Surface Impedance of Type-I Superconductors: Calculation of the Effect of a Static Magnetic Field*

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The microwave surface impedance of pure type-I superconductors in a static magnetic field is calculated at frequencies in the range $10^{-5}\lesssim\hbar\omega/\Delta\lesssim3$. In the theory developed, the effect of the static field is to change the quasiparticle energies by an amount $\mathbf{p}\cdot\mathbf{v}$, where $\mathbf{p}$ is the Fermi momentum and $\mathbf{v}$ is the drift velocity associated with the Meissner current. The results are in general agreement with the high-frequency ($\hbar\omega/\Delta\sim2$) experiments, and show the "anomalous" decrease in both surface resistance and reactance that occurs in the experiments at lower frequencies ($0.01\lesssim\hbar\omega/\Delta\lesssim0.2$). Finally, the results lead to a simple quadratic field dependence of the surface reactance in the low-frequency limit ($\hbar\omega/\Delta<0.01$), where the form of the calculation is similar to both the experimental results and the predictions of the Ginzburg-Landau theory.

I. INTRODUCTION

THE effect of a static magnetic field on the microwave surface impedance of superconductors was first studied by Pippard[1] in tin at 9.4 GHz. Since this early work, a number of investigators have extended these measurements to cover the frequency range from about 10^{-3} to 10^{3} GHz on a number of different metals[2–18]; and several attempts have been made to explain the behavior observed.[13,19–23] The experimental results in the frequency range 1 to 25 GHz (most of the early work was in this range) are very complex, having frequency, static-magnetic-field, temperature, and geometry (orientation of static and alternating fields relative to each other and the metal crystal) dependences which eluded a theoretical explanation for many years. All of the early theories failed in one way or another in giving any insight into the origin of the complex results. In particular, the theories had great difficulty in explaining the origin of the "anomalous" decrease in the absorptivity for some geometries over part of the temperature range between absolute zero and the superconducting transition temperature T_c. Pippard[24,25] made a generalization of the known experimental results up to 1963, listing the main features of the low-frequency surface impedance which he thought should be explained by the theory. This generalization is in itself rather complex. We can further simplify it as follows.

The anomalous decrease which occurs at temperatures above about $t=0.7$ is most pronounced at the lowest microwave frequencies, with the alternating and static magnetic fields parallel. The change in surface resistance becomes positive for high frequencies and in the geometry where the two fields are perpendicular.

Richards' experiments[6] demonstrated that the anomalous decrease changed over to an increase as the electron free path was decreased by the addition of impurities.

The theory must also explain the results of experiments at high frequency,[12–15] i.e., those frequencies where the photon energy $\hbar\omega$ is of the same order of magnitude as the energy gap 2Δ. In these latter experiments a static magnetic field leads to a drastic decrease in the energy of the absorption edge associated with gap jumping; that is, in a static magnetic field, absorption is observed to occur at photon energies considerably less than the zero-magnetic-field energy gap.

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Basing their argument on the above results, Budzinski and Garfunkel[12–15] proposed that a bulk superconductor in the presence of a static magnetic field has the observed excitation energy changed by the quantity $\mathbf{p}\cdot\mathbf{v}$, where $\mathbf{p}$ is the one-particle momentum near the Fermi surface and $\mathbf{v}$ $(=e\mathbf{A}/mc$, where $\mathbf{A}$ is the vector potential) is the drift velocity of electrons which make up the Meissner current. They then attributed this change in excitation energy to a shift in quasiparticle energies.[26,27] Pincus[22] has developed a theory using the $\mathbf{p}\cdot\mathbf{v}$ term as a potential well in the superconducting penetration depth to show that there are surface states whose existence make possible absorption at photon energies below the gap. Maki[21] tried to account for the anomalous low-frequency behavior by an oversimplified model, using a $\mathbf{p}\cdot\mathbf{v}$ correction to the quasiparticle energy spectrum. Finally, Budzinski and Garfunkel[14] proposed a model in which the effect of the $\mathbf{p}\cdot\mathbf{v}$ term on the Bardeen-Cooper-Schrieffer[28] (BCS) density-of-states terms is the source of the low-frequency anomalous magnetic field dependence. It is the purpose of the present paper to elaborate on this model and to present the results of the calculation of the surface impedance of superconductors in enough detail to show the origin of the most prominent features of the magnetic field dependence of both the high-frequency absorption and the temperature variation of the low-frequency surface impedance. Some relevant mechanisms, particularly the changes in Δ caused by a magnetic field,[29] have been treated by other authors[4,9,19] and are not included here, although the effects will be indicated qualitatively in the discussion.

