ground state. The number-conserving PQTD and PQSTD excitation operators are then applied to PBCS and orthonormalized by a Schmidt procedure. The final exactly spurion-free solution of the secular problem represents mixing of shell-model configurations of seniority $v=0$, 2, and 4 in a Hilbert space spanned on QTD and QSTD modes, thus of dimensions reduced enormously with respect to the original "exact" shell model. The dimensions are, in fact, independent of the number of nucleons; only the matrix elements depend on a given isotope (or isotone). The PQSTD model should clear up the question of the role played by the higher-order spurions not projected out in QSTD.

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# Overlapping Compound-Nucleus Resonances

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A simple model to study two overlapping compound-nucleus resonances is constructed. Expressions have been derived for the joint probability distribution of the spacing and the widths of the collision matrix for the elastic and inelastic scattering. In the inelastic case, we have considered only a two-channel problem. It is shown that the unitarity gives rise to the statistical correlation between the width and the spacing of the collision matrix. The model is also used to check which of the relations between the averages of the parameters of the statistical collision matrix, obtained using the ensemble of random complex orthogonal matrices, are consistent with the constraint of unitarity.

## I. INTRODUCTION

THE statistical properties of the resonance parameters of the low-energy collision matrix have been fairly well studied for the case of well-separated resonances.[1] During the last couple of years, there has been much interest in the study of the fluctuations of nuclear cross sections,[2] which occur in the region of overlapping resonances. In the derivation of the expressions for various average cross sections and their fluctuations we need the statistical properties of the resonance parameters in the region of overlapping resonances. Only a few attempts have been made for such a study.[3,4]

In Ref. 3 it was shown that the random-matrix hypothesis can be used to study the statistical properties of the statistical collision matrix introduced by Moldauer.[5] It was shown[3] that a number of relations between the parameters of the statistical collision matrix can be obtained without a complete knowledge of the weight function, which had to be introduced to make the normalization integral converge. Since the collision matrix must be unitary, there must be relations among the parameters of the collision matrix. The question we raise now is as follows: Which of the relations obtained using the ensemble of random complex orthogonal-matrices are consistent with the constraint of unitarity?

The statistical properties of the resonance parameters in Ref. 4 are obtained in two ways: (a) use of large-scale numerical computations. The numerical values of the parameters of the statistical collision matrix are obtained by diagonalizing a complex symmetric-level matrix, which is constructed using the parameters of the real-boundary-value problem. These numerical calculations are helpful in indicating certain trends in the behavior of the parameters only. (b) construction of unitary models. Some unitary models are constructed starting from $R$-matrix theory of nuclear reactions.[6] In most of these models the usual statistical distribution of the $R$-matrix parameters cannot be used.

A formulation using the ensemble of random unitary symmetric matrices is also developed by Krieger[7] to calculate the averages and the fluctuations of the cross sections. This formulation does not attempt to study the

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statistical properties of the usual resonance parameters separately but deals directly with the ensemble averages of the collision matrix.

We present a simple model in Secs. II and III to study the statistical properties of the resonance parameters where the resonances may be interfering quite strongly. In Sec. II we develop the model for elastic scattering, and in Sec. III we extend it to include inelastic processes.

## II. ELASTIC SCATTERING

### A. Description of the Model

According to the $R$-matrix theory of nuclear reactions,[6] the connection between the unitary collision function $U$ and the $R$ function is

$$U=\Omega^2(1-L^{0*}R)/(1-L^0R)\,, \quad (1)$$

where the diagonal matrices $\Omega$ and $L^0$ are defined in Ref. 6. Let us consider a simple model of two interfering resonances in the presence of some background scattering, then the $R$ function can be written as

$$R=R^0+\sum_{\lambda=1}^{2}\frac{\gamma_\lambda^2}{E_\lambda-E}\,, \quad (2)$$

where $R^0$ gives rise to background scattering, $\gamma_\lambda$ are the reduced-width amplitudes and $E_\lambda$ are the eigenvalues of the compound-nucleus Hamiltonian.

