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Existence of the Trineutron

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The possibility of the existence of a trineutron bound state has been investigated by using a separable N-N interaction in the p wave. Two different sets of central and spin-orbit forces of the $(\mathbf{L}\cdot\mathbf{S})$ and $(\mathbf{L}\cdot\mathbf{S})^2$ types have been used to obtain a detailed fit to the ${}^3P_J$ scattering phase shifts. The calculations have been performed for the $J^P=\frac{1}{2}^+$ state, which was shown by Mitra and Bhasin to be the best possible candidate for such a bound state. The solution of the coupled one-dimensional integral equations, as an eigenvalue problem in the "inverse strength parameter" of the n-n interaction, shows that the requisite eigenvalue for zero binding energy for the n^3 system has a sufficiently wide margin over the corresponding quantity obtained from the p-p phase shifts. This confirms the earlier conclusion of Mitra and Bhasin based on effective central p-wave forces, but is contradictory to the results of variational approaches with conventional potentials given by Okamoto and Davies, and by Barbi. The possible significance of the difference is discussed.

I. INTRODUCTION

THE possibility of the existence of a trineutron bound state has engaged the attention of many physicists for quite some time. There have been conflicting claims regarding the existence of n^3, both on experimental and theoretical grounds. Ajdacić *et al.*[1] surmised the existence of n^3 with a binding energy of about 1 MeV from the study of the reaction $\mathrm{H}^3(n,p)3n$. Kim *et al.*[2] have studied the mirror reaction $\mathrm{He}^3(p,n)3p$ with similar conclusions. On the other hand, the experiments of Thornton *et al.*[3] and Debertin and Rossel[4] on $\mathrm{H}^3(n,p)3n$, and of Anderson *et al.*[5] and Cookson[6] on $\mathrm{He}^3(p,n)3p$, do not indicate any evidence in favor of the n^3 or p^3 bound state, respectively. However, because of the paucity of data on such reactions, no positive conclusion can at present be reached, thus keeping the question of trineutron bound state still open from an experimental point of view.[7]

The situation on the theoretical side seems to be equally fluid. Okamoto and Davies,[8] using the Pease-Feshbach potential, found the n^3 state to be unbound by about 10 MeV in a standard variational calculation. However, as pointed out by Barbi,[9] an extrapolation of the Pease-Feshbach potential to p states is not justified. So, in the absence of a more detailed p-wave variational wave function, an approach free from the uncertainties of trial wave function is very much called for. Barbi,[9] on the other hand, attacked this problem, using the techniques of direct numerical solution of Euler-Lagrange variational equations, but reached the same conclusions about the n^3 bound state as Okamoto and Davies. More explicitly, he found that in the ${}^4P_{1/2}$ state the potential depth required for an n^3 bound state is less than that for n^2, but is nevertheless far greater than what is required for a fit to Bryan and Scott's[10] ${}^3P_0$ phase-shift data. Further, he found that the inclusion of the hard core improves the prospects of an n^3 bound state, though only slightly.

Earlier, Mitra and Bhasin[11] had investigated this problem with separable interactions and concluded that an n^3 bound state was very much within the realm of possibility. With only an s-wave potential of Yamaguchi type, the kernel of the (single) integral equation is repulsive, thus showing the importance of p-wave forces for this system. Their analysis was carried out with (i) a central p-wave interaction, in which case there is only one integral equation and (ii) both s- and p-wave interactions, which yield two-coupled equations. The strength parameter of the p-wave potential for the case of zero binding was computed and found to give a rough agreement with the ${}^3P_0$ N-N phase shifts of Bryan and Scott.[10] The effect of the s-wave force in the presence of the p-wave force was, however, found to be almost negligible ($\sim 0.3\%$).[12]

As the question of the n^3 bound state still seems to be open, we feel that a more realistic analysis of this problem is in order. In this regard the analyses of previous authors[8,9,11] all suffer from the defect that only

[1] V. Ajdačić, M. Cerineo, B. Lalović, G. Paić, I. Šlaus, and P. Tomaš, Phys. Rev. Letters **14**, 444 (1965).
[2] C. C. Kim, S. M. Bunch, D. W. Devins, and H. H. Forster, Phys. Letters **22**, 314 (1966).
[3] S. T. Thornton, J. K. Bair, C. M. Jones, and H. B. Willard, Phys. Rev. Letters **17**, 701 (1966).
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[5] J. D. Anderson, G. Wong, J. W. McClure, and B. A. Pohl, Phys. Rev. Letters **15**, 66 (1965).
[6] J. A. Cookson, Phys. Letters **22**, 612 (1966).
[7] B. Antolković, M. Cerineo, G. Paić, P. Tomaš, V. Ajdačić, B. Lalović, W. T. H. Van Oers, and I. Šlaus, Phys. Letters **23**, 477 (1966).
[8] K. Okamoto and B. Davies, Phys. Letters **24B**, 18 (1967).
[9] M. Barbi, Nucl. Phys. **A99**, 522 (1967).
[10] Ronald A. Bryan and Bruce Scott, Phys. Rev. **135**, B434 (1964).
[11] A. N. Mitra and V. S. Bhasin, Phys. Rev. Letters **16**, 523 (1966). Referred to as MB in what follows.
[12] This is analogous to a similar situation in Heitler-London Theory of hydrogen molecule. See e.g., John C. Slater, *Quantum Theory of Molecules and Solids* (McGraw-Hill Book Co., New York, 1963), Chap. 3.

central interactions have been considered. As is clear from various analyses,[10,13] there is considerable splitting in the ^3P phase shifts for various J states. Thus a proper representation of these phase shifts will require an appreciable contribution from the noncentral terms, such as $\mathbf{L}\cdot\mathbf{S}$ and/or tensor forces.

We have extended the earlier work of Mitra[14] and of Mitra and Bhasin[11] on the three-neutron bound state, using more realistic potentials, where the full effect of $\mathbf{L}\cdot\mathbf{S}$ and $(\mathbf{L}\cdot\mathbf{S})^2$ forces has been incorporated to obtain a reasonably good fit to the phenomenological ^3P_J phase shifts. An exhaustive p-wave formalism for a three-body system, which is especially tuned to an analysis of n^3, has been given in MP3N. It is found there that of all the states, of positive as well as negative parity, the best chance for binding n^3 is provided by one of the states $(1,\frac{3}{2},\frac{1}{2})^+$, $(1,\frac{3}{2},\frac{3}{2})^+$, and $(1,\frac{3}{2},\frac{5}{2})^+$ [in the representation—$(L,S,J)^P$], which are all degenerate in the absence of noncentral forces. Also, if the sign of the spin-orbit force be so adjusted as to yield a larger phase shift in $^3P_0$ than in $^3P_2$, then the state $(1,\frac{3}{2},\frac{1}{2})^+$ is somewhat more attractive than the other two, and hence the most favorable candidate for observation.

