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Soft-Photon Theorem for Bremsstrahlung in a Potential Model*

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The bremsstrahlung matrix element calculated from the Schrödinger equation for two particles interacting with each other via a local potential and with the electromagnetic field in the standard gauge-invariant manner is shown to satisfy the soft-photon theorem. That is, the first two terms of the expansion of the radiative matrix element in powers of the photon's energy are calculable from a knowledge of the nonradiative matrix element. The proof is given first without, then with, spin for arbitrary order of perturbation theory in the strong interaction. While the derivation is very different from the one which Low used for a relativistic theory, the final formulas are similar in appearance and agree in the nonrelativistic limit. For low particle momenta, it is found that there is a relation between the on- and off-energy-shell derivatives of the nonradiative T matrix to the leading order in the momentum. Furthermore, the p-wave contribution to the internal-emission matrix element is of the same order in the momentum as the s-wave part. For particle momenta much less than the reciprocal of the s-wave scattering length, the second term of the expansion of the bremsstrahlung matrix element in powers of the photon's energy is negligible compared with the first term.

I. INTRODUCTION

THE Low theorem was originally derived by him using the Lippmann-Schwinger equation,[1] but the published version[2] is given in terms of relativistic quantum field theory. Since several calculations of nucleon-nucleon bremsstrahlung have been performed using a potential model, it was considered worthwhile to publish the explicit derivation of the theorem according to that model. This can also serve as a check on those calculations.

The idea of the theorem is that it is possible to write the matrix element for the radiative process

$$p_1+p_2\to p_1'+p_2'+\gamma \qquad (1)$$

(with the symbols standing for particle labels and momenta) in the form

$$A/\gamma+B+O(\gamma), \qquad (2)$$

where both A and B are given in terms of matrix elements for the nonradiative process

$$k_1+k_2\to k_1'+k_2' \qquad (3)$$

at a set of momenta k_i which are related to the momenta p_i. This relation will be made clear during the derivation.

II. DERIVATION FOR SPINLESS PARTICLES

In the general problem the contributions to the bremsstrahlung matrix are classified according to whether the photon comes off an external line or an internal line. See Fig. 1, where particle 1 is charged and 2 is neutral. Low[2] wrote the external-emission contribution [Figs. 1(a) and 1(b)] as (various factors times) $\epsilon_\mu M_\mu^{(1)}$, with the four-vector $M_\mu^{(1)}$ given explicitly in terms of an invariant off-mass-shell T matrix, for process (3), as

$$M_\mu^{(1)}=e_1[(p_{1\mu}'/p_1'\cdot\gamma)\langle p_1'+\gamma,p_2'|T|p_1,p_2\rangle -\langle p_1',p_2'|T|p_1-\gamma,p_2\rangle p_{1\mu}/p_1\cdot\gamma] \qquad (4)$$

and ϵ_μ is the polarization vector of the photon. While no explicit formula can be written for the internal-emission contribution $M_\mu^{(2)}$ [Fig. 1(c)] in this general theory, gauge invariance requires that

$$\gamma_\mu M_\mu\equiv\gamma_\mu(M_\mu^{(1)}+M_\mu^{(2)})=0.$$

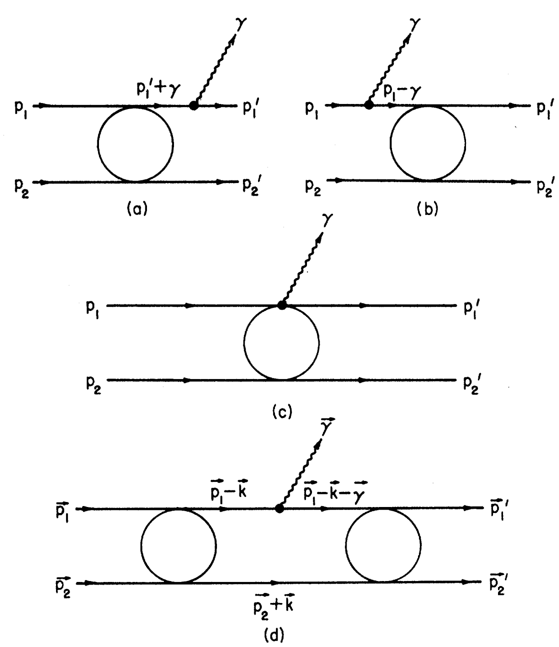

FIG. 1. In a general theory, bremsstrahlung matrix elements are classified as external emission (a) and (b) and internal emission (c), the circles indicating the strong interaction. If particle 2 also interacted with the electromagnetic field, there would be additional external diagrams (a′) and (b′). For the Schrödinger equation with a local potential the only possible internal emission is shown in (d), where the potential acts before and after the radiation.

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[1] F. E. Low (private communication).
[2] F. E. Low, Phys. Rev. **110**, 974 (1958).

This is sufficient to yield the result[2] that the first two terms of an expansion of M_μ in powers of γ can be obtained in terms of the *physical* T matrix for process (3). It was assumed that as $\gamma_\mu \to 0$, $M_\mu^{(2)} \to$ const, independent of the direction of the photon.

Even though the Schrödinger equation has no Lorentz-invariance properties, the existence of a charge-conservation law[3] with (current/density)$\approx \mathbf{p}/m$ means that (neglecting p^2/m^2 compared with unity) Eq. (4) and the entire Low derivation can be carried over to the potential problem. We shall proceed in a different manner that provides some insight into the way charge conservation produces the result.

In practice one chooses a particular gauge, $\varphi=0$ with $\boldsymbol{\varepsilon}\cdot\boldsymbol{\gamma}=0$, and writes the bremsstrahlung matrix element as

$$-(2\pi/\gamma)^{1/2}\boldsymbol{\varepsilon}\cdot(\mathbf{M}^{(1)}+\mathbf{M}^{(2)}), \qquad (5)$$

where $\mathbf{M}^{(1)}$ arises, as before, from emissions which either follow or precede all the scattering due to the local potential.[4] $\mathbf{M}^{(1)}$ is called the single-scattering term and is represented by the diagrams in Figs. 1(a) and 1(b), which are now not Feynman diagrams.

$$\begin{aligned}\mathbf{M}^{(1)}=(e_1/m_1)\{&\mathbf{p}_1'[E-E_1(\mathbf{p}_1'+\boldsymbol{\gamma})-E_2(\mathbf{p}_2')]^{-1}\\ &\times\langle\mathbf{p}_1'+\boldsymbol{\gamma},\,\mathbf{p}_2'|T(E)|\mathbf{p}_1,\,\mathbf{p}_2\rangle\\ &+\langle\mathbf{p}_1',\,\mathbf{p}_2'|T(E')|\mathbf{p}_1-\boldsymbol{\gamma},\,\mathbf{p}_2\rangle\\ &\times[E'-E_1(\mathbf{p}_1-\boldsymbol{\gamma})-E_2(\mathbf{p}_2)]^{-1}\mathbf{p}_1\}, \qquad (6)\end{aligned}$$

where

$$E_i(\mathbf{p})\equiv p^2/2m_i, \quad E\equiv E_1(\mathbf{p}_1)+E_2(\mathbf{p}_2),$$

and

$$E'\equiv E_1(\mathbf{p}_1')+E_2(\mathbf{p}_2').$$

The (half) off-energy-shell T-matrix elements in Eq. (6) refer to elastic (nonradiative) scattering. While this equation looks like the spatial components of Eq. (4), neither the T matrices nor the energy denominators in Eq. (6) have any relativistic invariance properties.

