Theory for the Measurement of the Earth's Velocity through the 3°K Cosmic Radiation*

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The intensity of radiation for an observer moving through thermal radiation is described by an anisotropic temperature $T(\theta)=T(1+\beta\cos\theta)^{-1}$ or $(h\nu^3/c^2)d\Omega d\nu[\exp h\nu/kT(\theta)-1]^{-1}$, where $\beta=v/c$. Temperature-measuring antennas measure $T(\theta)$. Intensity measurements give an anisotropy between radiation along $\pi+\theta$ and θ of $\beta\cos\theta$ for $(h\nu/kT)\ll1$ (the same as the temperature anisotropy), and $(h\nu/kT)\beta\cos\theta$ for $(h\nu/kT)\gg1$. Examples of moving detectors and surfaces in the laboratory and in space are discussed.

AS discussed in a number of laboratories and in a recent paper,[1] the velocity of the earth's motion with respect to the rest frame of 3°K cosmic background radiation[2] can be determined by measuring the anisotropy of the radiation. Some care is needed in carrying out the Lorentz transformation between the two frames of reference. The paper[1] was incorrect, and the subsequent erratum does not emphasize the character of the spectral distribution. Some historical aspects of the problem are given in this paper, and some of the possible measurements are considered.

The interest in problems of this type has a long history, and Pauli[3] noted in his discussion of blackbody radiation in a moving cavity in 1921 that "This case is of historical interest, since it can be treated entirely on the basis of electrodynamics, without relativity. When this is done, one comes to the inevitable conclusion that momentum, and thus inertial mass, must be ascribed to the moving radiation energy." He further notes that relativity permits a determination of radiation pressure, momentum, energy, and entropy, and the dependence of the spectral distribution on temperature and direction for a stationary observer. The problem of current interest is the motion of an observer relative to the background radiation, and much of the earlier discussion applies.

At a given point in space an observer can measure the power crossing a surface in a given direction and in a given frequency range and this quantity is usually referred to as the intensity. Intensity and the square of the amplitude of a plane wave are related by

$$I(\nu\theta\varphi p)d\Omega d\nu=\text{const}A^2{}_{\nu\theta\varphi p}\,, \tag{1}$$

where θ,φ is the direction of the wave with respect to a set of axes, A is the amplitude of the electric vector, and p is the polarization index of the traveling wave. A limiting process can be given to make this definition precise.

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[1] J. J. Condon and M. Harwit, Phys. Rev. Letters **20**, 1309 (1968); **21**, 58E (1968).

[2] R. B. Partridge and David T. Wilkinson, Phys. Rev. Letters **18**, 557 (1967).

[3] W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958), p. 139.

Consider now a source of electromagnetic radiation in reference frame S' very far from the observer S. Let S be moving with velocity v along the $+\hat{z}$ direction away from S' as shown in Fig. 1 and let (θ,φ) be the angular coordinates of the radiation. This is the problem considered by Einstein[4] in one of his first papers on special relativity. A Lorentz transformation between the two frames of reference yields the frequency relationship

$$\nu=\nu'\frac{(1-\beta\cos\theta')}{(1-\beta^2)^{1/2}}=\nu'\frac{(1-\beta^2)^{1/2}}{(1+\beta\cos\theta)}\,, \tag{2}$$

where

$$\beta=v/c\,. \tag{3}$$

θ' is the angle with the z axis in the reference frame S', and θ is the angle in the reference frame S. The angles are related by

$$\cos\theta=(\cos\theta'-\beta)/(1-\beta\cos\theta')\,. \tag{4}$$

The wave amplitude transforms as

$$(A'/\nu')^2=(A/\nu)^2 \tag{5}$$

and A/ν is an invariant for a plane wave. A transformation law for intensities follows immediately from Eq. (5), and the intensities in reference frames S and S' are related by

$$I(\nu\theta\varphi p)d\Omega d\nu/\nu^2=I'(\nu'\theta'\varphi'p')d\Omega'd\nu'/\nu'^2\,. \tag{6}$$

Further analysis requires the functional form of the

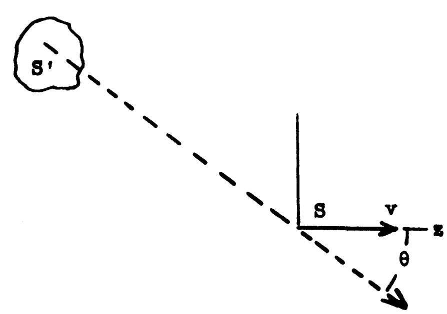

FIG. 1. Reference frame S is moving relative to reference frame S' with velocity v along $+z$, and radiation from S' makes an angle θ with the z axis.