In view of these earlier results we have restricted our analysis to only the $J^P=\frac{1}{2}^+$ case. In this case the problem essentially reduces to a solution of 12 coupled (one-dimensional) integral equations in terms of the well-known "spectator functions." A complete numerical analysis of this problem is, of course, out of the question at present. However, the problem can be somewhat simplified by an examination of certain features of the potential that emerge from a numerical fit to the p-p ^3P_J phase shifts. For example, the range parameters for the central and $(\mathbf{L}\cdot\mathbf{S})^2$ forces are found to be nearly equal (within about 4%). As a first approximation, one can assume them to be *identical*, which reduces the number of integral equations to *eight*. Further approximations based on the attractive and repulsive character of the various kernels reduce the problem to merely *three* coupled equations. This approximation, which is indeed very rough, can be partly justified, since it is known from the work of MB that repulsive kernels depress the eigenvalue only by a small amount. Further, since we are *not* interested in finding the exact binding energy of n^3, but only the likelihood or otherwise of its existence, we expect that such an approximation is not going to change the qualitative nature of our results.

In Sec. II, we obtain an expression for the ^3P_J phase shifts with the complete potential, comprising the central, $(\mathbf{L}\cdot\mathbf{S})$, and $(\mathbf{L}\cdot\mathbf{S})^2$ parts. Two sets of parameters, both fitting ^3P_J phase shifts reasonably well up to about 250 MeV, are determined. Certain features of these two sets are discussed. In Sec. III, the *exact* coupled integral equations for the three-body system are derived for the state $J^P=\frac{1}{2}^+$. The reduced form of these equations under certain approximations is also given. Section IV contains the numerical results of the present analysis as well as a comparison with the work of other authors.

II. TWO-NEUTRON p-WAVE INTERACTION

In this section, we shall derive the ^3P_J phase shifts using a separable p-wave interaction, central as well as noncentral, following closely the formalism of MP3N. The purely central factorable p-wave interaction between two neutrons is of the form

$$-M\langle \mathbf{p}|V_C|\mathbf{p}'\rangle = 3\lambda P_\sigma^+ P_\tau^+ u_1(p)u_1(p')(\mathbf{p}\cdot\mathbf{p}')\,, \quad (2.1)$$

where P_σ^+ and P_τ^+ are the triplet projection operators in spin- and isospin-space, respectively, and λ is the strength parameter. The operator P_τ^+ may be dropped since it leads to a symmetric isospin function, which can therefore be absorbed in the total wave function without affecting the internal dynamics. For any two neutrons i and j, the spin projection operator $P_\sigma^+(ij)$ is related to the corresponding permutation operator[15] $(ij)_\sigma$ through

$$P_\sigma^+(ij) = \tfrac{1}{2}[1+(ij)_\sigma]. \quad (2.2)$$

The p-wave spin-orbit force between two neutrons i and j can be represented in a factorable form as

$$-M\langle \mathbf{p}|V_{LS}|\mathbf{p}'\rangle = 3\lambda(2\mathbf{L}\cdot\mathbf{S})u_2(p)u_2(p')(\mathbf{p}\cdot\mathbf{p}')\,, \quad (2.3\text{a})$$

or equivalently as[16]

$$-M\langle \mathbf{p}|V_{LS}|\mathbf{p}'\rangle = 3i\lambda(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}^i+\boldsymbol{\sigma}^j)u_2(p)u_2(p')\,. \quad (2.3\text{b})$$

Also, the p-wave quadratic spin-orbit force has the structure

$$-M\langle \mathbf{p}|V_{(LS)^2}|\mathbf{p}'\rangle = 3\lambda(2\mathbf{L}\cdot\mathbf{S})^2 u_3(p)u_3(p')(\mathbf{p}\cdot\mathbf{p}') \quad (2.4\text{a})$$

or

$$-M\langle \mathbf{p}|V_{(LS)^2}|\mathbf{p}'\rangle = 3\lambda\{4(\mathbf{p}\cdot\mathbf{p}')+i(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}^i+\boldsymbol{\sigma}^j)+2(\boldsymbol{\sigma}^i\cdot\boldsymbol{\sigma}^j)(\mathbf{p}\cdot\mathbf{p}')-(\boldsymbol{\sigma}^i\cdot\mathbf{p})(\boldsymbol{\sigma}^j\cdot\mathbf{p}')-(\boldsymbol{\sigma}^i\cdot\mathbf{p}')(\boldsymbol{\sigma}^j\cdot\mathbf{p})\}u_3(p)u_3(p')\,, \quad (2.4\text{b})$$

where

$$\mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}^i+\boldsymbol{\sigma}^j)\,, \quad (2.5)$$

$$\mathbf{J} = \mathbf{L}+\mathbf{S}\,. \quad (2.6)$$

The ^3P_J phase shifts are then easily worked out through

[13] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966); Ryozo Tamagaki and Wataro Watari, Progr. Theoret. Phys. (Kyoto) Suppl. **39**, 23 (1967).

[14] A. N. Mitra, Phys. Rev. **150**, 839 (1966). Referred to as MP3N in what follows.

[15] M. Verde, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 170.

[16] A. N. Mitra and V. L. Narasinham, Nucl. Phys. **14**, 407 (1959–1960).

the two-body Schrödinger equation

$$(p^2-k^2)\psi(p)=-M\int d\mathbf{p}'\langle\mathbf{p}|V|\mathbf{p}'\rangle\psi(p')\,, \quad (2.7)$$

$$V=V_C+V_{LS}+V_{(LS)^2}\,, \quad (2.8)$$

and the boundary condition

$$\psi(p)=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})+4\pi f_k(p)3(\mathbf{p}\cdot\mathbf{k})\times(p^2-k^2-i\epsilon)^{-1}\,, \quad (2.9)$$

where V_C, V_{LS}, and $V_{(LS)^2}$ are given by Eqs. (2.1), (2.3a), and (2.4a), respectively, and $\mathbf{k}$ is the momentum in the c.m. frame. The general structure for the off-diagonal scattering amplitude $f_k(p)$ can be read off from Eqs. (2.7) and (2.9), and is

$$f_k(p)=A(k)u_1(p)+(2\mathbf{L}\cdot\mathbf{S})B(k)u_2(p)+(2\mathbf{L}\cdot\mathbf{S})^2C(k)u_3(p)\,. \quad (2.10)$$

Now, in a two-body analysis we can replace the operators $(2\mathbf{L}\cdot\mathbf{S})$ and $(2\mathbf{L}\cdot\mathbf{S})^2$ by their appropriate eigenvalues, Γ and Γ^2, respectively. The ^3P_J phase shifts can then be expressed in the form

$$\mathrm{Re}f^{-1}(k)=\mathfrak{D}_R/\mathfrak{N}_R=k^3\cot\delta_J\,, \quad (2.11)$$

where

$$f(k)=f_k(p)|_{\mathbf{p}=\mathbf{k}}\,. \quad (2.12)$$

$\mathfrak{D}_R$ and $\mathfrak{N}_R$ are defined in terms of certain integrals I_α and $I_{\alpha\beta}$ $(\alpha,\beta=1,2,3)$ and are given in Appendix A along with their evaluation for the following forms[17] of u_α:

$$u_1(k)=ck^2(k^2+\beta_1^2)^{-2}\,, \quad (2.13)$$

$$u_2(k)=(k^2+\beta_2^2)^{-1}\,, \quad (2.14)$$

$$u_3(k)=-bk^2(k^2+\beta_3^2)^{-2}\,, \quad (2.15)$$

where β_α are the inverse range parameters and (c,b) give the relative strengths of central and quadratic spin-orbit forces, respectively.