The internal-emission contribution $\mathbf{M}^{(2)}$, also called the double-scattering term, is represented by the diagram in Fig. 1(d). Whereas in the general derivation[2] one did not have an explicit formula for the internal emission, here one does,[4] and we shall make use of it to prove the following theorem:

$$\begin{aligned}\mathbf{M}^{(2)}=\frac{e_1}{m_1}\sum_{\mathbf{k}}&\langle\mathbf{p}_1',\mathbf{p}_2'|T(E')|\mathbf{p}_1-\mathbf{k}-\boldsymbol{\gamma},\,p_2+\mathbf{k}\rangle\\ &\times[E'-E_1(\mathbf{p}_1-\mathbf{k}-\boldsymbol{\gamma})-E_2(\mathbf{p}_2+\mathbf{k})+i\epsilon]^{-1}(\mathbf{p}_1-\mathbf{k})\\ &\times[E-E_1(\mathbf{p}_1-\mathbf{k})-E_2(\mathbf{p}_2+\mathbf{k})+i\epsilon]^{-1}\\ &\times\langle\mathbf{p}_1-\mathbf{k},\,\mathbf{p}_2+\mathbf{k}|T(E)|\mathbf{p}_1,\,\mathbf{p}_2\rangle. \qquad (7)\end{aligned}$$

[3] See, e.g., E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Chap. 4.

[4] For the two-potential formalism, see M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953); for the application to bremsstrahlung, see M. I. Sobel and A. H. Cromer, *ibid.* **132**, 2698 (1963).

It is clear that $\mathbf{M}^{(1)}$ diverges as $\gamma\to 0$, but $\mathbf{M}^{(2)}$ remains finite. Therefore $\mathbf{M}^{(1)}$ contributes to both A and B in Eq. (2), but $\mathbf{M}^{(2)}$ contributes only to B.

The heart of the derivation is the following expression for $\mathbf{M}^{(2)}$:

$$\mathbf{M}^{(2)}=e_1\nabla_{p_1}\langle\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\,\mathbf{p}_2'|T(e)|\mathbf{p}_1,\,\mathbf{p}_2\rangle|_{e=\bar{E}}+O(\gamma). \qquad (8)$$

The meaning of this statement is to take the completely off-energy-shell T-matrix element with an energy variable e that is unrelated to the momenta $\mathbf{p}_i$, take its gradient with respect to the external momentum of the charged particle, and then set the energy variable equal to $\bar{E}$, which can be taken to be the initial particle energy E in the bremsstrahlung process or the final energy E', or any energy that differs from these by an amount of order γ. This difference will be reflected in the error term in Eq. (8) which is itself $O(\gamma)$. We shall prove this result for arbitrary order of perturbation theory in the potential V. (The electromagnetic interaction is treated to first order throughout.)

Using the Lippmann-Schwinger equation

$$T(e)=V+VG^{(+)}(e)T(e), \qquad (9)$$

with $G^{(+)}(e)=(e+i\epsilon-H_0)^{-1}$ and H_0 the kinetic energy operator for the two particles, an arbitrary order of the perturbation expansion of this equation is given as

$$VG^{(+)}(e)VG^{(+)}(e)\cdots G^{(+)}(e)V$$

and the corresponding contribution to the matrix element $\langle\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\,\mathbf{p}_2'|T(e)|\mathbf{p}_1,\,\mathbf{p}_2\rangle$ is represented by the diagram shown in Fig. 2(a) (for the case of third order). Since the matrix elements of V depend only on the momentum transfer, they are not functions of $\mathbf{p}_1$,[5] and this variable occurs only in the energy denominators. For example, in third order,

$$\begin{aligned}T^{(3)}&\equiv\langle\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\,\mathbf{p}_2'|T(e)|\mathbf{p}_1,\,\mathbf{p}_2\rangle^{(3)}\\ &=(V\text{-matrix elements that are independent of }\mathbf{p}_1)\\ &\quad\times[e-E_1(\mathbf{p}_1-\mathbf{k}_1)-E_2(\mathbf{p}_2+\mathbf{k}_1)]^{-1}\\ &\quad\times[e-E_1(\mathbf{p}_1-\mathbf{k}_2)-E_2(\mathbf{p}_2+\mathbf{k}_2)]^{-1}. \qquad (10)\end{aligned}$$

Taking the gradient gives

$$\nabla_{p_1}T^{(3)}=\frac{T^{(3)}}{m_1}\left(\frac{\mathbf{p}_1-\mathbf{k}_1}{e-E_1(\mathbf{p}_1-\mathbf{k}_1)-E_2(\mathbf{p}_2+\mathbf{k}_1)}+\frac{\mathbf{p}_1-\mathbf{k}_2}{e-E_1(\mathbf{p}_1-\mathbf{k}_2)-E_2(\mathbf{p}_2+\mathbf{k}_2)}\right). \qquad (11)$$

By writing out the perturbation expansion of both T-matrix elements occurring in Eq. (7), it is seen that for any diagram which contributes to the T-matrix

[5] This is true for a local potential. I am indebted to J. L. Gammel for several valuable discussions concerning this derivation, for the main idea of looking at an arbitrary order of perturbation theory, and for pointing out the simplification that occurs if the external momentum $\mathbf{p}_1$ is carried along the charged-particle line.

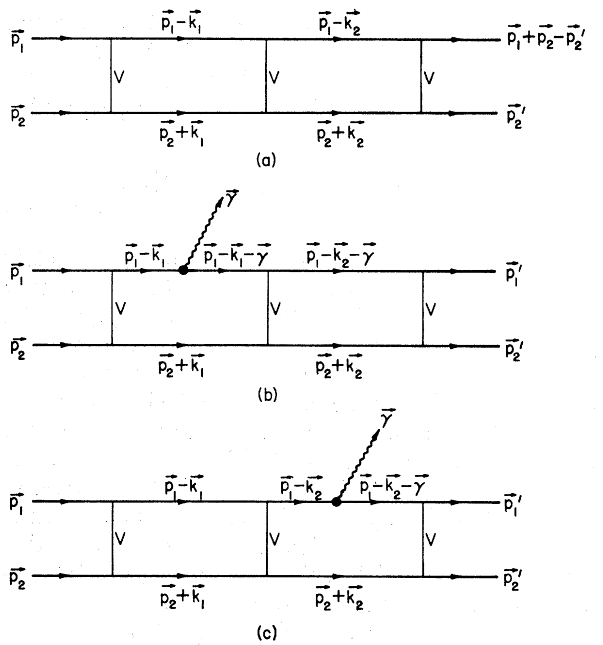

FIG. 2. For any diagram that contributes to the nonradiative T matrix in the Lippmann-Schwinger equation such as the third-order one shown in (a), there is a set of corresponding diagrams (b) and (c) (two in third order) that contribute to the internal-emission bremsstrahlung matrix element. If particle 2 also interacted with the electromagnetic field, there would be additional diagrams (b′) and (c′). Charge conservation (for a local potential) requires that there be a contribution to the bremsstrahlung matrix element from *each* segment of the $\mathbf{p}_1$ line. This is the essence of the proof of Eq. (8).