Substituting Eq. (2) in Eq. (1) and after some simplification we get

$$U=U^0\left(1-i\sum_{\lambda=1}^{2}\frac{g_\lambda^2}{E-\epsilon_\lambda+\frac{1}{2}i\Gamma_\lambda}\right), \quad (3)$$

where $U^0$ is a unit-modulus quantity, which gives the background scattering, $\epsilon_1$, $\epsilon_2$, $\Gamma_1$, $\Gamma_2$ are the observable resonance parameters. In terms of these resonance parameters, the amplitudes $g_\lambda^2$ are given by

$$g_\lambda^2=\Gamma_\lambda[(\epsilon_1-\epsilon_2)+(-1)^\lambda\tfrac{1}{2}i(\Gamma_1+\Gamma_2)]\times[(\epsilon_1-\epsilon_2)-\tfrac{1}{2}i(\Gamma_1-\Gamma_2)]^{-1}. \quad (4)$$

Equations (3) and (4) are valid both for sharp as well as interfering resonances. For the case of sharp resonances, we note from Eq. (4) that $g_\lambda^2\sim\Gamma_\lambda$, as it should be.

The connection between $U^0$, $\epsilon_1$, $\epsilon_2$, $\Gamma_1$, $\Gamma_2$ with the earlier parameters used in Eqs. (1) and (2) can be expressed by the following relations:

$$U^0=\Omega^2(1-L^{0*}R^0)/(1-L^0R^0)\,, \quad (5a)$$

$$\Gamma_1+\Gamma_2=X_1^2+X_2^2\,, \quad (5b)$$

$$\epsilon_1+\epsilon_2=E_1+E_2-\tfrac{1}{2}(X_1^2+X_2^2)\,, \quad (5c)$$

$$\epsilon_1\Gamma_2+\epsilon_2\Gamma_1=X_1^2E_2+X_2^2E_1\,, \quad (5d)$$

$$\epsilon_1\epsilon_2-\tfrac{1}{4}\Gamma_1\Gamma_2=E_1E_2-\tfrac{1}{2}\omega(X_1^2E_2+X_2^2E_1)\,, \quad (5e)$$

where we have written

$$X_\lambda=\{2\,\mathrm{Im}[L^0(1-L^0R^0)^{-1}]\}^{1/2}\gamma_\lambda\,,$$
$$\omega=\{\mathrm{Re}[L^0(1-L^0R^0)^{-1}]\}\{\mathrm{Im}[L^0(1-L^0R^0)^{-1}]\}^{-1}.$$

The statistical distributions of the quantities $\gamma_\lambda$ and $E_\lambda$ have been very well studied in the past.[1] Here, we would like to study the statistical properties of the resonance parameters of the collision function $U$, using the relations (5).

Before we proceed further, we make the following two simplifications: (1) We choose the origin of the energy such that $E_1+E_2=0$ and write $E_1=-\frac{1}{2}S$, $E_2=\frac{1}{2}S$, where $S$ is the spacing between the poles of $R$. This choice of the origin of the energy simply means a translation along the energy axis. (2) We choose the boundary condition such that $\mathrm{Re}[L^0(1-L^0R^0)^{-1}]=0$, which allows us to write $\epsilon_1=-\frac{1}{2}\epsilon$, $\epsilon_2=\frac{1}{2}\epsilon$, where $\epsilon$ is the spacing of the real parts of the poles of $U$.

### B. Distribution of the Parameters of the Collision Function

In this section we shall derive an expression for the joint distribution of the spacing $\epsilon$ and the widths $\Gamma_1$ and $\Gamma_2$ of the collision function $U$. The distributions of the width amplitudes $X_1$ and $X_2$, and the spacing $S$ of the $R$ function, using an ensemble of $2\times2$ real-symmetric Hamiltonian matrices, are given by[8,9]

$$P(X_1,X_2)=(2\pi\langle X^2\rangle)^{-1}\delta[1-(X_1^2+X_2^2)/2\langle X^2\rangle]\,, \quad (6)$$

$$P(S)=\pi(\tfrac{1}{2}S\langle S\rangle^2)\exp(-\tfrac{1}{4}\pi S^2/\langle S\rangle^2)\,, \quad (7)$$

where $\langle X\rangle$ and $\langle S\rangle$ are the ensemble averages of $X$ and $S$, respectively. The spacing distribution $P(S)$ is the same as Wigner's distribution.[10]

