$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t\,,\quad \nabla\times\mathbf{B}=4\pi\mathbf{J}-K\mathbf{B}+\partial\mathbf{E}/\partial t\,,$$

where we have set

$$K^\mu=(K_F,\mathbf{0})$$

and

$$K=\frac{e^2G}{\sqrt{2}}\frac{K_F{}^3}{9\pi^4}\cong 10^{-27}K_F{}^3\quad (K_F\text{ in eV}).$$

Now we proceed as in any elementary text and solve these equations in every possible way. Looking at the static equations first, we see that it is only the magnetostatic equations that are affected by the anomolous term. These equations are

$$\nabla\times\mathbf{B}+K\mathbf{B}=4\pi\mathbf{J}\,,\quad \nabla\cdot\mathbf{B}=0.$$

The solution is readily found by making the substitution $\mathbf{B}=\nabla\times\mathbf{A}-K\mathbf{A}$. The equation for $\mathbf{A}$ is $-\nabla^2\mathbf{A}-K^2\mathbf{A}=4\pi\mathbf{J}$, with the condition $\nabla\cdot\mathbf{A}=0$. The solution is

$$\mathbf{A}(\mathbf{x}')=\int d^3x\,\frac{\cos K|\mathbf{x}'-\mathbf{x}|}{4\pi|\mathbf{x}'-\mathbf{x}|}\mathbf{J}(\mathbf{x}).$$

The solution for $\mathbf{B}$, to first order in G, is then

$$\mathbf{B}=\nabla\times\mathbf{A}_0-K\mathbf{A}_0\,,$$

where

$$\mathbf{A}_0(\mathbf{x}')=\int\frac{d^3x\,\mathbf{J}(\mathbf{x})}{4\pi|\mathbf{x}'-\mathbf{x}|}.$$

This solution presents the interesting result that the magnetic field from a localized static current should have a term which drops off as $1/r$ instead of the usual $1/r^2$. To estimate the distance from the source where such a term would be observable we simply assume that the first term in B goes like $1/r^2$ and the second like K/r. When the terms are equal $r=1/K$. To discuss the feasibility of doing this experiment, we take a conservative value (compared to Weinberg's estimates) of $K_F=1$ eV. This gives a result of $r=10^{12}$ light years. An experiment based on this would be out of the question, since the age of the universe is believed to be only about 10^{10} years.

Another possibility which suggests itself is that of setting up an experiment in which the two parts of $\mathbf{B}$, namely, $\nabla\times\mathbf{A}_0$ and $K\mathbf{A}_0$, are perpendicular to each other. For example, one might set up a current in a wire along the z axis, as in Fig. 2. Then $\mathbf{A}_0$ will be in the z direction and $\nabla\times\mathbf{A}_0$ in the (x,y) plane. The fact that $\nabla\times\mathbf{A}_0$ and $K\mathbf{A}_0$ are perpendicular might allow one to measure $K\mathbf{A}_0$. Unfortunately, this is again impossible, as is seen by comparing the two terms

$$K|A_0|\simeq 10^{-22}\ \text{cm}^{-1}|A_0|\,,$$
$$|\nabla\times\mathbf{A}_0|\simeq|A_0|/L,$$

where L is some laboratory-sized dimension, say, $L=1$ cm. Thus

$$K|A_0|/(|A_0|/L)\simeq 10^{-22}.$$

Thus we would be required to measure fields which are 10^{-22} times smaller than laboratory fields.

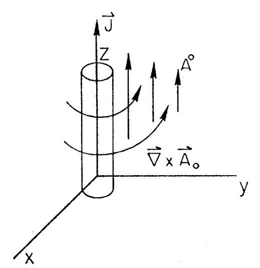

FIG. 2. Situation in which the feasibility of measuring K_F in the laboratory is discussed.