in Eq. (8), such as Fig. 2(a), there is a set of corresponding diagrams which contribute to $\mathbf{M}^{(2)}$, obtained by attaching a photon of momentum $\boldsymbol{\gamma}$ to *each* of the internal charged lines in Fig. 2(a). One of these is shown in Fig. 2(b). This particular diagram is obtained from Eq. (7) by expanding $\langle \mathbf{p}_1-\mathbf{k},\, \mathbf{p}_2+\mathbf{k} | T(E) | \mathbf{p}_1,\, \mathbf{p}_2\rangle$ to first order (with $\mathbf{k}=\mathbf{k}_1$) and $\langle \mathbf{p}_1',\, \mathbf{p}_2' | T(E') | \mathbf{p}_1-\mathbf{k}-\boldsymbol{\gamma},\, \mathbf{p}_2+\mathbf{k}\rangle$ to second order. The contribution of this diagram to $\mathbf{M}^{(2)}$ has precisely the same V-matrix elements as occurred in Eq. (10), multiplying

$$[E-E_1(\mathbf{p}_1-\mathbf{k}_1)-E_2(\mathbf{p}_2+\mathbf{k}_1)]^{-1}(\mathbf{p}_1-\mathbf{k}_1)\\ \times[E'-E_1(\mathbf{p}_1-\mathbf{k}_1-\boldsymbol{\gamma})-E_2(\mathbf{p}_2+\mathbf{k}_1)]^{-1}\\ \times[E'-E_1(\mathbf{p}_1-\mathbf{k}_2-\boldsymbol{\gamma})-E_2(\mathbf{p}_2+\mathbf{k}_2)]^{-1}.$$

Comparing this result with the first term in Eq. (11), and using $E-E'=\gamma$ and $E_i(\mathbf{p}-\boldsymbol{\gamma})-E_i(\mathbf{p})=O(\gamma)$, demonstrates that the respective contributions to the two sides of Eq. (8) do indeed differ by a term of order γ. The second term in Eq. (11) corresponds to Fig. 2(c), which radiates from the only other internal charged line in third order. It is clear that this correspondence applies in any order of perturbation theory, and if the perturbation series converges, the proof of Eq. (8) is complete. If the perturbation series only converges for the potential $\lambda V, |\lambda|<\lambda_0<1$, then it is still very likely true that the two sides of Eq. (8) can be analytically continued to $\lambda=1$.

We now expand Eq. (6) for $\mathbf{M}^{(1)}$ in powers of γ, keeping two terms, and will then combine the result with Eq. (8) for $\mathbf{M}^{(2)}$. First, it is necessary to parametrize an arbitrary T-matrix element $\langle \mathbf{k}_1',\, \mathbf{k}_2' | T(e) | \mathbf{k}_1,\, \mathbf{k}_2\rangle$ in terms of scalar variables. Since the motion of the c.m. merely provides a momentum-conserving δ function $\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_1'-\mathbf{k}_2')$, one introduces initial and final momenta in the c.m. system

$$\mathbf{q}_i\equiv\mu\left(\frac{\mathbf{k}_1}{m_1}-\frac{\mathbf{k}_2}{m_2}\right),\quad \mathbf{q}_f\equiv\mu\left(\frac{\mathbf{k}_1'}{m_1}-\frac{\mathbf{k}_2'}{m_2}\right),\quad \frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2},$$

and defines four scalar variables to be

$$\begin{aligned} \nu&\equiv\tfrac{1}{2}(q_i^2+q_f^2)/2\mu,\\ t&\equiv(\mathbf{q}_f-\mathbf{q}_i)^2=(\mathbf{k}_2'-\mathbf{k}_2)^2=(\mathbf{k}_1'-\mathbf{k}_1)^2,\\ \Delta_i&\equiv e-(k_1^2/2m_1+k_2^2/2m_2)=e_{\mathrm{c.m.}}-q_i^2/2\mu,\\ \Delta_f&\equiv e-(k_1'^2/2m_1+k_2'^2/2m_2)=e_{\mathrm{c.m.}}-q_f^2/2\mu. \end{aligned}\tag{12}$$

These play the same role as the corresponding relativistic invariants used in Ref. 2. ν is the average of the initial and final (kinetic) energies in the c.m. system, t is the square of the momentum transfer, Δ_i represents the amount that the initial state is off the energy shell, and Δ_f is the same thing for the final state. Expressing the T matrix in terms of these variables, one has

$$\langle \mathbf{k}_1',\, \mathbf{k}_2' | T(e) | \mathbf{k}_1,\, \mathbf{k}_2\rangle\equiv T(\nu,t,\Delta_i,\Delta_f).\tag{13}$$

Returning to Eq. (6) and defining some quantities for the bremsstrahlung process,

$$\begin{aligned} \mathbf{q}&\equiv\mu(\mathbf{p}_1/m_1-\mathbf{p}_2/m_2),\quad \mathbf{q}'\equiv\mu(\mathbf{p}_1'/m_1-\mathbf{p}_2'/m_2),\\ \bar{q}^2&\equiv\tfrac{1}{2}(q^2+q'^2),\\ \Delta&\equiv E_1(\mathbf{p}_1)+E_2(\mathbf{p}_2)-E_1(\mathbf{p}_1'+\boldsymbol{\gamma})-E_2(\mathbf{p}_2')\\ &=\gamma-\boldsymbol{\gamma}\cdot\mathbf{p}_1'/m_1-\gamma^2/2m_1,\\ \Delta'&\equiv E_1(\mathbf{p}_1')+E_2(\mathbf{p}_2')-E_1(\mathbf{p}_1-\boldsymbol{\gamma})-E_2(\mathbf{p}_2)\\ &=-\gamma+\boldsymbol{\gamma}\cdot\mathbf{p}_1/m_1-\gamma^2/2m_1, \end{aligned}\tag{14}$$

gives

$$\mathbf{M}^{(1)}=\frac{e_1}{m_1}\left[\mathbf{p}_1'\frac{1}{\Delta}T\left(\frac{\bar{q}^2}{2\mu}+\frac{\boldsymbol{\gamma}\cdot\mathbf{q}'}{2m_1}+\frac{\mu\gamma^2}{4m_1^2},\ (\mathbf{p}_2'-\mathbf{p}_2)^2,\ 0,\ \Delta\right)\right.\\ \left.+\mathbf{p}_1\frac{1}{\Delta'}T\left(\frac{\bar{q}^2}{2\mu}-\frac{\boldsymbol{\gamma}\cdot\mathbf{q}}{2m_1}+\frac{\mu\gamma^2}{4m_1^2},\ (\mathbf{p}_2'-\mathbf{p}_2)^2,\ \Delta',\ 0\right)\right].\tag{15}$$