TABLE I. Two sets of potential parameters as found from a comparison with ^3P_J phase shifts in the $S=1$, $T=1$ state. β_1, β_2, β_3, and λ are all expressed in units of the pion mass m_π.

Set	λ	c	b	β_1	β_2	β_3
I	0.06	4.71	0.71	4.4	1.7	4.5
II	0.025	7.87	1.22	5.5	1.3	5.6

Table I gives the two sets of parameters which have been found to yield a reasonably good fit to the ^3P_J scattering phase shifts for the $T=1$ and $S=1$ state with the above potential forms. The phase shifts, as found with these sets, are given in Table II, along with those of Bryan and Scott,[10] and Tamagaki and Watari[13] for comparison.

First of all, we notice that the three potentials—central, $\mathbf{L}\cdot\mathbf{S}$, and $(\mathbf{L}\cdot\mathbf{S})^2$—are of different forms. Whereas u_2 $[(\mathbf{L}\cdot\mathbf{S})]$ does not vanish at $k^2=0$, u_1 and u_3 [central and $(\mathbf{L}\cdot\mathbf{S})^2$, respectively] do. These forms for the potentials were found to be quite unique, i.e., no good fit to the phase shifts could be obtained by either taking *all* the potentials of the form (2.13) or by any interchange in the forms of various potentials.[18] Also, the $(\mathbf{L}\cdot\mathbf{S})^2$ term has been found to be repulsive. As can be seen from the figures given in Table I, the $(\mathbf{L}\cdot\mathbf{S})^2$ force is small compared to the central part of the interaction. Though the relative strength of the $(\mathbf{L}\cdot\mathbf{S})$ force is also small, its contribution is appreciable since its corresponding potential form factor u_2 is dominant at low energies as it does not vanish at $k^2=0$. Also, the ranges of the central and $(\mathbf{L}\cdot\mathbf{S})^2$ forces are nearly equal, which is a very important feature, since it helps to bring down the number of integral equations from 12 to eight, as we shall see in Sec. III. In both sets, the $(\mathbf{L}\cdot\mathbf{S})$ force is found to have a

TABLE II. The ^3P_J nucleon-nucleon phase shifts as found with the two sets of potential parameters listed in Table I. The experimental values based on phenomenological analyses of Bryan and Scott (BS) and of Tamagaki and Watari (TW) are also given for comparison.

$E_{\mathrm{Lab.}}$	δ_0				δ_1				δ_2			
(MeV)	BS	TW	Set I	Set II	BS	TW	Set I	Set II	BS	TW	Set I	Set II
10	3.16	4.56	2.57	2.85	−1.79	−2.37	−2.17	−2.00	0.54	0.74	1.78	2.29
20	6.30	8.47	5.90	6.06	−3.62	−4.49	−5.03	−4.30	1.55	1.98	4.22	4.96
30	8.47	···	8.68	8.45	−5.14	···	−7.76	−6.25	2.72	···	6.46	7.06
40	9.78	12.19	10.65	10.01	−6.44	−7.52	−10.22	−7.84	3.95	4.73	8.29	8.57
60	10.73	12.79	12.48	11.36	−8.65	−9.79	−14.30	−10.15	6.35	7.32	10.68	10.27
80	10.29	11.96	12.39	11.27	−10.56	−11.70	−17.34	−11.62	8.49	9.57	11.78	10.92
100	9.11	10.43	11.19	10.41	−12.29	−13.41	−19.49	−12.50	10.31	11.45	12.05	11.00
120	7.51	8.57	9.38	9.09	−13.92	−15.0	−20.90	−12.97	11.81	12.97	11.82	10.78
160	3.73	4.42	4.79	5.64	−16.95	−17.96	−21.85	−13.08	14.02	15.08	10.61	9.93
200	−0.31	0.16	−0.37	1.51	−19.78	−20.72	−20.85	−12.48	15.44	16.23	9.00	8.89
240	−4.35	−4.02	−5.79	−3.15	−22.44	−23.35	−18.46	−11.47	16.31	16.66	7.34	7.83

[17] We shall use, in this paper, the natural units $\hbar=c=m_\pi=1$.

[18] At least one of the potential form factors u_α has to be of the form $(k^2+\beta^2)^{-1}$ to yield the correct threshold behavior of the scattering amplitude.

much *larger* range than the purely central part of the potential, contrary to the usual idea of short-range nature of the spin-orbit force.[16] But this result is in agreement with that of Tamagaki and Watari,[13] who also conclude that the spin-orbit force should have a larger range than those proposed by phenomenological analyses.

The general features of the two sets of potentials are quite similar. However, for set I, both the central and the $(\mathbf{L}\cdot\mathbf{S})^2$ forces have somewhat smaller strength but larger range than for set II, whereas the behavior of the $(\mathbf{L}\cdot\mathbf{S})$ term is just the opposite.