An expression for the distribution of the spacing $\epsilon$ of the collision function has been given earlier.[11] The joint distribution of the spacing $\epsilon$ and the widths $\Gamma_1$ and $\Gamma_2$ of the collision function $U$, using Eqs. (5b), (5d), (5e), (6), and (7) and some mathematical manipulations, can be expressed as

$$P(\epsilon,\Gamma_1,\Gamma_2)=(8\langle S\rangle^2)^{-1}\delta[(\Gamma_1+\Gamma_2)-2\langle X^2\rangle]\times[4\epsilon^2+(\Gamma_1-\Gamma_2)^2][\Gamma_1\Gamma_2(\epsilon^2+\langle X^2\rangle^2)]^{-1/2}\times\exp[(-\tfrac{1}{4}\pi/\langle S\rangle^2)(\epsilon^2+\Gamma_1\Gamma_2)]. \quad (8)$$

An inspection of Eq. (8) shows that the spacing $\epsilon$ and the widths $\Gamma_1$ and $\Gamma_2$ of the $U$ function are not distributed independently of each other, even though we had started from an independent distribution of the spacing $S$ and the width-amplitudes $X_1$ and $X_2$ of the $R$ function. It is an easy matter to calculate the correlation coefficients of the widths $\Gamma_1$ and $\Gamma_2$ or the spacing $\epsilon$ and

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one of the widths $\Gamma_\mu$. The correlation coefficient of the widths $\Gamma_1$ and $\Gamma_2$ turns out to be $-1$, while the correlation coefficient of the spacing $\epsilon$ and the width $\Gamma_1$ or $\Gamma_2$ turns out to be zero. Even though the correlation coefficient of the spacing $\epsilon$ and one of the widths $\Gamma_\mu$ is zero, the correlation coefficients between higher powers of $\Gamma_\mu$ and $\epsilon$ are nonvanishing, as expected from the dependent distribution given by Eq. (8). By integrating out $\epsilon$ and one of the widths in expression (8) we find that the distribution of the single width $\Gamma=\Gamma_\lambda/\langle\Gamma_\lambda\rangle$ is given by

$$P(\Gamma)=(\mu\pi)^{-1}[\exp(\mu)][\Gamma(2-\Gamma)]^{-1/2}[K_0(\mu)+K_1(\mu)-2\Gamma(2-\Gamma)K_0(\mu)]\exp[-2\mu\Gamma(2-\Gamma)], \quad (9)$$

where $\mu=\frac{1}{8}\pi(\langle X^2\rangle/\langle S\rangle)^2$, and $K_n(\mu)$ is a modified Bessel function of the second kind.[12]

A relation can be obtained using Eq. (8), which connects the parameter $\mu$ with the ratio of the average spacing $\langle\epsilon\rangle$ to the average width $\langle\Gamma_\lambda\rangle$ of the collision function. It is given by

$$\langle\epsilon\rangle/\langle\Gamma_\lambda\rangle=(\tfrac{1}{8}\pi/\mu)^{1/2}[\exp(\mu)]\{I_0(\mu)-2\mu[I_0(\mu)-I_1(\mu)]\}\times[1-\Phi((2\mu)^{1/2})]+[\exp(-\mu)]I_0(\mu),$$

where $I_n(\mu)$ is a Bessel function of pure imaginary argument[12] and $\Phi$ is the error function.

The distribution $P(\Gamma)$ given by Eq. (9) depends on the ratio $\mu$, and depending on the value of $\mu$ it can differ from the two-dimensional Porter-Thomas distribution, which has the form

$$P(y)=\pi^{-1}[y(2-y)]^{-1/2},$$

where $y$ is the dimensionless width.