Expanding the T functions in powers of γ, for fixed $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$, $\mathbf{p}_2'$, one obtains

$$\mathbf{M}^{(1)}=\frac{e_1}{m_1}\left[\left(\frac{\mathbf{p}_1'}{\Delta}+\frac{\mathbf{p}_1}{\Delta'}\right)T+\frac{1}{2m_1}\left(\frac{\mathbf{p}_1'}{\Delta}\boldsymbol{\gamma}\cdot\mathbf{q}'-\frac{\mathbf{p}_1}{\Delta'}\boldsymbol{\gamma}\cdot\mathbf{q}\right)\frac{\partial T}{\partial\nu}\right.\\ \left.+\mathbf{p}_1'\frac{\partial T}{\partial\Delta_f}+\mathbf{p}_1\frac{\partial T}{\partial\Delta_i}\right]+O(\gamma).\tag{16}$$

In Eq. (16), the function T and its derivatives are evaluated at $\nu=\bar{q}^2/2\mu$, $t=(\mathbf{p}_2'-\mathbf{p}_2)^2$, $\Delta_i=0$, and $\Delta_f=0$.

What is the physical meaning of $T(\bar{q}^2/2\mu,\ (\mathbf{p}_2'-\mathbf{p}_2)^2,\ 0,0)$? According to Eqs. (12) and (13), it represents a T-matrix element for *elastic* scattering, since both the initial and final states are on the energy shell. Furthermore, from Eq. (14) the energy of this scattering in the c.m. system is equal to the *average* of the initial energy of the two particles taking part in the bremsstrahlung process in their initial c.m. system, and their final energy in their final c.m. system. Since the latter quantity differs from their final energy in their initial c.m. system by a term of order γ^2, it is sufficient to refer both energies to the over-all c.m. system. The momentum transfer squared must be described as belonging to particle 2, the one that is not radiating, since this differs from the corresponding quantity for particle 1 by a term of order γ. One could evaluate T in Eq. (16) at some other energy, e.g., the initial energy in the bremsstrahlung process, and compensate by changing the coefficient of $\partial T/\partial\nu$. This is equivalent to choosing a different energy variable ν' in Eq. (12).

Now consider $\langle \mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\ \mathbf{p}_2' | T(e) | \mathbf{p}_1,\ \mathbf{p}_2\rangle$, the T-matrix in Eq. (8), for which the parameters have the values

$$\nu=\frac{\mu}{4}\left[\left(\frac{\mathbf{p}_1}{m_1}-\frac{\mathbf{p}_2}{m_2}\right)^2+\left(\frac{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2'}{m_1}-\frac{\mathbf{p}_2'}{m_2}\right)^2\right],$$

$$t=(\mathbf{p}_2'-\mathbf{p}_2)^2,\quad \Delta_i=e-\frac{p_1^2}{2m_1}-\frac{p_2^2}{2m_2},$$

$$\Delta_f=e-\frac{(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')^2}{2m_1}-\frac{p_2'^2}{2m_2}\Bigg).\qquad(17)$$

Taking the gradient gives

$$\nabla_{p_1}\langle \mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\ \mathbf{p}_2'|T(e)|\mathbf{p}_1,\ \mathbf{p}_2\rangle$$

$$=\frac{1}{2m_1}\left(\mathbf{q}+\mathbf{q}'+\frac{\mu}{m_1}\boldsymbol{\gamma}\right)\frac{\partial T}{\partial\nu}-\frac{\mathbf{p}_1}{m_1}\frac{\partial T}{\partial\Delta_i}-\frac{\mathbf{p}_1'+\boldsymbol{\gamma}}{m_1}\frac{\partial T}{\partial\Delta_f}.\qquad(18)$$

Now setting $e=\bar{E}$ as described after Eq. (8), and expanding all terms in powers of γ, gives

$$\mathbf{M}^{(2)}=\frac{e_1}{m_1}\left(\tfrac{1}{2}(\mathbf{q}+\mathbf{q}')\frac{\partial T}{\partial\nu}-\mathbf{p}_1\frac{\partial T}{\partial\Delta_i}-\mathbf{p}_1'\frac{\partial T}{\partial\Delta_f}\right)+O(\gamma),\qquad(19)$$

where again the derivatives of T are evaluated at $\nu=\bar{q}^2/2\mu$, $t=(\mathbf{p}_2'-\mathbf{p}_2)^2$, $\Delta_i=0$, and $\Delta_f=0$. Observe that $\mathbf{M}^{(2)}$ does approach a constant as $\boldsymbol{\gamma}\to 0$, independent of its direction. Combining Eqs. (16) and (19) gives

$$\mathbf{M}^{(1)}+\mathbf{M}^{(2)}=\frac{e_1}{m_1}\Bigg[\left(\frac{\mathbf{p}_1'}{\Delta}+\frac{\mathbf{p}_1}{\Delta'}\right)T$$
$$+\left(\tfrac{1}{2}(\mathbf{q}+\mathbf{q}')+\frac{\mathbf{p}_1'}{\Delta}\frac{\boldsymbol{\gamma}\cdot\mathbf{q}'}{2m_1}-\frac{\mathbf{p}_1}{\Delta'}\frac{\boldsymbol{\gamma}\cdot\mathbf{q}}{2m_1}\right)\frac{\partial T}{\partial\nu}\Bigg]+O(\gamma),\qquad(20)$$

in which the off-energy-shell derivatives of T have disappeared precisely as in the original derivation.[2] This completes the proof that the coefficients of γ^{-1} and γ^0 in the expansion of the bremsstrahlung matrix element are completely determined by the elastic scattering T matrix and its derivative with respect to energy. Equation (20) is the analog of Eq. (2.16) in Ref. 2.