Another interesting case is provided by the propagation of light in a source-free region. The appropriate equations are

$$\nabla\cdot\mathbf{E}=0,\qquad \nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t,\quad \nabla\times\mathbf{B}=\partial\mathbf{E}/\partial t-K\mathbf{B}.$$

A solution is easily obtained by assuming a wave to be propagating in the z direction with frequency ω and wave number p. The divergence equations ensure that $\mathbf{E}$ and $\mathbf{B}$ are in the x-y plane. The curl equations yield the following results: There are two nondegenerate eigensolutions, namely, the left- and right-handed polarization modes. The usual degeneracy is destroyed by the parity-violating term in the modified Maxwell equations, so that each mode has its own dispersion relation:

$$\mathbf{E}_R=A[\hat{e}_x+i\hat{e}_y]e^{i(pz-\omega_R t)}\,,\quad \mathbf{B}_R=-(ip/\omega)\mathbf{E}_R\,,$$
$$\omega_R{}^2=p^2+Kp\,,$$
$$\mathbf{E}_L=A[\hat{e}_x-i\hat{e}_y]e^{i(pz-\omega_L t)}\,,\quad \mathbf{B}_L=(ip/\omega)\mathbf{E}_L\,,$$
$$\omega_L{}^2=p^2-Kp\,.$$

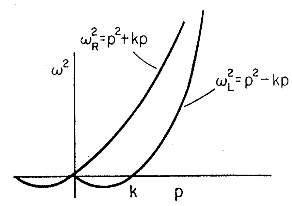

FIG. 3. Dispersion relations for the right- and left-handed polarization modes of free electromagnetic propagation.

FIG. 3

Illustrated in Fig. 3 are the dispersion relations for the two propagation modes. We see that ω_L is imaginary for $p<K$ and so the left-handed polarization mode cannot propagate for $p<K$. This cutoff wavelength with $K_F=1$ eV is $\lambda_c\sim 1/p\sim 1/K\sim 10^{12}$ light years. Any attempt to put meaningful limits on K_F by observing the above dispersion relations would require measurements of wavelengths the same order of magnitude as λ_c and thus seems to be out of the question.

The fact that the two rotational modes have different dispersion relations leads to the suggestion that one may be able to observe the rotation of linear polarization. For example, if we polarize a beam along the x direction at $t=0$, its subsequent behavior will be

$$\mathbf{E}=A[\hat{e}_x\cos\tfrac{1}{2}(\omega_R-\omega_L)t+\hat{e}_y\sin\tfrac{1}{2}(\omega_R-\omega_L)t]\times\cos[\tfrac{1}{2}(\omega_R+\omega_L)t-pz].$$

After a time $T=\pi/(\omega_R-\omega_L)$, the polarization vector will be in the y direction. For measureable wavelengths we have $K\ll p$, so that

$$\omega_R=(p^2+Kp)^{1/2}\simeq p+\tfrac{1}{2}K\,,$$
$$\omega_L=(p^2-Kp)^{1/2}\simeq p-\tfrac{1}{2}K.$$

Therefore,

$$\omega_R-\omega_L=K\quad\text{and}\quad T\simeq\pi/K\sim 10^{12}\ \text{years}.$$

Again, this is far beyond experimental feasibility.

The solution for the time-dependent Green's function is presented in the Appendix. The extra term in Maxwell's equations affects the low frequencies in the propagator through the same dispersion relation found in source-free propagation. Thus, radiation from the low frequencies in the source might exhibit properties differing from the propagation with the usual dispersion relation $\omega^2=p^2$. However, the problem is, as before, that of detecting such low-frequency radiation.

Up to this point we have been concerned with the photon or electron propagating through a stable Fermi gas of neutrinos: i.e., $|F\rangle_{\text{in}}=|F\rangle_{\text{out}}$. We would now like to consider the scattering of the photon off of the sea. We will have an initial "in" state

$${}_{\text{in}}\gamma_{K\lambda}{}^{\dagger}|F\rangle$$

consisting of a photon and the "vacuum" and a final "out" state

$${}_{\text{out}}a_{p_1s_1}{}^{\dagger}\ {}_{\text{out}}c_{p_2s_2}{}^{\dagger}\ {}_{\text{out}}\gamma_{K'\lambda'}{}^{\dagger}|F\rangle\,,$$

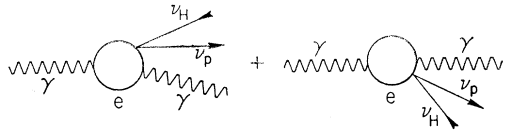

FIG. 4. Feynman diagrams for knocking a neutrino out of the neutrino sea by a photon.

which consists of a photon, a neutrino, a hole, and the vacuum. This is just the physical process of a photon knocking a neutrino out of the Fermi sea. We can calculate a lifetime for the photon this way by calculating the total cross section to see how long it takes the photon to scatter completely out of the initial beam. The relevant Feynman diagram is presented in Fig. 4.