II. MODEL

The regime of long electron free path and short electromagnetic skin depth that is characteristic of the measurements in pure metals at low temperatures corresponds to the case of the anomalous skin effect,[30,31] the condition for which, in normal metals, is characterized by

$$\delta_{\rm cl}/l \ll (1+\omega^2\tau^2)^{-3/4}, \tag{1}$$

where $\delta_{\rm cl}$, the classical skin depth, is given by $\delta_{\rm cl}=c/(2\pi\omega\sigma_n)^{1/2}$; l is the electron free path; τ is the electron relaxation time; and σ_n is the conductivity in the normal metal. In the case of a superconductor, the anomalous skin effect is further defined by the condition

$$\lambda/\xi \ll 1, \tag{2}$$

where λ is the superconducting penetration depth and ξ is the coherence length. In practice, inequality (1) is easily satisfied and corresponds to the conditions which we are interested in for most measurements. However, inequality (2) is not generally satisfied. Of the materials that have been studied, aluminum is most nearly in the domain of the anomalous skin effect, since $\lambda/\xi \approx 0.03$, but tin, the most extensively studied material, has $\lambda/\xi \approx 0.2$. Nevertheless, for the features that we are interested in discussing, it is appropriate to limit our discussion to the extreme anomalous limit, i.e., the regime where inequalities (1) and (2) are both satisfied, and then to consider how the corrections[32] modify our conclusions.

In this limit we follow Mattis and Bardeen,[33] expressing the surface impedance Z $(=4\pi E_s/H_s$, where E_s and H_s are the magnitudes of the alternating electric and magnetic fields at the surface of the metal) in terms of the surface impedance of the normal state, Z_n:

$$Z/Z_n = (\sigma_1/\sigma_n - i\sigma_2/\sigma_n)^{-1/3}, \tag{3}$$

where σ_1 and σ_2 are the real and imaginary parts of the bulk conductivity in the superconductor. We define $s(\omega,t,h,\beta)\equiv s_1(\omega,t,h,\beta)-is_2(\omega,t,h,\beta)\equiv\sigma_1/\sigma_n-i\sigma_2/\sigma_n$, where t is the reduced temperature, h is the ratio of the applied static magnetic field H to some critical field at that temperature H_0, to be defined in Sec. III, and β is the angle between the static magnetic field and the alternating magnetic field. Then, in the limit of $h=0$ (the only case considered by Mattis and Bardeen), we have

$$\begin{aligned} s_1(\omega,t,0,0) = {}& (2/\hbar\omega)\int_\Delta^\infty [1+(\Delta^2/EE')][\rho(E)\rho(E')] \\ &\times[f(E)-f(E')]dE + (1/\hbar\omega)\int_{\Delta-\hbar\omega}^{-\Delta}[1+(\Delta^2/EE')] \\ &\times[\rho(E)\rho(E')][f(E)-f(E')]dE, \end{aligned} \tag{4}$$

where the second integral only applies if $\hbar\omega > 2\Delta$; and

$$\begin{aligned} s_2(\omega,t,0,0) = {}& (i/\hbar\omega)\int_{\Delta-\hbar\omega,-\Delta}^{\Delta}[1+(\Delta^2/EE')] \\ &\times[\rho(E)\rho(E')][1-2f(E')]dE, \end{aligned} \tag{5}$$

where the lower limit is the larger of $\Delta-\hbar\omega$ and $-\Delta$.

[26] The modification of the quasiparticle energy by $\mathbf{p}\cdot\mathbf{v}$ has been discussed in a short article by J. Bardeen, Phys. Rev. Letters **1**, 399 (1958). In particular, Bardeen notes that the Fermi function describing the thermal-excitation spectrum has the BCS energy plus the $\mathbf{p}\cdot\mathbf{v}$ term.