In the over-all c.m. system $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2'+\boldsymbol{\gamma}=0$, Eq. (20) becomes

$$\mathbf{M}^{(1)}+\mathbf{M}^{(2)}=\frac{e_1}{m_1}\Bigg[\frac{1}{\gamma}\left(\frac{\mathbf{p}_1'}{1-\boldsymbol{\gamma}\cdot\mathbf{p}_1'/\gamma m_1-\gamma/2m_1}\right.$$
$$\left.-\frac{\mathbf{p}_1}{1-\boldsymbol{\gamma}\cdot\mathbf{p}_1/\gamma m_1+\gamma/2m_1}\right)T(\bar{E}_{\text{c.m.}},t_2)$$
$$+\frac{1}{2}\left(\frac{\mathbf{p}_1'}{1-\boldsymbol{\gamma}\cdot\mathbf{p}_1'/\gamma m_1}+\frac{\mathbf{p}_1}{1-\boldsymbol{\gamma}\cdot\mathbf{p}_1/\gamma m_1}\right)$$
$$\times\frac{\partial T}{\partial\bar{E}_{\text{c.m.}}}(\bar{E}_{\text{c.m.}},t_2)\Bigg]+O(\gamma),\qquad(21)$$

where we have introduced $\bar{E}_{\text{c.m.}}\equiv\bar{q}^2/2\mu$, $t_2\equiv(\mathbf{p}_2'-\mathbf{p}_2)^2$, and T is now the elastic T matrix. Going to the static limit where $p/m\ll 1$ (γ/m is always much less than unity nonrelativistically),

$$\mathbf{M}^{(1)}+\mathbf{M}^{(2)}\simeq\frac{e_1}{m_1}\left(\frac{\mathbf{p}_1'-\mathbf{p}_1}{\gamma}T(\bar{E}_{\text{c.m.}},t_2)\right.$$
$$\left.+\tfrac{1}{2}(\mathbf{p}_1'+\mathbf{p}_1)\frac{\partial T}{\partial\bar{E}_{\text{c.m.}}}(\bar{E}_{\text{c.m.}},t_2)\right)+O(\gamma).\qquad(22)$$

Except for an incorrect relative sign in Eq. (1.7NR) of Ref. 2, there is agreement with Eq. (22) if it is remembered that in the leading (γ^{-1}) term, the momentum transfer must be taken equal to that of the nonradiating particle.

If particle 2 is also charged, one adds to Eqs. (20)–(22) additional terms obtained by interchanging all subscripts $1\leftrightarrow 2$ in these equations [also in $\mathbf{q}$ and $\mathbf{q}'$, according to Eq. (14)]. If the ratio e/m is the same for the two particles, then the γ^{-1} term is reduced by a factor p/m. At the same time, a term of order γ^0 is introduced that involves the derivative of T with respect to the momentum-transfer variable. If, in addition, the two particles are identical, then the entire amplitude must be symmetrized.

III. TWO SPIN-$\frac{1}{2}$ PARTICLES

Low[2] stated that the cancellation of the derivatives of the T matrix with respect to the off-shell variables is a very general result, not restricted to spinless particles. He gave an explicit proof also for one spin-0 and one spin-$\frac{1}{2}$ particle. Nyman[6] has done the same thing for two spin-$\frac{1}{2}$ particles, and has compared the cross section computed from the model-independent terms with existing proton-proton bremsstrahlung data. We now show how this derivation proceeds in a potential model. The essential point is that Eq. (8) is still correct with spin included.

The states of the particles are written $|\mathbf{k}_1,\mathbf{k}_2;sm\rangle$, using the total spin and its z-component representation. The general T-matrix element is a sum of terms

$$\langle \mathbf{k}_1',\mathbf{k}_2';s'm'|T(e)|\mathbf{k}_1,\mathbf{k}_2;sm\rangle =\sum_n T_n(\nu,t,\Delta_i,\Delta_f)\langle s'm'|A_n(\mathbf{L}_1,\mathbf{L}_2,\mathbf{L}_3)|sm\rangle, \quad (23)$$

where the variables occurring in the functions T_n are the same ones used in Eqs. (12) and (13); the three orthogonal vectors are taken to be

$$\mathbf{L}_1\equiv \mathbf{q}_f-\mathbf{q}_i, \quad \mathbf{L}_2\equiv \mathbf{q}_i\times\mathbf{q}_f, \quad \mathbf{L}_3\equiv \mathbf{L}_1\times\mathbf{L}_2, \quad (24)$$

and the operators A_n are of three types according to the number of spin operators that are present:

$$1;\quad \boldsymbol{\sigma}_a\cdot\mathbf{L}_i;\quad (\boldsymbol{\sigma}_1\cdot\mathbf{L}_i)(\boldsymbol{\sigma}_2\cdot\mathbf{L}_j). \quad (25)$$

If the invariances of space inversion and time reversal are assumed for the nucleon-nucleon interaction, then only six terms A_n are permitted for proton-proton scattering, and one of them vanishes for elastic scattering. For the proof of the theorem it is not necessary to impose these symmetries.

The electromagnetic interaction is also generalized to include a possible magnetic moment μ_1 for particle 1 (not to be confused with the reduced mass μ):

$$\langle \mathbf{k}_1-\boldsymbol{\gamma},\mathbf{k}_2;s'm'|H_{\rm em}|\mathbf{k}_1,\mathbf{k}_2;sm\rangle = -(2\pi/\gamma)^{1/2}\boldsymbol{\varepsilon}\cdot[(e_1/m_1)\mathbf{k}_1\delta_{ss'}\delta_{mm'} +i\mu_1\langle s'm'|\boldsymbol{\gamma}\times\boldsymbol{\sigma}_1|sm\rangle]. \quad (26)$$

As in the spinless case, radiation from particle 2 can be written at the end by interchanging all subscripts.

With the diagrams in Figs. 1 and 2 now carrying the additional labels sm for the initial state of the particles and $s'm'$ for their final state, Eq. (6) in the spinless case becomes

$$\mathbf{M}^{(1)}=\sum_{s''m''}[(e_1/m_1)\mathbf{p}_1'\delta_{s's''}\delta_{m'm''}+i\mu_1\langle s'm'|\boldsymbol{\gamma}\times\boldsymbol{\sigma}_1|s''m''\rangle]\Delta^{-1}\langle \mathbf{p}_1'+\boldsymbol{\gamma},\mathbf{p}_2';s''m''|T(E)|\mathbf{p}_1,\mathbf{p}_2;sm\rangle + \sum_{s''m''}\langle \mathbf{p}_1',\mathbf{p}_2';s'm'|T(E')|\mathbf{p}_1-\boldsymbol{\gamma},\mathbf{p}_2;s''m''\rangle(1/\Delta')[(e_1/m_1)\mathbf{p}_1\delta_{s''s}\delta_{m''m}+i\mu_1\langle s''m''|\boldsymbol{\gamma}\times\boldsymbol{\sigma}_1|sm\rangle]. \quad (27)$$