III. THREE-BODY FORMALISM

In this section we shall first derive the *exact* integral equations for the n^3 system with a binding energy α_T. Once again we follow closely the formalism of MP3N. The Schrödinger equation for a three-body system, in the over-all c.m. frame

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0\,, \tag{3.1}$$

becomes

$$D_E(\mathbf{p}_{ij},\mathbf{P}_k)\Psi=-M\sum_{i<j<k=1}^{3}\int d\mathbf{p}_{ij}'\langle\mathbf{p}_{ij}|V_{ij}|\mathbf{p}_{ij}'\rangle \times\Psi(\mathbf{p}_{ij}',\mathbf{P}_k)\,, \tag{3.2}$$

where

$$\mathbf{P}_k=-(\mathbf{P}_i+\mathbf{P}_j)\,,\quad 2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j;\quad i,\ j,\ k\ \text{cyclic}\,, \tag{3.3}$$

$$D_E(\mathbf{p}_{ij},\mathbf{P}_k)=p_{ij}^2+\tfrac{3}{4}P_k^2-EM\,, \tag{3.4}$$

and for a bound state

$$-EM=\alpha_T^2. \tag{3.5}$$

Writing Eq. (3.2) in full, we obtain

$$\begin{aligned}D_E(\mathbf{p}_{ij},\mathbf{P}_k)\Psi &=3\lambda\sum_{i<j<k=1}^{3}\int d\mathbf{p}_{ij}'[(\mathbf{p}_{ij}\cdot\mathbf{p}_{ij}')P_\sigma^{+}(ij)u_1(p_{ij})u_1(p_{ij}')\\&+i(\mathbf{p}_{ij}\times\mathbf{p}_{ij}')\cdot(\boldsymbol{\sigma}^i+\boldsymbol{\sigma}^j)u_2(p_{ij})u_2(p_{ij}')\\&+\{4(\mathbf{p}_{ij}\cdot\mathbf{p}_{ij}')+i(\mathbf{p}_{ij}\times\mathbf{p}_{ij}')\cdot(\boldsymbol{\sigma}^i+\boldsymbol{\sigma}^j)\\&+2(\mathbf{p}_{ij}\cdot\mathbf{p}_{ij}')(\boldsymbol{\sigma}^i\cdot\boldsymbol{\sigma}^j)-(\boldsymbol{\sigma}^i\cdot\mathbf{p}_{ij})(\boldsymbol{\sigma}^j\cdot\mathbf{p}_{ij}')\\&-(\boldsymbol{\sigma}^i\cdot\mathbf{p}_{ij}')(\boldsymbol{\sigma}^j\cdot\mathbf{p}_{ij})\}u_3(p_{ij})u_3(p_{ij}')]\Psi(\mathbf{p}_{ij}',\mathbf{P}_k)\,.\end{aligned} \tag{3.6}$$

As discussed in the Introduction, we shall restrict our analysis to the state $J^P=\frac{1}{2}^+$, as it affords the best chance of binding n^3. The wave functions for various $(LSJ)^P$ states for this system have been listed in Table I of MP3N. The properly normalized, totally antisymmetric wave function for the state $J^P=\frac{1}{2}^+$ may be written as

$$\Psi=(2\sqrt{2})^{-1}\{(\psi'\chi''-\psi''\chi')+(\psi_\mu'\chi_\mu''-\psi_\mu''\chi_\mu')\}+\tfrac{1}{2}(\psi_\mu^a\chi_\mu^s+\psi_{\mu\nu}^a\chi_{\mu\nu}^s)\,. \tag{3.7}$$

Here (ψ',ψ'') and (ψ_μ',ψ_μ'') are spatial wave functions of mixed symmetry for the cases $L=0$ and 1, respectively; ψ_μ^a and $\psi_{\mu\nu}^a$ are the spatial antisymmetric wave functions for $L=1$ and 2, respectively. The various scalar, vector, and tensor spin functions of different symmetries have been defined in Eqs. (3.1)–(3.6) of MP3N.

The three possible even-parity states of $L=0$, 1, and 2, corresponding to scalar, vector, and tensor products of the pair of available vectors $(\mathbf{p}_{ij},\mathbf{P}_k)$ for a p-wave pairwise interaction, are

$$\mathbf{p}_{ij}\cdot\mathbf{P}_k\,, \tag{3.8}$$

$$Q_\mu^a=(\mathbf{p}_{ij}\times\mathbf{P}_k)_\mu\,, \tag{3.9}$$

and the traceless tensorial product[19]

$$Q_{\mu\nu}(k,ij)=\tfrac{1}{2}\{p_{ij\mu}P_{k\nu}+p_{ij\nu}P_{k\mu}-\tfrac{2}{3}(\mathbf{p}_{ij}\cdot\mathbf{P}_k)\delta_{\mu\nu}\}\,. \tag{3.10}$$

As usual, in a separable interaction model one can express scalar, vector, and tensor functions in terms of the above quantities, viz., Eqs. (3.8)–(3.10), and the well-known "spectator functions."[20] Denoting the spectator functions by E, F, and G for the cases of $L=0$, 1, and 2, respectively, the wave functions have the following structure (with correct symmetry properties incorporated):

$$\begin{bmatrix}\psi' & \psi_\mu' & \psi_{\mu\nu}'\\ \psi'' & \psi_\mu'' & \psi_{\mu\nu}''\\ \psi^a & \psi_\mu^a & \psi_{\mu\nu}^a\end{bmatrix}=D_E^{-1}\begin{bmatrix}S' & A_\mu' & T_{\mu\nu}'\\ S'' & A_\mu'' & T_{\mu\nu}''\\ S^a & A_\mu^a & T_{\mu\nu}^a\end{bmatrix}\,, \tag{3.11}$$

where, for example, the scalar form factors are

$$S=S_1+S_2+S_3\,, \tag{3.12}$$

$$S'=-S_1+\tfrac{1}{2}(S_2+S_3)\,, \tag{3.13}$$

$$S''=\tfrac{1}{2}\sqrt{3}(S_2-S_3)\,, \tag{3.14}$$

$$S_k=(\mathbf{p}_{ij}\cdot\mathbf{P}_k)\sum_{\alpha=1}^{3}E_\alpha^m(P_k)u_\alpha(p_{ij})\,. \tag{3.15}$$

In Eq. (3.15), we have for a scalar function three form factors corresponding to the three different interaction terms. The superscripts a and m in Eqs. (3.11)–(3.15) denote antisymmetric and mixed symmetric combinations, respectively.

Similar definitions hold for the corresponding "form factors" A_μ and $T_{\mu\nu}$ in terms of the quantities $A_{k;\mu}$ and $T_{k;\mu\nu}$, respectively, where

$$A_{k;\mu}=Q_\mu^a\sum_{\alpha=1}^{3}F_\alpha^{a,m}(P_k)u_\alpha(p_{ij})\,, \tag{3.16}$$

$$T_{k;\mu\nu}=Q_{\mu\nu}(k,ij)\sum_{\alpha=1}^{3}G_\alpha^a(P_k)u_\alpha(p_{ij})\,. \tag{3.17}$$

[19] Throughout the paper the Latin indices i, j, and k will be used to label the particles and the Greek indices μ, ν, and λ for the (three-dimensional) tensorial character. α and β will always refer to the three different potentials and their corresponding spectator functions, etc.

[20] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).

This gives in all 12 different spectator functions, $E_\alpha{}^m$, $F_\alpha{}^{a,m}$, and $G_\alpha{}^a$. Substitution of Eqs. (3.11)–(3.17) in Eq. (3.6) leads in a standard manner[14,20] to the desired (one-dimensional) 12 coupled integral equations, the coupling to various L states coming from the noncentral interactions. These equations are given in Appendix B. Some of the more important spin relations employed in the derivation are listed in Appendix C.