[27] J. I. Gittleman, B. Rosenblum, T. Seidel, and A. W. Wicklund, in *Proceedings of the Eighth International Conference on Low-Temperature Physics, London, 1962*, edited by R. O. Davies (Butterworths Scientific Publications Ltd., London, 1963), p. 336; Phys. Rev. **137**, A527 (1965). In these papers, the authors considered the effect of the $\mathbf{p}\cdot\mathbf{v}$ term on the thermal excitation of quasiparticles as measured by the microwave reactance of thin films. The $\mathbf{p}\cdot\mathbf{v}$ term comes from an actual current in the film rather than from the Meissner effect that is being considered in the present paper. The short mean free path in thin films should change the character of their results from the present study of pure superconductors.

[28] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[29] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

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Conservation of energy gives $E'=E+\hbar\omega$. In the integrals of Eqs. (4) and (5) the first set of square brackets contains the superconducting coherence factors; the second set contains the product of the densities of initial and final states, with $\rho(E)=E/(E^2-\Delta^2)^{1/2}$; and the final set contains the Fermi functions, defined by $f(E)=[1+\exp(E/kT)]^{-1}$.

In the presence of a static magnetic field we must modify the above expressions for s_1 and s_2. This will be done in a manner analogous to the method of Pippard, in which he makes use of the "ineffectiveness concept."[34] In this approach only those carriers that contribute to the surface impedance need be considered in obtaining the conductivity. In the regime of the anomalous skin effect, these are the particles that travel almost parallel to the surface of the sample.

We shall assume that the scattering of carriers at the sample surface is diffuse; that is, the direction of motion is randomized without changing the energy. This means that momentum of the particle is not conserved (being freely exchanged with the surface of the metal). This is an important point, since the significant features of the results are largely consequences of this assumption. The justification for this assumption is primarily experimental.[31] However, it may not be necessary to make the assumption of diffuse reflection to get some of the features which we are after. In the theory of Pincus,[22] the problem of nonconservation of the component of electron momentum parallel to the surface is avoided, since the surface excitations that he obtains do not have the same energy-momentum relationship that are characteristic of the BCS excitations. This makes it possible to conserve momentum (the component parallel to the surface) and still have absorption at energies below the energy gap. At any rate, in our model we shall use the assumption of diffuse reflection. The recent work of Koch and Kuo[18] and of Koch and Pincus[23] seems to indicate that for very carefully prepared surfaces specular reflection may occur. In that case, the experimental results should (and do) differ from the results obtained on less ideal samples.

In calculating the energy of a particle, we must associate a drift velocity $\mathbf{v}$ with it. To a first approximation, we can average $\mathbf{v}$ $[=\mathbf{v}(\mathbf{r})=e\mathbf{A}(\mathbf{r})/mc]$ over the wave packet that represents the particle. It is well known that superconducting wave packets have a spatial extent equal to the coherence length ξ; this can be shown simply by an uncertainty-principle argument, recognizing that the momentum must be specified well enough so that the uncertainty in energy is less than the energy-gap parameter Δ. However, the usual argument specifies the wave packet to be of range ξ *in the direction of motion*. The wave packet can be much narrower at right angles to the motion, having a range given by[35] $\xi_t\approx(\xi\lambda_F)^{1/2}$, where λ_F is the wavelength of a particle at the Fermi surface. Thus there is no difficulty in localizing those particles traveling parallel to the surface within the penetration depth, while those particles traveling at an appreciable angle to the surface will have their wave packets extend well past the penetration depth into the body of the metal (in the limit of $\lambda\ll\xi$). This means that the average drift velocity is only appreciable for those particles traveling parallel to the surface, and it is only these particles that will have a $\mathbf{p}\cdot\mathbf{v}$ correction to their energy. These, of course, are precisely the same particles that contribute to the microwave surface impedance. The average drift velocity to be used for the energy of particles traveling parallel to the surface should vary with the angle which the trajectory makes with the surface and with the width of the wave packet. Relating the value of $\mathbf{v}$ precisely to the applied magnetic field is rather difficult in this semiclassical model. However, we note that the maximum value of $|\mathbf{v}(\mathbf{r})|$, namely, $v(0)$, is such that $pv(0)\sim\Delta$ when $H\sim H_c$.