Introducing the three vectors that characterize the bremsstrahlung process

$$\mathbf{Q}\equiv\mathbf{q}'-\mathbf{q}, \quad \mathbf{N}\equiv\mathbf{q}\times\mathbf{q}', \quad \mathbf{P}\equiv\mathbf{Q}\times\mathbf{N}, \quad (28)$$

the two T matrices in Eq. (27) can be written, according to Eq. (23), as

$$\langle \mathbf{p}_1'+\boldsymbol{\gamma},\mathbf{p}_2';s''m''|T(E)|\mathbf{p}_1,\mathbf{p}_2;sm\rangle =\sum_n T_n\left(\frac{\bar{q}^2}{2\mu}+\frac{\boldsymbol{\gamma}\cdot\mathbf{q}'}{2m_1},(\mathbf{p}_2'-\mathbf{p}_2)^2,0,\Delta\right)\langle s''m''|A_n\left(\mathbf{Q}+\frac{\mu}{m_1}\boldsymbol{\gamma},\mathbf{N}+\frac{\mu}{m_1}\mathbf{q}\times\boldsymbol{\gamma},\mathbf{P}+\frac{\mu}{m_1}[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}\times\boldsymbol{\gamma})]\right)|sm\rangle$$

and (29)

$$\langle \mathbf{p}_1',\mathbf{p}_2';s'm'|T(E')|\mathbf{p}_1-\boldsymbol{\gamma},\mathbf{p}_2;s''m''\rangle =\sum_n T_n\left(\frac{\bar{q}^2}{2\mu}-\frac{\boldsymbol{\gamma}\cdot\mathbf{q}}{2m_1},(\mathbf{p}_2'-\mathbf{p}_2)^2,\Delta',0\right)\langle s'm'|A_n\left(\mathbf{Q}+\frac{\mu}{m_1}\boldsymbol{\gamma},\mathbf{N}+\frac{\mu}{m_1}\mathbf{q}'\times\boldsymbol{\gamma},\mathbf{P}+\frac{\mu}{m_1}[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}'\times\boldsymbol{\gamma})]\right)|s''m''\rangle,$$

dropping terms of order γ^2. Expanding the functions T_n and the operators A_n in powers of γ gives

$$\langle \mathbf{p}_1'+\boldsymbol{\gamma},\mathbf{p}_2';s''m''|T(E)|\mathbf{p}_1,\mathbf{p}_2;sm\rangle=\sum_n\Big\{\langle s''m''|A_n|sm\rangle T_n+\langle s''m''|A_n|sm\rangle\left[\frac{\boldsymbol{\gamma}\cdot\mathbf{q}'}{2m_1}\frac{\partial T_n}{\partial\nu}+\Delta\frac{\partial T_n}{\partial\Delta_f}\right] + (\mu/m_1)\langle s''m''|\boldsymbol{\gamma}\cdot\nabla_Q A_n+(\mathbf{q}\times\boldsymbol{\gamma})\cdot\nabla_N A_n+[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}\times\boldsymbol{\gamma})]\cdot\nabla_P A_n|sm\rangle T_n\Big\}+O(\gamma^2)$$

and (30)

$$\langle \mathbf{p}_1',\mathbf{p}_2';s'm'|T(E')|\mathbf{p}_1-\boldsymbol{\gamma},\mathbf{p}_2;s''m''\rangle=\sum_n\Big\{\langle s'm'|A_n|s''m''\rangle T_n+\langle s'm'|A_n|s''m''\rangle\left[-\frac{\boldsymbol{\gamma}\cdot\mathbf{q}}{2m_1}\frac{\partial T_n}{\partial\nu}+\Delta'\frac{\partial T_n}{\partial\Delta_i}\right] + (\mu/m_1)\langle s'm'|\boldsymbol{\gamma}\cdot\nabla_Q A_n+(\mathbf{q}'\times\boldsymbol{\gamma})\cdot\nabla_N A_n+[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}'\times\boldsymbol{\gamma})]\cdot\nabla_P A_n|s''m''\rangle T_n\Big\}+O(\gamma^2),$$

[6] E. M. Nyman, Phys. Letters **25B**, 135 (1967); Phys. Rev. **170**, 1628 (1968).

where the functions T_n and their derivatives are again evaluated at $\nu=\bar{q}^2/2\mu$, $t=(\mathbf{p}_2-\mathbf{p}_2')^2$, $\Delta_i=0$, and $\Delta_f=0$, and the operators A_n and their derivatives are evaluated at $\mathbf{L}_1=\mathbf{Q}$, $\mathbf{L}_2=\mathbf{N}$, and $\mathbf{L}_3=\mathbf{P}$.

Proceeding to the double-scattering term, when Eq. (7) is rewritten to include spin and magnetic moment, the radiation due to the magnetic moment can be dropped since it is of order γ; and since the radiation due to the charge does not change the spin state, the method of proof of Eq. (8) applies just as before, with the result that

$$\mathbf{M}^{(2)}=e_1\nabla_{p_1}\langle\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2',\,\mathbf{p}_2';s'm'|T(e)|\mathbf{p}_1,\mathbf{p}_2;sm\rangle|_{e=E}+O(\gamma). \tag{31}$$

Now the gradient operates on the functions T_n and the operators A_n, giving in place of Eq. (19)

$$\mathbf{M}^{(2)}=\frac{e_1}{m_1}\sum_n\Bigg\{\langle s'm'|A_n|sm\rangle\left[\tfrac{1}{2}(\mathbf{q}+\mathbf{q}')\frac{\partial T_n}{\partial\nu}-\mathbf{p}_1\frac{\partial T_n}{\partial\Delta_i}-\mathbf{p}_1'\frac{\partial T_n}{\partial\Delta_f}\right]$$
$$+\mu\langle s'm'|\mathbf{Q}\times\nabla_N A_n+Q^2\nabla_P A_n-\mathbf{Q}(\mathbf{Q}\cdot\nabla_P A_n)|sm\rangle T_n\Bigg\}+O(\gamma), \tag{32}$$

with T_n and A_n evaluated as described after Eq. (30).

Combining Eqs. (27), (30), and (32) gives the spin generalization of Eq. (20)

$$\mathbf{M}^{(1)}+\mathbf{M}^{(2)}=\langle s'm'|\mathbf{B}|sm\rangle,$$

$$\mathbf{B}=\frac{e_1}{m_1}\Bigg[\left(\frac{\mathbf{p}_1'}{\Delta}+\frac{\mathbf{p}_1}{\Delta'}\right)\sum_n T_n\left(\frac{\bar{q}^2}{2\mu},\,(\mathbf{p}_2'-\mathbf{p}_2)^2,0,0\right)A_n(\mathbf{Q},\mathbf{N},\mathbf{P})+\left(\tfrac{1}{2}(\mathbf{q}+\mathbf{q}')+\frac{\mathbf{p}_1'}{\Delta}\frac{\boldsymbol{\gamma}\cdot\mathbf{q}'}{2m_1}-\frac{\mathbf{p}_1}{\Delta'}\frac{\boldsymbol{\gamma}\cdot\mathbf{q}}{2m_1}\right)$$
$$\times\sum_n\frac{\partial T_n}{\partial\nu}\left(\frac{\bar{q}^2}{2\mu},\,(\mathbf{p}_2'-\mathbf{p}_2)^2,0,0\right)A_n(\mathbf{Q},\mathbf{N},\mathbf{P})\Bigg]+\sum_n T_n\left(\frac{\bar{q}^2}{2\mu},\,(\mathbf{p}_2'-\mathbf{p}_2)^2,0,0\right)\{(e_1/m_1)\mu[\![\mathbf{Q}\times\nabla_N+\mathbf{Q}^2\nabla_P$$
$$-\mathbf{Q}(\mathbf{Q}\cdot\nabla_P)+(\mathbf{p}_1'/m_1\Delta)(\boldsymbol{\gamma}\cdot\nabla_Q+(\mathbf{q}\times\boldsymbol{\gamma})\cdot\nabla_N+[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}\times\boldsymbol{\gamma})]\cdot\nabla_P)+(\mathbf{p}_1/m_1\Delta')(\boldsymbol{\gamma}\cdot\nabla_Q+(\mathbf{q}'\times\boldsymbol{\gamma})\cdot\nabla_N$$
$$+[\boldsymbol{\gamma}\times\mathbf{N}+\mathbf{Q}\times(\mathbf{q}'\times\boldsymbol{\gamma})]\cdot\nabla_P)]\!]A_n(\mathbf{Q},\mathbf{N},\mathbf{P})+i\mu_1[\boldsymbol{\gamma}\times\boldsymbol{\sigma}_1A_n(\mathbf{Q},\mathbf{N},\mathbf{P})/\Delta+A_n(\mathbf{Q},\mathbf{N},\mathbf{P})\boldsymbol{\gamma}\times\boldsymbol{\sigma}_1/\Delta']\}+O(\gamma). \tag{33}$$