[4] A. Einstein, Ann. Phys. Leipzig **17**, 891 (1905).

intensity. It is convenient to note that the transformation law for solid angles is given by the invariant

$$\nu'^2 d\Omega' = \nu^2 d\Omega. \qquad (7)$$

Intensity is frequently written in the alternate forms

$$I(\nu\theta\varphi p)d\Omega d\nu = [n(\nu\theta\varphi p)h\nu c]\nu^2 d\Omega d\nu/c^3, \qquad (8a)$$

$$I(\nu\theta\varphi p)d\Omega d\nu = (h\nu^3/c^2)d\Omega d\nu/\{\exp[h\nu/kT(\nu\theta\varphi p)]-1\}. \qquad (8b)$$

By introducing ν^3 into the right-hand side of Eq. (8) and by noting that $\nu d\nu d\Omega$ is invariant to a Lorentz transformation, it follows from Eqs. (6) and (8a) that

$$n(\nu\theta\varphi p) = n'(\nu'\theta'\varphi' p') \qquad (9)$$

is invariant and is often called the photon number. The second form or Eq. (8b) permits the radiation field in S and S' to be related by

$$\nu/T(\nu\theta\varphi p) = \nu'/T'(\nu'\theta'\varphi' p'). \qquad (10)$$

This relationship is quite general[5] for radiation from incoherent sources.

In the same paper in which Eq. (5) was given, Einstein[4] emphasized that for a laterally bounded finite wave enclosed entirely in volume V', the enclosed energy E'/ν', $\nu' V'$, and the pressure p' are invariants.

If the radiation is thermal radiation in some reference frame S', then the temperature is a scalar and independent of frequency, or

$$T'(\nu'\theta'\varphi' p') = T'.$$

In reference frame S moving in the $+\hat{z}$ direction the radiation is described by an anisotropic temperature. Equation (10) simplifies and

$$T(\theta) = T/(1+\beta\cos\theta), \qquad (11)$$

where

$$T = (1-\beta^2)^{1/2} T'. \qquad (12)$$

$T(\theta)$ completely describes the anisotropy of the radiation field.

Equation (11) completes the description of the thermal radiation field of S' as seen by the observer in S. The spectral distribution in S is completely described by the equation

$$I(\nu\theta)d\Omega d\nu = (2h/c^2)\nu^3 d\Omega d\nu/ \{\exp[(h\nu/kT)(1+\beta\cos\theta)]-1\}. \qquad (13)$$

Since observations are made in reference frame S and detecting systems have their conventional properties in S, Eq. (13) gives a complete description of the measurements which can be made in reference frame S. Pauli[3] considered in detail the problem of a stationary observer and a moving cavity and differs from the above by $(\pi+\theta)$.

[5] L. Landau and E. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958), p. 178.

TEMPERATURE MEASUREMENTS

If the antenna and detector have a reasonably narrow response angle and frequency range $d\Omega d\nu$, and if the detecting element can come into equilibrium with the radiation field, the antenna measures the temperature $T(\theta)$ which is given by Eq. (11). This measurement is independent of the frequency. In order to avoid changing the definition of $I(\nu\theta)$ from its usual form, it should be noted that an antenna with direction (γ) from the $\hat{z}$ axis receives radiation which is denoted as $\theta=\pi+\gamma$. Thus $I(\nu\pi)$ refers to radiation received antiparallel to the direction of motion and $I(\nu 0)$ radiation received parallel to the direction of motion. The anisotropy in temperature between radiation along $\pi+\theta$ and radiation along θ is

$$[T(\pi+\theta)-T(\theta)]/[T(\pi+\theta)+T(\theta)] = \beta\cos\theta. \qquad (14)$$

If two identical antennas are pointed in opposite directions, the temperature difference between the two detectors gives a direct measure of the velocity of S through the thermal radiation. It is the velocity, since both β and θ can be measured. Orientation of the double antenna to the maximum difference yields the direction of motion. At a velocity of 100 km/sec, a temperature difference of the order of 2 mdeg occurs for a background temperature of 3°K.