A numerical solution of these 12 coupled integral equations at this stage is certainly not possible. Therefore, certain approximations have to be used to make the problem tractable. In this context, it can be seen from the numerical results presented in Sec. II, that the range of the central and $(\mathbf{L}\cdot\mathbf{S})^2$ forces is nearly the same, so that from Eqs. (2.13) and (2.15)

$$u_3(k)\approx(b/c)u_1(k)=\gamma u_1(k). \tag{3.18}$$

With this substitution in the integral equations, the number of *independent* spectator functions reduces to only *eight*, the relevant transformation being

$$X_1+\gamma X_3\rightarrow X_1, \tag{3.19}$$

where X represents any one of the spectator functions E^m, $F^{m,a}$, and G^a. These eight equations can be read off easily from the 12 equations given in Appendix B by simply dropping the kernels and spectator functions referring to index 3 and α running over 1, 2, only.

Next we make some assumptions based upon the results of numerical calculations performed in MB, viz., that the coupling to repulsive kernels depresses the eigenvalue only by a very small amount. Now, as can be seen from Appendix B, the equation for $E_2{}^m$ ($L=0$) has a *purely* negative kernel and is coupled only to states of $L=1$, 2. Also, equations for $F_1{}^m$, $F_2{}^m$, $G_1{}^a$, and $G_2{}^a$ have purely negative kernels. Thus if we neglect the effect of these five spectator functions, we are left with only the following three coupled equations:

$$\lambda^{-1}F_1{}^a(P)-[1+(16/3)\gamma^2]h_{1\beta}(P)F_\beta{}^a(P)+\tfrac{2}{3}\gamma^2h_{11}(P)E_1{}^m(P)=3\int q^2d\mathbf{q}[(1+(16/3)\gamma^2)(1-\cos^2\theta)K_{1\beta}(\mathbf{P},\mathbf{q})F_\beta{}^a(q)$$
$$-\tfrac{1}{6}\gamma\{1+4\cos^2\theta+(2\cos\theta/pq)(q^2+P^2)\}K_{11}(\mathbf{P},\mathbf{q})E_1{}^m(q)], \tag{3.20}$$

$$\lambda^{-1}F_2{}^a(P)-(5/3)h_{2\beta}(P)F_\beta{}^a(P)+\tfrac{2}{3}h_{21}(P)E_1{}^m(P)=5\int q^2d\mathbf{q}[(1-\cos^2\theta)K_{2\beta}(\mathbf{P},\mathbf{q})F_\beta{}^a(q)$$
$$-\tfrac{1}{10}\{1+4\cos^2\theta+(2\cos\theta/Pq)(q^2+P^2)\}K_{21}(\mathbf{P},\mathbf{q})E_1{}^m(q)], \tag{3.21}$$

$$\lambda^{-1}E_1{}^m(P)-[1+(16/3)\gamma^2]h_{11}(P)E_1{}^m(P)+(8/3)\gamma^2h_{1\beta}(P)F_\beta{}^a(P)$$
$$=\frac{3}{4}\int q^2d\mathbf{q}[(1+(16/3)\gamma^2)\{1+4\cos^2\theta+(2\cos\theta/Pq)(q^2+P^2)\}$$
$$\times K_{11}(\mathbf{P},\mathbf{q})E_1{}^m(q)-(32/3)\gamma^2(1-\cos^2\theta)K_{1\beta}(\mathbf{P},\mathbf{q})F_\beta{}^a(q)]. \tag{3.22}$$

IV. NUMERICAL RESULTS AND DISCUSSION

From a numerical point of view, the problem has been reduced to eigenvalue equations in terms of the strength parameter λ^{-1}. The usual procedure is to calculate the binding energy of the system, taking the value of all the six parameters in the theory, viz., λ, b, c, β_1, β_2, and β_3 as found from the two-body phase-shift analysis. Since we are interested *only* in the likelihood of existence or otherwise of n^3, we calculate the minimum value of λ ($\lambda_{\min}$) for *zero* binding energy keeping all other parameters fixed, which is a much simpler problem than the calculation of binding energy itself.

As has been shown in MP3N, the most attractive state is the one with quantum numbers $(1,\frac{3}{2},\frac{1}{2})^+$ which corresponds to the spectator functions $F_{1,2}{}^a$ in our formalism. Therefore, as a first approximation, we can even neglect $E_1{}^m$ in our equations, which reduces the problem to only *two* coupled equations. The effect of the coupling through $E_1{}^m$ can then be taken by solving all the three coupled equations *exactly*. If the change in $\lambda_{\min}$ produced by this additional coupling is small, then the *a priori* assumption of neglecting the other equations with negative kernels will at least be partially justified.

In Table III, we present the values of $\lambda_{\min}$ found

TABLE III. The strength parameter λ_3, for zero binding, as found from a solution of two- and three-coupled integral equations for both the sets of Table I. The corresponding two-body value λ_2 is also given for comparison.

	Set I		Set II	
Strength parameter	Three-coupled equations	Two-coupled equations	Three-coupled equations	Two-coupled equations
λ_3 (Three-body)	0.025	0.0247	0.0128	···
λ_2 (Two-body)	0.06		0.025	

from the solutions of both two-coupled as well as three-coupled equations, for either of the two sets of potential parameters given in Table I. The corresponding two-body values of λ are also listed for comparison. First we note that indeed, the solution of three-coupled equations changes the value of $\lambda_{\min}$ by about 1.0% over its two-coupled-equations value, which puts our approximation on quite a firm footing.

The value of $\lambda_{\min}$, as found from the three-body analysis, is smaller than its two-body value by as much as a factor of 2.5 and 2.0 for sets I and II of Table I, respectively, which is strongly in *favor* of the existence of n^3. Even though certain rough approximations have been made for numerical convenience, we do not feel that a more accurate numerical analysis will destroy a factor as large as 2.5. Mitra and Bhasin had predicted the possibility of an n^3 bound state with a very simple potential. Making a much more elaborate analysis of the problem, we have thus confirmed their qualitative result.