Using the formalism developed in the Appendix, the calculation is straightforward, but tedious. The expression for the total transition rate out of the initial state for low frequencies $\omega \ll m_e$ is

$$\Lambda = \frac{\omega}{\pi^4} \frac{K_F{}^4 G^2 e^4}{324}.$$

For visible light with $K_F = 1$ eV, the lifetime is $T = 1/\Lambda \simeq 10^{37}$ years.

If we assume the lifetime to be 10^{10} years, which corresponds to the most distant light sources observed, we get an upper limit on K_F:

$$K_F < 10^7 \text{ eV}.$$

Unfortunately, this limit does not tell us anything since much better limits have been established.

III. SUMMARY

We started with the assumption that the universe is filled with a degenerate Fermi gas of neutrinos at zero temperature. We have derived in two different ways a neutrino-sea-dependent term which modifies Maxwell's equations. The solutions which we studied lead to the conclusion that the neutrino-sea-dependent terms are too small to be observed. Any limit on K_F which follows from these solutions is much higher than limits already established.

ACKNOWLEDGMENT

I would like to take this opportunity to thank Professor S. D. Drell for suggesting this problem and for guiding the research with many helpful discussions.

APPENDIX A

In this Appendix, we explain the formalism and use it to derive contributions to the electron self-energy and to the photon polarization tensor.

1. Formalism

The only difference in the following formalism and the usual formalism is that we assume that all neutrino states with energy less than K_F are filled. These filled states are just the neutrinos which comprise the neutrino sea. We then define a new vacuum

$$|F\rangle = \prod_{\substack{\text{all } K<K_F \\ \text{all } s}} b_{Ks}{}^\dagger |0\rangle, \quad |F_{\text{in}}\rangle = |F_{\text{out}}\rangle = |F\rangle,$$

where $b_{Ks}{}^\dagger$ is the creation operator for a neutrino of momentum K and spin s.

We use the usual minimal electromagnetic interaction and the current-current form of the weak interaction. Although the weak interaction is CP-, T-, and CPT-invariant, our vacuum is not, because the neutrinos turn into antineutrinos. This asymmetry allows effects such as an e^--e^+ mass difference.

The neutrino field operator is

$$\hat{\psi}_\nu(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{E}\right)^{1/2} (U_{ps} e^{-ip\cdot x} b_{ps} + V_{ps} e^{ip\cdot x} d_{ps}{}^\dagger).$$

For convenience in calculation, we make a canonical transformation to neutrinos and holes, as follows.[7] We define two new operators:

$$a_{ps} \equiv b_{ps} \quad (p > K_F),$$
$$c_{ps} \equiv b_{ps}{}^\dagger \quad (p < K_F).$$

These new operators obey the usual fermion commutation rules and have the virtue of destroying the vacuum.

$$a_{ps}|F\rangle = c_{ps}|F\rangle = 0;$$

$a_{ps}{}^\dagger$ creates neutrinos and $c_{ps}{}^\dagger$ creates holes. $d_{ps}{}^\dagger$ as usual creates antineutrinos. In terms of these new operators, the neutrino field operator becomes

$$\hat{\psi}_\nu(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{E}\right)^{1/2} [U_{ps} e^{-ip\cdot x} a_{ps} \theta(p - K_F) + U_{ps} e^{-ip\cdot x} c_{ps}{}^\dagger \theta(K_F - p) + V_{ps} d_{ps}{}^\dagger e^{ip\cdot x}].$$

Using the above formalism, we calculate the neutrino propagator

$$iS_{F\nu}(x - x') = \langle F | T\psi_\nu(x) \bar{\psi}_\nu(x') | F\rangle = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{p - m + i\epsilon} - \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{2p_0} \times (p+m)\theta(K_F - p) 2\pi\delta(p_0 - E_p).$$

2. Electron Self-Energy

The weak interaction current-current Hamiltonian is

$$H_\omega(x) = (G/\sqrt{2}) \bar{\psi}_e(x) \gamma^\mu (1-\gamma_5) \psi_\nu(x) \bar{\psi}_\nu(x) \gamma^\mu (1-\gamma_5) \times \psi_e(x) + \text{H.c.}$$

[7] T. Kinoshito and Y. Nambu, Phys. Rev. 94, 598 (1954).