INTENSITY MEASUREMENTS

A detecting system which measures the intensity of the radiation in a small angle $d\Omega$ and frequency range $d\nu$, or a detecting system which has its response $g(\nu\theta)$ sharply peaked along the angle θ and near the frequency ν measures the intensity

$$I(\nu\theta) \cong \int I(\nu_1\theta_1) g(\nu\theta;\nu_1\theta_1) d\Omega_1 d\nu_1. \qquad (15)$$

It is assumed that the $g(\nu\theta)$ is sufficiently sharp that for each orientation of the detector $I(\nu\theta)$ can be taken outside the integral. An anisotropy for intensity can be defined as

$$F(\nu\theta;\beta) = \frac{I(\nu,\pi+\theta)-I(\nu,\theta)}{I(\nu,\pi+\theta)+I(\nu,\theta)} = \frac{e^x-e^y}{e^x+e^y-2}, \qquad (16)$$

where

$$x = (h\nu/kT)(1+\beta\cos\theta)$$

and

$$y = (h\nu/kT)(1-\beta\cos\theta). \qquad (17)$$

At low frequencies the anisotropy term is

$$F(\nu\theta;\beta)\cong\beta\cos\theta,\quad (h\nu/kT)\ll 1. \tag{18}$$

This is the Rayleigh-Jeans region and gives the same anisotropy as the temperature measurement. At high frequencies

$$F(\nu\theta;\beta)\cong(h\nu/kT)\beta\cos\theta,\quad (h\nu/kT)\gg 1 \tag{19}$$

and the anisotropy increases as $(h\nu/kT)$. At the intensity maximum for blackbody radiation, $(h\nu_m/kT)=2.822$ for $dI(\nu)/d\nu=0$, the anisotropy is

$$F(\nu_m\theta;\beta)\cong 3\beta\cos\theta. \tag{20}$$

Intensity measurements at higher frequencies enhance the anisotropy. The received power for the antenna or detector system pointing in the direction of motion always exceeds the received power from the detector system pointing opposite to the direction of motion. The anisotropy for $\theta=0$, $F(\nu\beta)$, is shown in Fig. 2 as a function of ν/ν_m.

TOTAL THERMAL POWER MEASUREMENTS

The total power received in direction $d\Omega$ is given by the integral of Eq. (13) over $d\nu$ and is

$$I(\theta)d\Omega=\frac{(\sigma/\pi)T^4}{(1+\beta\cos\theta)^4}d\Omega, \tag{21}$$

where $\sigma=2\pi^5k^4/15h^3c^2$ is the Stefan-Boltzmann constant. An object at temperature T_Σ emits $(\sigma/\pi)T_\Sigma{}^4d\Omega$ and equilibrium occurs for $T_\Sigma=(1+\beta\cos\theta)^{-1}T$. This is the same temperature as given by Eq. (11) for $T(\theta)$. If this energy is incident on a surface with a normal along $+\hat{z}$, then, introducing the cosine law for the size of the surface Σ, the thermal power incident on a square meter with a $+\hat{z}$ normal is

$$I_\pm=\sigma T^4(1\mp\tfrac{1}{3}\beta)/(1\mp\beta)^3. \tag{22}$$

The lower signs refer to the surface with a $-\hat{z}$ normal. A blackbody disk at temperature T_Σ radiates thermal power $2\sigma T_\Sigma{}^4$, receives power I_++I_-, and the net loss is quadratic in β. If the back side is highly reflecting, the net loss is $\sigma T_\Sigma{}^4-I_+$ and is linear in β. At $T=T_\Sigma$ this disk gains energy and this is due to the anisotropy in the incident radiation. Detailed balance does not follow for an element of surface $\Delta\Sigma$ at temperature T_Σ for each direction. If the surface normal of the element of surface $\Delta\Sigma$ makes angle γ with the $+z$ axis, then the power incident on $\Delta\Sigma$ is approximately given by

$$I\Delta\Sigma_\gamma\cong\sigma T^4\{1+(8/3)\beta\cos\gamma+\tfrac{5}{2}\beta^2[1+\cos^2\gamma]+\cdots\}\Delta\Sigma_\gamma. \tag{23}$$

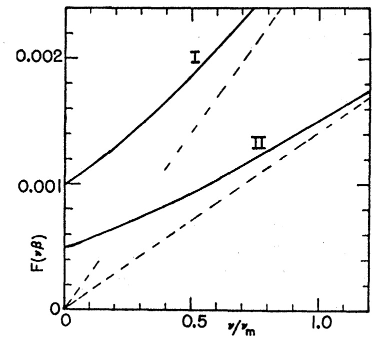

FIG. 2. $F(\nu\beta)$ is a measure of the anisotropy of the power received from directions antiparallel and parallel to the direction of motion and is given by Eq. (16). $F(\nu\beta)$ is plotted against ν/ν_m for $\beta=v/c=0.001$ for curve I and 0.0005 for curve II. ν_m is defined as the frequency at the peak intensity of the blackbody radiation, $h\nu_m/kT=2.822$.