Our results are in definite contradiction with those of Okamoto and Davies[8] and of Barbi,[9] which are based on some variational techniques employing conventional potentials. The accuracy of the variational approach, which makes use of a trial wave function, depends upon the complexity of the wave function employed. For p-wave structures, in particular, a variational calculation requires much more elaborate trial functions. Okamoto and Davies and Barbi have used wave functions with only one or two parameters which seem to be inadequate for the present calculation. Our approach is, of course, free from these uncertainties as we have employed *exact* wave functions. Also, (along with MB) these authors had considered only a central force, which is certainly not a good approximation as it cannot produce the splitting in various ^3P_J phase shifts, a splitting which experimentally is rather large. In our analysis we have removed this serious defect by the introduction of noncentral terms in the potential. A similar calculation with the inclusion of noncentral effects in local potentials is also in progress at Sussex, the results of which are yet being awaited.[21]

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APPENDIX A

In this Appendix we give the expressions for $\mathfrak{D}_R$ and $\mathfrak{N}_R$, occurring in Eq. (2.11) in the text, in terms of certain integrals, which shall also be defined here. Thus, we have

$$\mathfrak{D}_R=(I_1-1)(\Gamma I_2-1)(\Gamma^2I_3-1)-\Gamma I_{12}^2(\Gamma^2I_3-1)-\Gamma^2I_{13}^2(\Gamma I_2-1)-\Gamma^3I_{23}^2(I_1-1)+2\Gamma^3I_{12}I_{13}I_{23}\,,\quad\text{(A1)}$$

$$\begin{aligned}\mathfrak{N}_R=u_1^2\{\Gamma^3I_{23}^2-(\Gamma I_2-1)(\Gamma^2I_3-1)\}&+\Gamma u_2^2\{\Gamma^2I_{13}^2-(I_1-1)(\Gamma^2I_3-1)\}\\&+\Gamma^2u_3^2\{\Gamma I_{12}^2-(I_1-1)(\Gamma I_2-1)\}\\&+2\Gamma u_1u_2\{I_{12}(\Gamma^2I_3-1)-\Gamma^2I_{13}I_{23}\}\\&+2\Gamma^3u_2u_3\{I_{23}(I_1-1)-I_{13}I_{12}\}\\&+2\Gamma^2u_1u_3\{I_{13}(\Gamma I_2-1)-\Gamma I_{12}I_{23}\}\,,\end{aligned}\quad\text{(A2)}$$

where the integrals I_α and $I_{\alpha\beta}$ are given by

$$I_\alpha=(4\pi\lambda)P\int_0^\infty\frac{q^4dqu_\alpha^2(q)}{(q^2-k^2)}\,,\qquad I_{\alpha\beta}=(4\pi\lambda)P\int_0^\infty\frac{q^4dqu_\alpha(q)u_\beta(q)}{(q^2-k^2)}\,.\quad\text{(A3)}$$

For the explicit forms (2.13) to (2.15) for u_α, all these integrals can be easily evaluated by using the general result

$$P\int_0^\infty\frac{q^8dq}{\prod_{i=1}^4(q^2+\beta_i^2)(q^2-k^2)}=\tfrac{1}{2}\pi\sum_{i=1}^4\beta_i^7(\beta_i^2+k^2)^{-1}\prod_{\substack{j=1\\(j\neq i)}}^4(\beta_i^2-\beta_j^2)^{-1}.\quad\text{(A4)}$$

APPENDIX B

Here we derive the 12 coupled integral equations for the n^3 system in $J^P=\frac{1}{2}^+$ state. On substituting Eqs. (3.11)–(3.17) in Eq. (3.6) and equating the coefficients of $u_\alpha(p_{23})$ on both sides of Eq. (3.6), we find

$$-[(\mathbf{P}_1\cdot\mathbf{p}_{23})E_1^m(P_1)\chi''+(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_1^m(P_1)\chi_\mu''-(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_1^a(P_1)\chi_\mu^s-Q_{\mu\nu}(1,23)G_1^a(P_1)\chi_{\mu\nu}^s]=3\lambda\int d\mathbf{p}_{23}'(\mathbf{p}_{23}\cdot\mathbf{p}_{23}')u_1(p_{23}')P_\sigma^+(23)\Psi_1(\mathbf{p}_{23}',\mathbf{P}_1)\,,\quad\text{(B1)}$$

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[21] L. M. Delves, in *Few-Body Problems*, edited by G. Paić and I. Šlaus (Gordon and Breach Science Publishers, Inc., New York 1968).

$$-(\mathbf{P}_1\cdot\mathbf{p}_{23})E_2{}^m(P_1)\chi''-(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_2{}^m(P_1)\chi_\mu''+(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_2{}^a(P_1)\chi_\mu{}^s+Q_{\mu\nu}(1,23)G_2{}^a(P_1)\chi_{\mu\nu}{}^s$$

$$=3\lambda\int d\mathbf{p}_{23}'i(\mathbf{p}_{23}\times\mathbf{p}_{23}')_\lambda(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}^3)_\lambda u_2(p_{23}')\Psi_1(\mathbf{p}_{23}',\mathbf{P}_1)\,, \quad \text{(B2)}$$

$$-(\mathbf{P}_1\cdot\mathbf{p}_{23})E_3{}^m(P_1)\chi''-(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_3{}^m(P_1)\chi_\mu''+(\mathbf{p}_{23}\times\mathbf{P}_1)_\mu F_3{}^a(P_1)\chi_\mu{}^a+Q_{\mu\nu}(1,23)G_3{}^a(P_1)\chi_{\mu\nu}{}^s$$

$$=3\lambda\int d\mathbf{p}_{23}'\{4(\mathbf{p}_{23}\cdot\mathbf{p}_{23}')+2(\mathbf{p}_{23}\cdot\mathbf{p}_{23}')(\boldsymbol{\sigma}^2\cdot\boldsymbol{\sigma}^3)+(\mathbf{p}_{23}\times\mathbf{p}_{23}')_\lambda i(\boldsymbol{\sigma}^2+\boldsymbol{\sigma}^3)_\lambda-(\sigma_\rho{}^2\sigma_\lambda{}^3+\sigma_\rho{}^3\sigma_\lambda{}^2)p_{23\rho}p_{23\lambda}'\}$$
$$\times u_3(p_{23}')\Psi_1(\mathbf{p}_{23}',\mathbf{P}_1)\,, \quad \text{(B3)}$$

where Ψ_1 is the same expression as Ψ of Eq. (3.7), but expressed entirely in terms of the momentum pair $(\mathbf{P}_1,\mathbf{p}_{23})$. In Eqs. (B1)–(B3) we further equate different spin functions on both sides to obtain as many coupled integral equations as there are spectator functions $(E_\alpha{}^m,F_\alpha{}^m,\ F_\alpha{}^a,G_\alpha{}^a)$. As outlined in Ref. 20, we next perform the azimuthal integration on the right-hand side of Eqs. (B1)–(B3), after making a suitable transformation in order to make spectator functions (under integration) independent of angular coordinates. The final equations for spectator functions are then expressible as

$$\lambda^{-1}E_1{}^m(P)-h_{1\alpha}(P)E_\alpha{}^m(P)=3\int d\mathbf{q}\ q^2K_{1\alpha}(\mathbf{P},\mathbf{q})(\tfrac{1}{2}+P\cos\theta/q)(\tfrac{1}{2}+q\cos\theta/P)E_\alpha{}^m(q)\,, \quad \text{(B4)}$$