Just as in the spinless case, the derivatives with respect to off-energy-shell-variables are not present, the cancellation occurring in precisely the same way. We plan to evaluate Eq. (33) from the elastic phase shifts, and compare it with an exact calculation using Eq. (27) and the spin-magnetic-moment generalization of Eq. (7), for various potentials that fit the elastic data. Since these fits are approximate, one should compute the elastic T matrices directly from the potential under consideration. This will serve as a check on the exact calculation and will also demonstrate just how the $O(\gamma)$ term enters.

With the proton-proton elastic T matrix written in the most general form consistent with parity conservation and time-reversal invariance,

$$T=T_1+T_2(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{N}+T_3(\boldsymbol{\sigma}_1\cdot\mathbf{Q})(\boldsymbol{\sigma}_2\cdot\mathbf{Q})$$
$$+T_4(\boldsymbol{\sigma}_1\cdot\mathbf{N})(\boldsymbol{\sigma}_2\cdot\mathbf{N})+T_5(\boldsymbol{\sigma}_1\cdot\mathbf{P})(\boldsymbol{\sigma}_2\cdot\mathbf{P}), \tag{34}$$

there are only four nonvanishing gradients in Eq. (33):

$$\begin{aligned}\nabla_N A_2&=\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2,\\ \nabla_Q A_3&=(\boldsymbol{\sigma}_1\cdot\mathbf{Q})\boldsymbol{\sigma}_2+(\boldsymbol{\sigma}_2\cdot\mathbf{Q})\boldsymbol{\sigma}_1,\\ \nabla_N A_4&=(\boldsymbol{\sigma}_1\cdot\mathbf{N})\boldsymbol{\sigma}_2+(\boldsymbol{\sigma}_2\cdot\mathbf{N})\boldsymbol{\sigma}_1,\\ \nabla_P A_5&=(\boldsymbol{\sigma}_1\cdot\mathbf{P})\boldsymbol{\sigma}_2+(\boldsymbol{\sigma}_2\cdot\mathbf{P})\boldsymbol{\sigma}_1.\end{aligned} \tag{35}$$

IV. DISCUSSION

If the particle momenta are very small, then the T matrix is predominantly s wave. If it were purely s wave, the double-scattering term would contribute nothing in the c.m. system to bremsstrahlung because the only direction in the (spinless) problem is the photon's momentum, and $\boldsymbol{\varepsilon}\cdot\boldsymbol{\gamma}=0$. This is the well-known statement that $J=0\to0$ is forbidden. Under these conditions $\mathbf{M}^{(2)}$ would be of order γ (and in the direction $\boldsymbol{\gamma}$). How, then, could the cancellation of off-shell derivatives occur in the term of order γ^0 if $\mathbf{M}^{(2)}$ does not have such a term? From Eq. (19) it would appear that perhaps $\partial T/\partial\Delta_i$, $\partial T/\partial\Delta_f$, and $\partial T/\partial\nu$ are not separately zero, but that instead

$$\frac{1}{2}\frac{\partial T}{\partial\nu}-\frac{\partial T}{\partial\Delta_i}\quad\text{and}\quad\frac{1}{2}\frac{\partial T}{\partial\nu}-\frac{\partial T}{\partial\Delta_f}$$

vanish, thereby making the term of order γ^0 vanish. (Recall that in the c.m. system, $\mathbf{q}=\mathbf{p}_1$ and $\mathbf{q}'=\mathbf{p}_1'+\mu\boldsymbol{\gamma}/m_2$.) While these relations among the derivatives of T are certainly not exact, there is some approximate validity to them, as will now be shown.

Although the p-wave contribution to the T-matrix *for a local potential* is only of order $\bar{q}^2$ (and the s-wave of order unity) at low energies, they make comparable contributions to $\mathbf{M}^{(2)}$! This can be seen from an examination of the half-off-shell T matrices

$$T_R(q_i,q_f,\cos\theta)\equiv T(\nu,t,0,\Delta_f)=4\pi\sum_l(2l+1)P_l(z)\times\int_0^\infty\frac{F_l(q_fr)}{q_f}V(r)\frac{U_l(q_i,r)}{q_i}dr\,,$$
$$T_L(q_i,q_f,\cos\theta)\equiv T(\nu,t,\Delta_i,0)=4\pi\sum_l(2l+1)P_l(z)\times\int_0^\infty\frac{U_l(q_f,r)}{q_f}V(r)\frac{F_l(q_ir)}{q_i}dr\,,\qquad(36)$$

where

$$z=\cos\theta\equiv\hat{q}_i\cdot\hat{q}_f\,,\quad F_l(x)\equiv xj_l(x)\,,\quad H_l^{(+)}(x)\equiv ixh_l^{(1)}(x)\,,$$

and U_l is the regular solution of the partial-wave Schrödinger equation with the asymptotic behavior

$$U_l\to F_l(kr)+e^{i\delta_l(k)}\sin\delta_l(k)H_l^{(+)}(kr)\,.$$

Expanding in powers of the momentum,[7]

$$\begin{aligned}
U_0(k,r)/k&=v_0^{(0)}(r)+kv_0^{(1)}(r)+k^2v_0^{(2)}(r)+O(k^3)\,,\\
U_1(k,r)/k&=kv_1^{(1)}(r)+O(k^3)\,,\\
U_2(k,r)/k&=O(k^2)\,,\\
F_0(kr)/k&=r-\tfrac{1}{6}k^2r^3+O(k^4)\,,\\
F_1(kr)/k&=\tfrac{1}{3}kr^2+O(k^3)\,,\\
F_2(kr)/k&=O(k^2)\,,
\end{aligned}\qquad(37)$$

which gives

$$\begin{aligned}
T_R&=A+\alpha q_i+\tfrac{1}{2}\beta q_i^2+\tfrac{1}{2}\beta' q_f^2+\lambda q_iq_f\cos\theta+O(q^3),\\
\partial T_R/\partial q_i&=\alpha+\beta\bar{q}+\lambda\bar{q}\cos\theta+O(\bar{q}^2)\,,\\
\partial T_R/\partial q_f&=\beta'\bar{q}+\lambda\bar{q}\cos\theta+O(\bar{q}^2)\,,\\
\partial T_R/\partial\cos\theta&=\lambda\bar{q}^2+O(\bar{q}^4)\,,
\end{aligned}\qquad(38)$$

with all derivatives evaluated at $q_i=q_f=\bar{q}$. The coefficient λ comes from the p state.