The thermal power emitted by the element of surface is $\sigma T_\Sigma{}^4\Delta\Sigma$. Thermal equilibrium occurs for the entire object when the total incident power and the total emitted thermal power are equal. Thus the shape of the object and the angular dependence of the absorption coefficient are important.

Suppose a large system is constructed of two thermally isolated sheets with normals along the $\pm\hat{z}$ directions. Let each sheet have a thermally black side toward the surrounding radiation and a highly reflecting side toward each other to maintain thermal isolation between the sheets. Each sheet comes to a temperature which is determined by $\sigma T_\Sigma{}^4=I_\pm$, and the temperature difference between the two sheets is

$$\Delta T\cong\tfrac{4}{3}\beta T. \tag{24}$$

This is of the order of 1 mdeg for a system moving at 100 km/sec. These calculations are in the short-wavelength limit or the size is much larger than any of the important wavelengths. At 3°K, "large" refers to objects larger than a few centimeters.

DRAG FORCE

The drag force on an arbitrary object can be determined from the anisotropic radiation field which carries momentum

$$c^{-1}I(\nu\theta)d\Omega d\nu$$

per second across a surface with normal along θ, and the momentum per second carried away by thermal radiation. For convenience assume that a flat disk with a $\hat{z}$ normal is reflecting on both surfaces. The change in momentum per second at the reflecting surface gives rise to a force per unit area along the z normal of

$$F_z/\Sigma=-2c^{-1}\int I(\nu\theta)\cos^2\theta d\Omega d\nu$$

$$=\mp(4\sigma/3c)T^4(1\mp\beta)^{-3}. \tag{25}$$

The lower sign refers to the force on the perfectly reflecting $-\hat{z}$ side of the disk. The net drag force per unit area or approximate pressure difference across the moving mirror is

$$\Delta p \cong \Delta F_z/\Sigma \cong -(8\sigma/c)\beta T^4 \qquad (26)$$

and is linear in β. At 3°K its magnitude for a velocity of 100 km/sec is 4.1×10^{-17} N/m^2. The energy loss per second as the moving disk does work on the radiation field is approximately $8\sigma T^4\beta^2$ or 4.1×10^{-12} W/m^2 for the above example. This may be compared with a total energy of 10^4 J for an object of the order of 1 mm in diameter. Such forces are negligible in slowing down an object in 10^{10} years.[6] Since this effect is so small, examples with nonreflecting surfaces are not considered. The maximum effect occurs for reflecting surfaces. For completeness, the momentum per second crossing a surface element $\Delta\Sigma$ with a surface normal making angle γ with the $+\hat{z}$ axis is given by

$$-(2\sigma/3c)T^4\{1+3\beta\cos\gamma+\beta^2(2+4\cos^2\gamma)+\cdots\}\Delta\Sigma. \qquad (27)$$

The minus sign indicates that the flow of momentum is opposite to the surface normal and only the flow opposite to the normal is included. Pressure has a rather special meaning and is not used to denote this surface force.

At surface $\Delta\Sigma$ the radiation can be partially absorbed and reflected. The radiation field does adiabatic work on the perfectly reflecting mirror and decreases the mirror's velocity. An absorbing surface does work and exchanges heat with the radiation field.

[6] If the 3°K radiation is a relic of the primeval fireball, then the character of the radiation at earlier stages must be taken into account.

It may also be noted that the absorption coefficient of an atom is proportional to $I(\nu\theta)$ and is anisotropic. The stimulated emission would also be anisotropic but spontaneous emission is isotropic in the rest frame of the atom. On the average there is a loss of translational energy for a nonthermal group of atoms or molecules. Again the effect is small and the details are not given.

SUMMARY

Equation (13) gives the angular and spectral distribution of the radiation received by an observer in reference frame S moving through a thermal radiation field. Subsequent equations relate temperature measurements of ideal antennas and surfaces with their velocity through this thermal radiation. Most of the current interest is in measuring the motion of the earth through the 3°K cosmic radiation. The difficulties of such measurements are not considered in this paper. Some idealized experiments on objects well away from the earth are given and temperature differences of the order mdeg occur. The same equations apply to laboratory experiments, but the increase in the thermal background temperature is more than compensated for by the decrease in velocity, and temperature differences of $\frac{1}{10}$ mdeg would be difficult to produce in the laboratory. The anisotropy increases as $(h\nu/kT)$, and this might assist a laboratory measurement in the same manner as suggested for 3°K radiation. Although this subject has a long history, no experimental test has as yet been made.

ACKNOWLEDGMENT

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