A quantity of interest in the theory of the distribution of random variables is the mean-square deviation. We can easily calculate its values for the dimensionless width $\Gamma$ of the $U$ function and the quantity $y$ having a Porter-Thomas distribution. We are then immediately led to the conclusion that the distribution $P(\Gamma)$ is always broader than the two-dimensional Porter-Thomas distribution, except when the parameter $\mu$ is very small.

Some of the results which have been discussed in this section are generalized to the case of an arbitrary number of interfering resonances and are intended to be published shortly.[13]

All the parameters that are used in the complex-boundary-value problem[5] can now be expressed in terms of $\epsilon$, $\Gamma_1$, and $\Gamma_2$; e.g., the normalization constant $N_\lambda$ can be written as

$$N_\lambda=[4\epsilon^2+(\Gamma_1+\Gamma_2)^2]^{1/2}[4\epsilon^2+(\Gamma_1-\Gamma_2)^2]^{-1/2}. \quad (10)$$

Using Eqs. (4), (8), and (10), it can be easily shown that

$$|\langle g_\lambda^2\rangle|/\langle|g_\lambda^2|\rangle=\langle N_\lambda\rangle^{-1}, \quad (11)$$

$$\langle\Gamma_\lambda\rangle=\langle|g_\lambda|^2\rangle/\langle N_\lambda\rangle, \quad (12)$$

the relations which were found earlier using the ensemble of random complex orthogonal matrices.[3]

A relation which is not found to be exactly satisfied is

$$\langle|g_\lambda|^4\rangle/[\langle|g_\lambda|^2\rangle]^2=\tfrac{1}{2}[1+2\langle N_\lambda^2\rangle](\langle N_\lambda\rangle)^{-2},$$

except when the parameter $\mu\ll 1$.

An interesting result which can be obtained using Eqs. (1) and (2) is that for any finite number of resonances included in the $R$ function, the collision function $U$ has the form given by Eq. (3) and that

$$\sum_{\lambda=1}^{n} g_\lambda^2=\sum_{\lambda=1}^{n} X_\lambda^2. \quad (13)$$

Since the ensemble average $J$ of the width amplitude $X_\lambda^2$ is real, it follows that

$$\langle(g_\lambda^{\text{real}})^2\rangle-\langle(g_\lambda^{\text{imag}})^2\rangle=J, \quad (14a)$$

$$\langle g_\lambda^{\text{real}}g_\lambda^{\text{imag}}\rangle=0, \quad (14b)$$

a result which was found earlier using the ensemble of random complex orthogonal matrices.[3]

## III. INELASTIC SCATTERING

### A. Description of the Model

We would now like to include the inelastic scattering in our model. Lane and Thomas[6] have given the following expression for the collision matrix

$$U=U^0+2i\Omega P^{1/2}[\sum_{\lambda\mu}(\alpha_\lambda\times\alpha_\mu)A_{\lambda\mu}]P^{1/2}\Omega, \quad (15)$$

where $U^0$ is the background matrix,

$$\alpha_\lambda=(1-R^0L^0)^{-1}\gamma_\lambda,$$

$$(A^{-1})_{\lambda\mu}=(E_\lambda-E)\delta_{\lambda\mu}-\xi_{\lambda\mu},$$

and

$$\xi_{\lambda\mu}=(L^0[1-R^0L^0]^{-1}\gamma_\lambda,\gamma_\mu).$$

As earlier, we consider the model of two interfering resonances $\lambda=1$ and 2, and restrict ourselves to a two channel problem, the channels being denoted by $C1$ and $C2$. As in Sec. II B, we choose $B_{C1}$ and $B_{C2}$ by imposing the condition that

$$\text{Re}[L_C^0(1-R_{CC}^0L_C^0)^{-1}]=0, \quad C=C1, C2.$$

Defining, the phase factors

$$\exp(-2i\phi_C)=\Omega_C^2(1-L_C^{0*}R_{CC}^0)/(1-L_C^0R_{CC}^0), \quad C=C1, C2, \quad (16)$$

similar to Eq. (5a) and a quantity $t$

$$t=(P_{C1}P_{C2})^{1/2}[|1-R_{C1C1}^0L_{C1}^0|\times|1-R_{C2C2}^0L_{C2}^0|]^{-1}R_{C1C2}^0, \quad (17)$$