The first-order weak correction to the electron propagator is

$$iS_F'(x'-x)=-i\int d^4y\langle F|T\psi_e(x')\bar{\psi}_e(x)H_\omega(y)|F\rangle .$$

Using Wick's theorem, we find that to first order in the weak interaction, the self-energy is

$$\Sigma(p)=\frac{G}{\sqrt{2}}\frac{K_F^3}{3\pi^2}\gamma_0(1-\gamma_5).$$

The Feynman diagram for this process is shown in Fig. 5. This self-energy produces an electron-positron mass difference which is seen as follows: We take the expectation value of the modified electron Hamiltonian between electron states and between positron states at rest.

$$H'=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m+\beta\Sigma(p),$$

$$\langle e^-|H'|e^-\rangle=m+\Delta E,$$

$$\langle e^+|H'|e^+\rangle=-m+\Delta E.$$

Therefore,

$$\delta m=(m+\Delta E)+(-m+\Delta E)=2\Delta E,$$

where

$$\Delta E=\frac{G}{\sqrt{2}}\frac{K_F^3}{3\pi^2}\simeq\frac{10^{-23}}{3\pi^2}K_F^3\ \ (K_F \text{ in eV}).$$

This result is not surprising in view of the asymmetry in our boundary conditions, which means that *CPT* is not a good symmetry.

3. Photon Polarization Tensor

To calculate the photon polarization tensor we will use the modified electron propagator in place of the usual electron propagator. This procedure can be justified by using Wick's theorem and perturbation theory. We will keep terms which are of first order in the weak interaction. The diagrams which contribute to the first-order weak correction are shown in Fig. 1.

$$i\pi_{\mu\nu}(q)=-e^2\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}[\gamma_\mu S_F^e(p)\gamma_\nu S_F^e(q-p)],$$

where

$$iS_F^e(p)=\frac{i}{p-m}+\frac{i}{p-m}\left(\frac{-iK_F^3G}{\sqrt{2}3\pi^2}\gamma_0(1-\gamma_5)\right)\frac{i}{p-m}.$$

If we insert $iS_F^e(p)$ and keep only first-order weak terms, then

$$i\pi'_{\mu\nu}(q^2)=-C\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\left[\gamma_\mu\frac{1}{p-m}\gamma_\nu\frac{1}{p-q-m}\gamma_0(1-\gamma_5)\right.$$
$$\left.\times\frac{1}{p-q-m}+\gamma_\mu\frac{1}{p-m}\gamma_0(1-\gamma_5)\frac{1}{p-m}\gamma_\nu\frac{1}{p-q-m}\right],$$

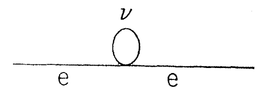

FIG. 5. Feynman diagrams for the first-order neutrino-sea contribution to the electron propagator.

where

$$C=e^2(G/\sqrt{2})K_F^3/3\pi^2.$$

We can use the charge-conjugation operator to simplify this to

$$i\pi'_{\mu\nu}(q^2)=2C\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\left[\gamma_\mu\frac{1}{p-m+i\epsilon}\gamma_0\gamma_5\right.$$
$$\left.\times\frac{1}{p-m+i\epsilon}\gamma_\nu\frac{1}{p-q-m+i\epsilon}\right].$$