$$\lambda^{-1}E_2{}^m(P)-\tfrac{4}{3}h_{2\alpha}(P)(2F_\alpha{}^m(P)-\sqrt{2}F_\alpha{}^a(P))=-4\int d\mathbf{q}\ q^2K_{2\alpha}(\mathbf{P},\mathbf{q})\sin^2\theta\{F_\alpha{}^m(q)+\sqrt{2}F_\alpha{}^a(q)\}\,, \quad \text{(B5)}$$

$$\lambda^{-1}E_3{}^m(P)-\tfrac{4}{3}h_{3\alpha}(P)(4E_\alpha{}^m(P)+2F_\alpha{}^m(P)-\sqrt{2}F_\alpha{}^a(P)-(5/\sqrt{3})G_\alpha{}^a(P))$$
$$=\int d\mathbf{q}\ q^2K_{3\alpha}(\mathbf{P},\mathbf{q})[16(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)-4\sin^2\theta\{F_\alpha{}^m(q)+\sqrt{2}F_\alpha{}^a(q)\}$$
$$-(8/\sqrt{3})\{(5/4)(1+\cos^2\theta)+(\cos\theta/Pq)(q^2+P^2)\}G_\alpha{}^a(q)]\,, \quad \text{(B6)}$$

$$\lambda^{-1}F_1{}^m(P)-h_{1\alpha}(P)F_\alpha{}^m(P)=-\frac{3}{2}\int d\mathbf{q}\ q^2K_{1\alpha}(\mathbf{P},\mathbf{q})\sin^2\theta F_\alpha{}^m(q)\,, \quad \text{(B7)}$$

$$\lambda^{-1}F_2{}^m(P)-h_{2\alpha}(P)\{\tfrac{4}{3}E_\alpha{}^m(P)+\tfrac{4}{3}F_\alpha{}^m(P)+\tfrac{1}{3}\sqrt{2}F_\alpha{}^a(P)+\tfrac{1}{4}(5\sqrt{3})G_\alpha{}^a(P)\}$$
$$=\int d\mathbf{q}\ q^2K_{2\alpha}(\mathbf{P},\mathbf{q})[4(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)-2\sin^2\theta\{F_\alpha{}^m(q)-\tfrac{1}{2}\sqrt{2}F_\alpha{}^a(q)\}$$
$$-4\sqrt{3}\{\tfrac{1}{8}\sin^2\theta+\tfrac{1}{3}(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)\}G_\alpha{}^a(q)]\,, \quad \text{(B8)}$$

$$\lambda^{-1}F_3{}^m(P)-\tfrac{4}{3}h_{3\alpha}(P)\{E_\alpha{}^m(P)+5F_\alpha{}^m(P)-\sqrt{2}F_\alpha{}^a(P)+(5/\sqrt{3})G_\alpha{}^a(P)\}$$
$$=\int d\mathbf{q}\ q^2K_{3\alpha}(\mathbf{P},\mathbf{q})[4(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)-\sin^2\theta\{10F_\alpha{}^m(q)+4\sqrt{2}F_\alpha{}^a(q)\}$$
$$-(1/\sqrt{3})\{10+4\cos^2\theta+8q\cos\theta/P+5P\cos\theta/q\}G_\alpha{}^a(q)]\,, \quad \text{(B9)}$$

$$\lambda^{-1}F_1{}^a(P)-h_{1\alpha}(P)F_\alpha{}^a(P)=3\int d\mathbf{q}\ q^2K_{1\alpha}(\mathbf{P},\mathbf{q})F_\alpha{}^a(q)\sin^2\theta\,, \quad \text{(B10)}$$

$$\lambda^{-1}F_2{}^a(P)+h_{2\alpha}(P)\{\tfrac{1}{3}(2\sqrt{2})E_\alpha{}^m(P)-\tfrac{1}{3}\sqrt{2}F_\alpha{}^m(P)-(5/3)F_\alpha{}^a(P)+(5/3\sqrt{6})G_\alpha{}^a(P)\}$$
$$=\int d\mathbf{q}\ q^2K_{2\alpha}(\mathbf{P},\mathbf{q})[-2\sqrt{2}(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)-(1/\sqrt{2})\sin^2\theta\{F_\alpha{}^m(q)-5\sqrt{2}F_\alpha{}^a(q)\}$$
$$+(2\sqrt{6})\{\tfrac{1}{8}\sin^2\theta+\tfrac{1}{3}(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)\}G_\alpha{}^a(q)]\,, \quad \text{(B11)}$$

$$\lambda^{-1}F_3{}^a(P)+\tfrac{1}{3}(2\sqrt{2})h_{3\alpha}(P)\{E_\alpha{}^m(P)+2F_\alpha{}^m(P)-4\sqrt{2}F_\alpha{}^a(P)+(5/\sqrt{3})G_\alpha{}^a(P)\}$$

$$=\int d\mathbf{q}\, q^2K_{3\alpha}(\mathbf{P},\mathbf{q})[-2\sqrt{2}(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)+2\sqrt{2}\sin^2\theta\{F_\alpha{}^m(q)+4\sqrt{2}F_\alpha{}^a(q)\}$$
$$+(\sqrt{2}/2\sqrt{3})\{10+4\cos^2\theta+8q\cos\theta/P+5P\cos\theta/q\}G_\alpha{}^a(q)], \quad \text{(B12)}$$

$$\lambda^{-1}G_1{}^a(P)-h_{1\alpha}(P)G_\alpha{}^a(P)=-3\int d\mathbf{q}\, q^2K_{1\alpha}(\mathbf{P},\mathbf{q})\sin^2\theta(1+q\cos\theta/P)G_\alpha{}^a(q), \quad \text{(B13)}$$

$$\lambda^{-1}G_2{}^a(P)-(1/\sqrt{3})h_{2\alpha}(P)\{2F_\alpha{}^m(P)-\sqrt{2}F_\alpha{}^a(P)+\sqrt{3}G_\alpha{}^a(P)\}$$

$$=\int d\mathbf{q}\, q^2K_{2\alpha}(\mathbf{P},\mathbf{q})[-\sqrt{3}\sin^2\theta\{F_\alpha{}^m(q)+\sqrt{2}F_\alpha{}^a(q)\}-6\{\tfrac{1}{4}(1+9\cos^2\theta)+(\cos\theta/Pq)(q^2+P^2)\}G_\alpha{}^a(P)], \quad \text{(B14)}$$

$$\lambda^{-1}G_3{}^a(P)-\tfrac{1}{3}4\sqrt{3}h_{3\alpha}(P)\{E_\alpha{}^m(P)+2F_\alpha{}^m(P)-(7/4)\sqrt{2}F_\alpha{}^a(P)+(8/\sqrt{3})G_\alpha{}^a(P)\}$$

$$=\int d\mathbf{q}\, q^2K_{3\alpha}(\mathbf{P},\mathbf{q})[4\sqrt{3}(\tfrac{1}{2}+q\cos\theta/P)(\tfrac{1}{2}+P\cos\theta/q)E_\alpha{}^m(q)-4\sqrt{3}\sin^2\theta\{F_\alpha{}^m(q)+\tfrac{1}{4}7\sqrt{2}F_\alpha{}^a(q)\}$$
$$-2\{11-\cos^2\theta+(10-6\cos^2\theta)(q\cos\theta/P)+4P\cos\theta/q\}G_\alpha{}^a(q)]. \quad \text{(B15)}$$