To obtain the derivatives of T with respect to ν and Δ, invert Eq. (12),

$$q_i^2=\mu(2\nu+\Delta_f)\,,\quad q_f^2=\mu(2\nu-\Delta_f)\,,$$
$$\cos\theta=(2\nu-t/2\mu)/(4\nu^2-\Delta_f^2)^{1/2}\,,$$

and use Eq. (38); the result is

$$\frac{\partial T(\nu,t,0,0)}{\partial\nu}=\mu\left[\frac{\alpha}{\bar{q}}+(\beta+\beta'+2\lambda)\right]+O(\bar{q})$$

and

$$\frac{\partial T(\nu,t,0,0)}{\partial\Delta_f}=\tfrac{1}{2}\mu\left[\frac{\alpha}{\bar{q}}+(\beta-\beta')\right]+O(\bar{q})\,.\qquad(39)$$

[7] These properties of the functions U_l can be obtained from an exact expression for the Jost function such as Eq. (5-12) in R. G. Newton, *The Complex j-Plane* (W. A. Benjamin, Inc., New York, 1964). They follow directly from the expansions of $F_l(x)$ and $H_l^{(+)}(x)$ in powers of x.

Using the symmetry from Eq. (36),

$$T_R(u,v,x)=T_L(v,u,x)\,,$$

one finds that

$$\partial T(\nu,t,0,0)/\partial\Delta_i=\partial T(\nu,t,0,0)/\partial\Delta_f\,.\qquad(40)$$

Observe that the p-wave contribution to $\partial T/\partial\nu$ is only one power of $\bar{q}$ weaker than the s-wave at low energies. Now form the combination that occurs in $\mathbf{M}^{(2)}$ (in the c.m. system)

$$\tfrac{1}{2}\partial T/\partial\nu-\partial T/\partial\Delta=\mu(\beta'+\lambda)+O(\bar{q})\,;\qquad(41)$$

the p- and s-wave contributions are of the same order in $\bar{q}$. The resolution of the apparent paradox stated at the beginning of this section resides in the fact that there is a relation between the on- and off-shell derivatives of T to the leading order in $\bar{q}$. The only consistent way to make the approximation of only s-wave scattering is to neglect *all* the terms in T that are quadratic in $\bar{q}$, and then $\mathbf{M}^{(2)}$ would indeed be zero. The more accurate calculation given above shows that $\partial T/\partial\nu$ and $\partial T/\partial\Delta$ are both of order $\bar{q}^{-1}$, but the combination that occurs in $\mathbf{M}^{(2)}$ is order $\bar{q}^0$.

The coefficient α in Eq. (39) can be obtained from the s-wave scattering length a, since the elastic T matrix can be written

$$\begin{aligned}
T&=-\frac{2\pi}{\mu}\frac{e^{i\delta_0}\sin\delta_0}{\bar{q}}+O(\bar{q}^2)\\
&=-\frac{2\pi}{\mu}\frac{1}{-a^{-1}-i\bar{q}}+O(\bar{q}^2)
\end{aligned}\qquad(42)$$

and

$$\partial T/\partial\bar{E}_{\text{c.m.}}=-2\pi ia^2/\bar{q}+O(1)\,,$$

with $\bar{E}_{\text{c.m.}}=\bar{q}^2/2\mu$. Returning to Eq. (22), the ratio of the second term to the first term is

$$\left|\frac{\gamma}{T}\frac{\partial T}{\partial\bar{E}_{\text{c.m.}}}\right|=\frac{\gamma\mu a}{\bar{q}}<\bar{q}a\,.\qquad(43)$$

At sufficiently low particle energies, the second term is negligible compared with the first term, no matter what the photon energy is.

If a portion of the potential V is not simply a function of $\mathbf{r}$, as is true in the nucleon-nucleon system where there is momentum dependence, charge exchange, and possibly explicit nonlocality, then the derivation used in this paper does not work. First of all, additional electromagnetic interaction must be introduced to make the Schrödinger equation gauge-invariant. This has the effect of destroying the identity of single scatters ing with external emission where single scattering mean- (as it did before) to pick out of the bremsstrahlung matrix element

$$^{(-)}\langle\mathbf{p}_1',\mathbf{p}_2'|H_{\text{em}}|\mathbf{p}_1,\mathbf{p}_2\rangle^{(+)}$$

the free-wave part of one of the states and the scattered-wave part of the other state. In addition, there is now a zero scattering contribution $\mathbf{M}^{(0)}$ where one takes the free-wave part of both states. External emission still has the form of the right-hand side of Eqs. (6) or (27), and all the new bits of radiation are of the internal-emission type. According to the general result of Ref. 2, we expect that our Eqs. (20) and (21) or (33) will still be valid for the total bremsstrahlung matrix element.

The main result of this work—that bremsstrahlung calculated from the Schrödinger equation satisfies the Low theorem—is contained in a paper by Feshbach and Yennie,[8] although it is not obvious from their Eq. (21). That equation contains gradients of half-off-shell T matrices with respect to the relative momentum of the off-shell state, and individually these gradients depend upon the off-shell behavior of the T matrices. A direct way to demonstrate that these off-shell properties cancel out of the equation is to use the parametrization of the present paper.

While the gradient of T_R with respect to q_f is expected to be smaller than the gradient with respect to q_i, the latter being proportional to Γ^{-1}, where Γ is the width of a nearby resonance, $\nabla_{q_f} T_R$ may still be large in absolute value. The rough qualitative argument which says that it should be simply the range of the force times T_R can be incorrect. For example, near a narrow resonance of a δ-function potential, $\nabla_{q_f} T_R \propto \Gamma^{-1/2}$.

The point about which Feshbach and Yennie[8] expand the T matrix is probably the best one insofar as the evaluation of the γ^{-1} term in the bremsstrahlung matrix element is concerned; but, as their discussion indicates, the contribution of the internal emission to the γ^0 term, as obtained from a general formula that neglects terms $O(\gamma)$, can be unreliable near a narrow resonance.[9] The Low theorem is probably not useful under these circumstances.

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[8] H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).

[9] See also S. Barshay and T. Yao, Phys. Rev. **171**, 1708 (1968).