Using the identity

$$\frac{1}{p-m+i\epsilon}=-i(p+m)\int_0^\infty dz\, e^{iz(p^2-m^2+i\epsilon)}$$

to elevate the denominators and making the substitution $l=p-(z_3/z)q$, which completes the square in the exponent, we have

$$i\pi'_{\mu\nu}(q)=2iC\int_0^\infty\int_0^\infty\int_0^\infty dz_1dz_2dz_3\left(\frac{4\epsilon_{0\mu\nu\alpha}q^\alpha}{16\pi^2z^2}\right)$$
$$\times\left[\frac{-i[2z_3-(z_1+z_2)]}{z^2}+\frac{m^2(2z_3+z_1+z_2)}{z}\right.$$
$$\left.+\frac{q^2z_3^2(z_1+z_2)}{z^3}\right]\exp\{i[q^2(z_1+z_2)z_3/z-z(m^2-i\epsilon)]\},$$

where $z=z_1+z_2+z_3$.

We can subtract 0 from this expression in the form

$$0\equiv 2iC\int_0^\infty\int_0^\infty\int_0^\infty dz_1dz_2dz_3\left(\frac{4\epsilon_{0\mu\nu\alpha}q^\alpha}{16\pi^2z^2}\right)$$
$$\times\left(\frac{-i[2z_3-(z_1+z_2)]}{z^2}\right)e^{-iz(m^2-i\epsilon)}.$$

Now introduce

$$1=\int_0^\infty\frac{d\lambda}{\lambda}\delta\left(1-\frac{z}{\lambda}\right)$$

under the integrand and scale all the z's by the substitution $z_i\to\lambda z_i$. Then we do the λ integral:

$$i\pi'_{\mu\nu}=\frac{iC}{2\pi^2}\epsilon_{0\mu\nu\alpha}q^\alpha\int_0^\infty\int_0^\infty\int_0^\infty dz_1dz_2dz_3\delta(1-z)$$
$$\times\left\{\frac{m^2(2z_3+z_1+z_2)+q^2z_3^2(z_1+z_2)}{i[m^2-i\epsilon-z_3(z_1+z_2)q^2]}+\frac{2z_3-z_1-z_2}{i}\right.$$
$$\left.\times\ln\frac{m^2-i\epsilon}{m^2-i\epsilon-z_3(z_1+z_2)q^2}\right\}.$$

To zeroth order in q^2/m^2, we have

$$i\pi'_{\mu\nu}(q)=e^2\frac{G}{\sqrt{2}}K_F{}^3\frac{1}{9\pi^4}\epsilon_{0\mu\nu\alpha}q^\alpha .$$

To first order, this result is independent of the electron mass and is the same as the result for the muon in a μ-neutrino sea.

APPENDIX B

The solution for the time-dependent Green's function of the modified Maxwell's equations is now explained. We start with the equations

$$\partial_\mu F^{\nu\mu}=4\pi J^\nu-\tfrac{1}{2}K\epsilon^{0\nu\alpha\beta}F_{\alpha\beta}, \tag{B1}$$

$$\epsilon_{\alpha\beta\delta\gamma}\partial^\beta F^{\delta\gamma}=0. \tag{B2}$$

Equation (B2) permits us to introduce a vector potential A^μ, where $F^{\mu\nu}=\partial^\nu A^\mu-\partial^\mu A^\nu$. Equation (B1) becomes

$$\partial_\mu\partial^\mu A^\nu-\partial^\nu(\partial_\mu A^\mu)=4\pi J^\nu-\tfrac{1}{2}K\epsilon^{0\nu\alpha\beta}(\partial_\beta A_\alpha-\partial_\alpha A_\beta).$$

Now we choose to solve this in the Lorentz gauge $\partial_\mu A^\mu=0$. The equations are then

$$\nu=0,\qquad \partial_\mu\partial^\mu A_0=4\pi J_0; \tag{B3}$$

$$\nu=1,2,3,\qquad \partial_\mu\partial^\mu\mathbf{A}=4\pi\mathbf{J}-K(\nabla\times\mathbf{A}). \tag{B4}$$

These equations are solved by Fourier-transforming the equations and inverting the operators. The solution of Eq. (B3) is well known:

$$A_0(x)=4\pi\int d^4x'G(x-x')J_0(x'),$$

where

$$G(x)=-\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ik\cdot x}}{p^2}.$$