Starting with trajectories almost parallel to the surface, we see that a diffuse reflection will scatter most particles into trajectories making a large angle with the surface. In our rough calculation, we make two approximations: (1) Those particles that are involved in absorption (i.e., traveling almost parallel to the surface) all have the same average drift velocity $\mathbf{v}$, while those scattered out of the surface have negligible average drift velocity; (2) those particles that travel almost parallel to the surface both before and after scattering at the surface are of negligible number and so can be neglected. This latter approximation will lead to serious error at very low temperatures and low frequency, because in this regime almost all the absorption would come from the neglected particles. (The contribution from this group is just that calculated by Maki.[21])

The only electrons which are "effective" in both the normal and the superconducting states are those traveling almost parallel to the surface either before or after scattering from the surface. In obtaining a generalization of Eqs. (4) and (5) for s_1 and s_2, it is only necessary to consider those traveling parallel to the surface before collision, since the inclusion of those traveling parallel after collision merely doubles the conductivity in both

[34] A. B. Pippard, Proc. Roy. Soc. (London) **A191**, 385 (1947).

[35] This can be seen by requiring the uncertainty in energy to be of the order of the energy-gap parameter Δ. Thus, for an uncertainty in momentum δp perpendicular to the motion of a particle at the Fermi surface, we write $2m\Delta=[p^2+(\delta p)^2]-p^2=(\delta p)^2$. The transverse coherence length ξ_t is then defined as the width of the wave packet associated with δp, so that $\xi_t\sim\hbar/(\delta p)$. Putting this back in the above equation, we get $\hbar^2/\xi_t^2\sim 2m\Delta$ or $\xi_t=\hbar/(2m\Delta)^{1/2}$. With some manipulation, this becomes $\xi_t\sim(\xi\lambda_F)^{1/2}$, where $\lambda_F\equiv\hbar/p$ is the wavelength of a particle at the Fermi surface, and $\xi\sim\hbar v_F/\Delta$ is the usual coherence length (to within a constant factor of order unity). The transverse coherence length ξ_t as defined above is related to the localization found by P. G. de Gennes [Rev. Mod. Phys. **36**, 225 (1964)] in superconductor–normal-conductor sandwiches.

the normal *and* superconducting states with no effect on the ratios s_1 and s_2.

Since Eqs. (4) and (5) are valid only in the absence of a drift velocity, the integration variable E is unambiguous. In a static magnetic field, however, there is a nonzero drift velocity, and the variable E in the various brackets in a generalization of Eqs. (4) and (5) must be reexamined. In the BCS theory, the matrix elements and the density of states are given in terms of the electron center-of-mass system, so that the energy E must be in a system moving at a velocity $\mathbf{v}$ relative to the crystal surface. Since the final states are not for electrons moving parallel to the surface, we neglect the drift-velocity correction to their energies. Conservation of energy therefore requires that $E'=E+\hbar\omega+\mathbf{p}\cdot\mathbf{v}$, where $\mathbf{p}$ is the momentum associated with the initial state. (A diffuse reflection of the particles at the surface only conserves energy in the frame at rest relative to the sample, not in any other reference frame.) The Fermi functions in the final brackets of Eqs. (4) and (5), however, should be written in terms of the energy relative to the thermal bath (i.e., the laboratory or crystal system), $E+\mathbf{p}\cdot\mathbf{v}+\hbar\omega$ for final states.

We have implicitly assumed in the above discussion that absorption of a photon and scattering of the quasiparticle from the surface occur simultaneously. This should not cause any difficulty, since the time spent in the skin depth is so small that a particle may be in a virtual state between the absorption of a photon and the scattering from the surface. Nevertheless, it would be desirable to have a proper microscopic theory to justify this assumption, and to establish the limit of its validity.