Equations (B4)–(B15) are the desired 12 coupled one-dimensional integral equations for $J^P=\frac{1}{2}^+$. In all these equations, summation over the index α is implied. Further, the various kernels appearing in these equations are defined as follows:

$$h_{\alpha\beta}(P)=\int\frac{q^2d\mathbf{q}\,\mathrm{u}_\alpha(q)\mathrm{u}_\beta(q)}{(q^2+\frac{3}{4}P^2+\alpha_T{}^2)}, \quad \text{(B16)}$$

$$K_{\alpha\beta}(\mathbf{P},\mathbf{q})=\frac{\mathrm{u}_\alpha(\mathbf{P}+\frac{1}{2}\mathbf{q})\mathrm{u}_\beta(\frac{1}{2}\mathbf{P}+\mathbf{q})}{(P^2+q^2+\mathbf{P}\cdot\mathbf{q}+\alpha_T{}^2)}, \quad \text{(B17)}$$

where

$$\cos\theta=(\hat{P}\cdot\hat{q}). \quad \text{(B18)}$$

APPENDIX C

Here we give a list of some of the more important results of spinology used in Appendix B:

$$\sigma_\mu{}^3\sigma_\nu{}^3\sigma_\mu{}^3\equiv-\sigma_\nu{}^3, \quad \text{(C1)}$$

$$(\sigma_\rho{}^3\sigma_\mu{}^3\sigma_\lambda{}^3+\sigma_\lambda{}^3\sigma_\mu{}^3\sigma_\rho{}^3)\equiv2(\delta_{\mu\lambda}\sigma_\rho{}^3+\delta_{\mu\rho}\sigma_\lambda{}^3-\delta_{\lambda\rho}\sigma_\mu{}^3), \quad \text{(C2)}$$

$$i(\sigma^2+\sigma^3)_\lambda\chi_\mu{}''=\tfrac{1}{3}\sqrt{2}[-2\sqrt{2}\epsilon_{\lambda\mu\nu}\chi_\nu{}''+\epsilon_{\lambda\mu\nu}\chi_\nu{}^s-2\sqrt{2}\delta_{\mu\lambda}\chi''-\sqrt{6}\chi_{\lambda\mu}{}^s], \quad \text{(C3)}$$

$$i(\sigma^2+\sigma^3)_\lambda\chi_\mu{}^s=\tfrac{1}{3}\sqrt{2}\epsilon_{\lambda\mu\nu}\chi_\nu{}''-(5/3)\epsilon_{\lambda\mu\nu}\chi_\nu{}^s-\tfrac{2}{3}\sqrt{2}\delta_{\mu\lambda}\chi''-\sqrt{\tfrac{2}{3}}\chi_{\lambda\mu}{}^s, \quad \text{(C4)}$$

$$i(\sigma^2+\sigma^3)_\lambda\chi_{\mu\nu}{}^s=\epsilon_{\mu\lambda\rho}\chi_{\nu\rho}{}^s+\epsilon_{\nu\lambda\rho}\chi_{\mu\rho}{}^s+(1/\sqrt{3})\{\delta_{\lambda\nu}[\chi_\mu{}''+(1/\sqrt{2})\chi_\mu{}^s]+\delta_{\lambda\mu}[\chi_\nu{}''+(1/\sqrt{2})\chi_\nu{}^s]$$
$$-\tfrac{2}{3}\delta_{\mu\nu}[\chi_\lambda{}''+(1/\sqrt{2})\chi_\lambda{}^s]\}, \quad \text{(C5)}$$

$$(\sigma_\rho{}^2\sigma_\lambda{}^3+\sigma_\lambda{}^2\sigma_\rho{}^3)\chi''=\tfrac{2}{3}(\delta_{\rho\lambda}\chi''+2\sqrt{3}\chi_{\rho\lambda}{}^s), \quad \text{(C6)}$$

$$(\sigma_\rho{}^2\sigma_\lambda{}^3+\sigma_\lambda{}^2\sigma_\rho{}^3)\chi_\mu{}^s=\tfrac{4}{3}\delta_{\rho\lambda}\chi_\mu{}^s-\tfrac{1}{3}(2\sqrt{2})\delta_{\rho\lambda}\chi_\mu{}''+\delta_{\mu\rho}(\sqrt{2}\chi_\lambda{}''-\chi_\lambda{}^s)+\delta_{\mu\lambda}(\sqrt{2}\chi_\rho{}''-\chi_\rho{}^s)+(\sqrt{\tfrac{2}{3}})(\epsilon_{\mu\rho\tau}\chi_{\tau\lambda}{}^s+\epsilon_{\mu\lambda\tau}\chi_{\tau\rho}{}^s), \quad \text{(C7)}$$

$$(\sigma_\rho{}^2\sigma_\lambda{}^3+\sigma_\lambda{}^2\sigma_\rho{}^3)\chi_{\mu\nu}{}^s=(2/\sqrt{3})(\delta_{\lambda\nu}\delta_{\mu\rho}+\delta_{\lambda\mu}\delta_{\nu\rho}-\tfrac{2}{3}\delta_{\mu\nu}\delta_{\rho\lambda})\chi''-(1/\sqrt{3})(\delta_{\lambda\nu}\epsilon_{\mu\rho\tau}+\delta_{\lambda\mu}\epsilon_{\nu\rho\tau}+\delta_{\rho\nu}\epsilon_{\mu\lambda\tau}+\delta_{\rho\mu}\epsilon_{\nu\lambda\tau})$$
$$\times[\chi_\tau{}''+(1/\sqrt{2})\chi_\tau{}^s]+(\tfrac{4}{3}\delta_{\mu\nu}\chi_{\rho\lambda}{}^s+2\delta_{\rho\lambda}\chi_{\mu\nu}{}^s-\delta_{\lambda\nu}\chi_{\mu\rho}{}^s-\delta_{\lambda\mu}\chi_{\nu\rho}{}^s-\delta_{\rho\nu}\chi_{\mu\lambda}{}^s-\delta_{\rho\mu}\chi_{\nu\lambda}{}^s). \quad \text{(C8)}$$