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1962), p. 373.
[13] Nazakat Ullah (unpublished).

we find that the matrix $U^0$ can be expressed as

$$U^0=\begin{bmatrix} \frac{1-t^2}{1+t^2}\exp(-2i\phi_{C1}) & \frac{2it}{1+t^2}\exp[-i(\phi_{C1}+\phi_{C2})] \\ \frac{2it}{1+t^2}\exp[-i(\phi_{C1}+\phi_{C2})] & \frac{1-t^2}{1+t^2}\exp(-2i\phi_{C2}) \end{bmatrix}. \quad (18)$$

We write the resonance part of the collision matrix given by Eq. (15) as

$$(U-U^0)_{CC'}=-ie^{-i(\phi_C+\phi_{C'})}\times\sum_{\lambda=1}^{2} g_{\lambda C}g_{\lambda C'}/(E-\epsilon_\lambda+\tfrac{1}{2}i\Gamma_\lambda), \quad (19)$$

where the resonance parameters $\epsilon_\lambda$ and $\Gamma_\lambda$ of the collision matrix $U$ are related to the parameters of the $R$ matrix by the following relations:

$$\Gamma_1+\Gamma_2=(1+t^2)^{-1}(X_{1C1}^2+X_{2C1}^2+X_{1C2}^2+X_{2C2}^2), \quad (20a)$$

$$\epsilon_1+\epsilon_2=t(1+t^2)^{-1}(X_{1C1}X_{1C2}+X_{2C1}X_{2C2}), \quad (20b)$$

$$\epsilon_1\epsilon_2-\tfrac{1}{4}\Gamma_1\Gamma_2=-\tfrac{1}{4}S^2+\tfrac{1}{2}St(1+t^2)^{-1}(X_{1C1}X_{1C2}-X_{2C1}X_{2C2})-\tfrac{1}{4}(1+t^2)^{-1}(X_{1C1}X_{2C2}-X_{1C2}X_{2C1})^2, \quad (20c)$$

$$\epsilon_1\Gamma_2+\epsilon_2\Gamma_1=\tfrac{1}{2}S(1+t^2)^{-1}\times(X_{1C1}^2+X_{1C2}^2-X_{2C1}^2-X_{2C2}^2). \quad (20d)$$

The quantity $S$ has been defined in Sec. II B as the spacing of the poles of $R$ matrix, and the quantities $X_{\lambda C}$'s, which are like the width amplitudes of the $R$ matrix are defined by

$$X_{\lambda C}=(2P_C)^{1/2}(|1-R_{CC}^0L_C^0|)^{-1}\gamma_{\lambda C}.$$

### B. Distribution of the Width and Spacing of the Collision Matrix

The problem we would like to discuss in this section is to find an expression for the joint probability distribution of the parameters of $U$ matrix starting from the known distribution of the parameters of $R$ matrix and using the relations (20). The joint probability density function of the amplitudes $X_{\lambda C}$'s using the ensemble of real-symmetric 2×2 Hamiltonian matrices is given by[8]

$$P(X_{1C1},X_{1C2},X_{2C1},X_{2C2})=(2\pi|\Sigma|)^{-1}\delta[\tfrac{1}{2}(X_1,\Sigma^{-1}X_1)-1]\times\delta[\tfrac{1}{2}(X_2,\Sigma^{-1}X_2)-1]\delta[(X_1,\Sigma^{-1}X_2)], \quad (21)$$

where $X_\lambda$ denotes a vector in the channel space, $\Sigma$ is the covariance matrix, the diagonal elements of which give the variances of $X_{\lambda C}$'s and the off-diagonal element is related to the correlation of $X_{\lambda C1}$ and $X_{\lambda C2}$; $|\Sigma|$ denotes the determinant of the matrix $\Sigma$.