The solution of Eq. (B4) is a little more laborious. If we assume that the solution is of the form

$$A_i(x)=4\pi\int d^4x'G_{ij}(x-x')J_j(x'),$$

then $G_{ij}(x)$ satisfies

$$(\partial_\mu\partial^\mu\delta_{ij}+K\epsilon_{imj}\nabla_m)G_{jk}(x)=\delta_{ik}\delta^4(x)$$

or

$$(-p^2\delta_{ij}+iK\epsilon_{imj}p_m)G_{jk}(p)=\epsilon_{ik}.$$

Now we must invert $-p^2\delta_{ij}+iK\epsilon_{imj}p_m$ considered as a matrix in ij.

$$m_{ij}=\begin{bmatrix}-p^2 & -iKp_3 & +iKp_2\\ +iKp_3 & -p^2 & -iKp_1\\ -iKp_2 & +iKp_1 & -p^2\end{bmatrix}_{ij},$$

$$G_{ij}(p)=(m^{-1})_{ij}=\frac{-p^2\delta_{ij}-iK\epsilon_{imj}p_m}{(p^2)^2-K^2|\mathbf{p}|^2}+\frac{K^2p_ip_j}{p^2[(p^2)^2-K^2|\mathbf{p}|^2]},$$

and

$$G_{ij}(x)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}G_{ij}(p).$$

Therefore

$$\mathbf{A}(x')=4\pi\int\frac{d^4p\,d^4x}{(2\pi)^4}\times e^{-ip\cdot(x'-x)}\left[\frac{-p^2\mathbf{J}(x)-iK\mathbf{p}\times\mathbf{J}(x)}{(p^2)^2-K^2|\mathbf{p}|^2}+\frac{K^2\mathbf{pp}\cdot\mathbf{J}(x)}{p^2[(p^2)^2-K^2|\mathbf{p}|^2]}\right]$$

and

$$A_0(x')=4\pi\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x'-x)}}{(-p^2)}J_0(x).$$

We can make this a little simpler by making a gauge transformation:

$$\Lambda(x')=+K^2\nabla_{x'}\cdot\int\frac{\mathbf{J}(x)e^{-ip\cdot(x'-x)}}{+p^2[(p^2)^2-K^2|\mathbf{p}|^2]},$$

$$\mathbf{A}'(x)=\mathbf{A}(x)+\nabla\Lambda(x),$$

$$A_0'(x)=A_0(x)-\frac{\partial\Lambda(x)}{\partial t}.$$

Then

$$\mathbf{A}'(x')=4\pi\int\frac{dp^4dx^4}{(2\pi)^4}e^{-ip\cdot(x'-x)}\left[\frac{-p^2\mathbf{J}(x)-iK\mathbf{p}\times\mathbf{J}(x)}{(p^2)^2-K^2|\mathbf{p}|^2}\right],$$

$$A_0'(x')=4\pi\int\frac{d^4p\,d^4x}{(2\pi)^4}e^{-ip\cdot(x'-x)}\left[\frac{-K^2-p^2}{(p^2)^2-K^2|\mathbf{p}|^2}\right].$$

APPENDIX C

We now show the equivalence of the two derivations of the modified Maxwell equations; we do this by showing that they both have the same dispersion relation. From Appendix B, we have the dispersion relation for the first derivation:

$$p^2[(p^2)^2-K^2|\mathbf{p}|^2]=0.$$

To find the dispersion relation for the second derivation, we must find the poles of the modified photon propagator. We use the fact that

$$(D^{-1})_{\mu\nu}=(D_0{}^{-1})_{\mu\nu}-\pi'_{\mu\nu}.$$

The dispersion relation is given by the determinant of the matrix $(D_0{}^{-1})_{\mu\nu}-\pi'_{\mu\nu}$,

$$\det\begin{bmatrix}p^2 & 0 & 0 & 0\\ 0 & -p^2 & -iKp_3 & iKp_2\\ 0 & iKp_3 & -p^2 & -iKp_1\\ 0 & -iKp_2 & iKp_1 & -p^2\end{bmatrix}=0,$$

$$\Rightarrow\quad p^2[(p^2)^2-K^2|\mathbf{p}|^2]=0,$$

which is identical to the dispersion relation obtained in the first derivation.