The distribution of $S$, the spacing of the poles of $R$ matrix is the Wigner's distribution.[10]

To derive an expression which is similar to the earlier expression (8), which gives the joint probability density function of the spacing and total widths of the $U$ matrix, we take $t=0$ and, as in Sec. II B, write $\epsilon_1=-\frac{1}{2}\epsilon$ and $\epsilon_2=\frac{1}{2}\epsilon$. Let us introduce three variables $v_1$, $v_2$, and $v_3$, defined by the relations

$$v_1=X_{1C1}^2+X_{2C1}^2+X_{1C2}^2+X_{2C2}^2, \quad (22a)$$

$$v_2=S^2+(X_{1C1}X_{2C2}-X_{1C2}X_{2C1})^2, \quad (22b)$$

$$v_3=S(X_{1C1}^2+X_{1C2}^2-X_{2C1}^2-X_{2C2}^2). \quad (22c)$$

Since $\Sigma$ is a real symmetric matrix we can find a real orthogonal transformation which diagonalizes it. Under this orthogonal transformation relations (22) and the volume element in the space of $X_{\lambda C}$'s remain invariant. Let $D_{C1}$ and $D_{C2}$ be the eigenvalues of $\Sigma$ and let $X_{\lambda C}$'s now denote the transformed variables, then using Eqs. (7), (21), and (22) we can write the joint probability density function of the variables $v_1$, $v_2$, and $v_3$ as

$$P(v_1,v_2,v_3)=(4\langle S\rangle^2|\Sigma|)^{-1}\int\delta[v_1-(X_{1C1}^2+X_{2C1}^2+X_{1C2}^2+X_{2C2}^2)]\times\delta[v_2-S^2-(X_{1C1}X_{2C2}-X_{1C2}X_{2C1})^2]\delta[v_3-S(X_{1C1}^2+X_{1C2}^2-X_{2C1}^2-X_{2C2}^2)]\times[S\exp(-\tfrac{1}{4}\pi S^2/\langle S\rangle^2)]\delta[\tfrac{1}{2}(D_{C1}^{-1}X_{1C1}^2+D_{C2}^{-1}X_{1C2}^2)-1]\delta[\tfrac{1}{2}(D_{C1}^{-1}X_{2C1}^2+D_{C2}^{-1}X_{2C2}^2)-1]\times\delta[D_{C1}^{-1}X_{1C1}X_{2C1}+D_{C2}^{-1}X_{1C2}X_{2C2}]dS\,dX_{1C1}dX_{1C2}dX_{2C1}dX_{2C2}. \quad (23)$$

Integrating Eq. (23) and using the Eqs. (20) and (22) we find that the joint probability density function of $\epsilon$, $\Gamma_1$, and $\Gamma_2$ is given by

$$P(\epsilon,\Gamma_1,\Gamma_2)=(8\langle S\rangle^2)^{-1}\delta[(\Gamma_1+\Gamma_2)-2(D_{C1}+D_{C2})][(\Gamma_1\Gamma_2-4D_{C1}D_{C2})[\epsilon^2+(D_{C1}-D_{C2})^2]]^{-1/2}\times[4\epsilon^2+(\Gamma_1-\Gamma_2)^2]\exp[(-\tfrac{1}{4}\pi/\langle S\rangle^2)(\epsilon^2+\Gamma_1\Gamma_2-4D_{C1}D_{C2})]. \quad (24)$$

As expected, when one of the channels is switched off, Eq. (24) reduces to Eq. (8).

By integrating out $\epsilon$ and one of the total widths in Eq. (24) we find that the distribution of the single total width $\Gamma=\Gamma_\lambda/\langle\Gamma_\lambda\rangle$ is given by

$$P(\Gamma)=(\mu_3\pi^{-1})[\exp(\mu_1)][\Gamma(2-\Gamma)-\mu_2]^{-1/2}\times\{\mu_1\mu_3^{-1}[K_1(\mu_1)+K_0(\mu_1)]-2[\Gamma(2-\Gamma)-\mu_2]K_0(\mu_1)\}\times\exp\{-2\mu_3[\Gamma(2-\Gamma)-\mu_2]\}\,,\quad(25)$$

where

$$\mu_1=(\pi/8\langle S\rangle^2)(D_{C1}-D_{C2})^2\,,\quad \mu_2=4D_{C1}D_{C2}/(D_{C1}+D_{C2})^2\,,$$
$$\mu_3=(\pi/8\langle S\rangle^2)(D_{C1}+D_{C2})^2.$$

This should be compared with the total dimensionless width $y$ of the $R$ matrix which is given by

$$P(y)=(\pi^{-1})[y(2-y)-\mu_2]^{-1/2}.\quad(26)$$

To compare the distribution of the width $\Gamma$ given by Eq. (25) with the one given by Eq. (26), we again calculate the mean-square deviations of the quantities $\Gamma$ and $y$. As in Sec. II B, we find that the distribution of the width $\Gamma$ is always broader than the one given by Eq. (26), except when the quantities $D_{C1}/\langle S\rangle$ and $D_{C2}/\langle S\rangle$ are very small.

## IV. CONCLUDING REMARKS

The statistical study of the spacing and the widths of the unitary collision matrix described in Secs. II and III is somewhat like the statistical study of the eigenvalues and eigenvector components of the random $2\times2$ real-symmetric Hamiltonian matrices, which was done by Porter and Rosenzweig[11] in the early days of the statistical model. We have shown that the spacing and the widths of the unitary collision matrix can be correlated, while the correlation between the eigenvalues and eigenvector components of the real-symmetric Hamiltonian matrices was strictly zero. We have shown how the distribution of a single width of the unitary collision matrix, depending on the ratio of the average width to average spacing of the collision matrix, differs from the usual Porter-Thomas–type distributions.

The simple model has also been used to check which of the relations between the various ensemble averages of the parameters of the statistical collision matrix, obtained using the ensemble of random complex orthogonal matrices,[3] are consistent with the constraint of unitarity.

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# Neutron Spectroscopic Factors from Isobaric Analog States*

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The spectroscopic factors $S_n$ of bound neutron states are usually found from $(d,p)$ stripping reactions. An alternative method of finding $S_n$ for medium-to-heavy nuclei is to analyze isobaric analog resonances observed in $(p,p)$ scattering from these nuclei. The present analysis uses a modified $R$-matrix theory in which boundary matching is done within the optical-model potential region rather than directly onto the Coulomb potential region. A resonance mixing phase and an optical penetrability are introduced. Both single- and multilevel resonances are treated. The effects of compound elastic scattering and the energy dependence of the level shift are investigated. Formulas for the spreading width are obtained. The variation of $S_n$ with the value of the matching radius and the best choice of this radius are discussed. As examples of the method, analyses of the $s$-wave resonance in $^{92}Zr(p,p)^{92}Zr$ near 6.0-MeV bombarding energy and of $s$- and $d$-wave resonances in $^{90}Zr(p,p)^{90}Zr$ near 5.8 and 6.8 MeV are presented. The values of $S_n$ obtained are compared with those from $(d,p)$ experiments, and the reliability of the two methods is discussed.

## I. INTRODUCTION

In the elastic scattering of protons from medium-to-heavy nuclei it was found that large resonance states, which are the isobaric analogs of the bound neutron-plus-target states, are produced.[1] Such resonances may be described by an $R$-matrix theory in which the neutron-plus-target states are simply related to the analog states formed by the proton plus target.[2]

The parameters of the analog resonances can be obtained by analyses of differential cross-section excitation functions at energies near the resonances. Corresponding polarization excitation functions also are useful in finding the best resonance parameters. An optical-model potential which describes the nonresonant (background, or $T_<$ states) scattering may be determined by fitting angular distributions taken at energies off resonance, but in the same energy region. The best situation

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