In arriving at Eqs. (4) and (5), none of the factors in the integrands was assumed to depend on direction, so that the weighting of the contributions of electrons around the Fermi surface was integrated out of the result. The angular weighting factors must be put back now that we have made the energy a function of direction. From the theory of metals we know that the contribution of an electron having a momentum $\mathbf{p}$ to the conductivity is proportional to $\cos^2\alpha$, where α is the angle between $\mathbf{p}$ and the current. We now set $\alpha=\beta+\theta$, where β is also the angle between the alternating current and the steady current (or $\mathbf{v}$) and θ is the angle between $\mathbf{p}$ and $\mathbf{v}$. We have been assuming that $\mathbf{p}$ is the electron momentum near the Fermi surface. We now set p equal to the magnitude of the Fermi momentum and note that $\mathbf{p}\cdot\mathbf{v}\approx pv\cos\theta$.

Summing over the angle θ, we have for s_1

$$s_1(\omega,t,h,\beta)=(2/\pi\hbar\omega)\int_0^\pi\cos^2(\theta+\beta)\bigg\{\int_{\Delta;\Delta-\hbar\omega-pv\cos\theta}^{\infty}(1+\Delta^2/EE')[\rho(E)\rho(E')][f(E)-f(E+\hbar\omega)$$
$$+f(E+pv\cos\theta)-f(E+pv\cos\theta+\hbar\omega)]dE+\int_{\Delta-\hbar\omega-pv\cos\theta}^{-\Delta}(1+\Delta^2/EE')[\rho(E)\rho(E')]$$
$$\times[f(E+pv\cos\theta)-f(E+pv\cos\theta+\hbar\omega)]dE\bigg\}d\theta,\quad(4')$$

where $E'=E+pv\cos\theta+\hbar\omega$. The lower limit of the first integral over E is chosen as the larger of Δ and $\Delta-\hbar\omega-pv\cos\theta$. The second integral over E occurs only if $\hbar\omega+pv\cos\theta>2\Delta$. The quantity $2/\pi$ that appears in front of the right-hand side is a normalization constant for the integral over θ.

Similarly, we arrive at an expression for s_2, namely,

$$s_2(\omega,t,h,\beta)=(2i/\pi\hbar\omega)\int_0^\pi\cos^2(\theta+\beta)\int_{\Delta-\hbar\omega-pv\cos\theta;-\Delta;-\Delta}^{\Delta;-\Delta-\hbar\omega-pv\cos\theta;\Delta}(1+\Delta^2/EE')[\rho(E)\rho(E')]$$
$$\times[1-f(E+\hbar\omega)-f(E+\hbar\omega+pv\cos\theta)]dEd\theta,\quad(5')$$

where the limits on the integral over E are specified as follows:

For	Lower limit	Upper limit
$2\Delta>(\hbar\omega+pv\cos\theta)>0$	$\Delta-\hbar\omega-pv\cos\theta$	Δ
$0>(\hbar\omega+pv\cos\theta)>-2\Delta$	$-\Delta$	$-\Delta-\hbar\omega-pv\cos\theta$
$\lvert\hbar\omega+pv\cos\theta\rvert>2\Delta$	$-\Delta$	Δ.

In both Eqs. (4′) and (5′), there are temperature dependences in the Fermi functions, in Δ, and in the drift velocity v $[=(e/mc)A=(e/mc)\lambda(t)H]$. Furthermore, we must note that, in general, Δ should also be considered as a function of h (or v). But in the extreme anomalous limit that we are considering, $\lambda/\xi\ll1$, which makes Δ independent of h.[36] For materials such as tin, where λ/ξ is not very small, the magnetic field dependence of Δ might give an appreciable correction to the conductivity ratios s_1 and s_2.

[36] C. Caroli, Ann. Inst. Henri Poincaré 4, 159 (1966).

III. CALCULATION

The results that we wish to examine fall into two categories, and since the computation of the integrals in Eqs. (4′) and (5′) are rather lengthy, we have divided the problem into two cases: (A) the surface impedance at $t=0$ for high frequencies, where we only have gap jumping (or pair creation); and (B) the temperature dependence of the surface impedance at low frequency (absorption only by thermally excited quasiparticles). In (A), the double integrals reduce to one-dimensional integrals which are easily evaluated on a desk calculator, while in (B), both the two dimensionality of the integrals and the greater accuracy needed (the field dependence is smaller in this case) make the use of a high-speed computer desirable.

Since the result that we wish to compute is the *surface* impedance, we know that it must be representable as a 2×2 symmetric matrix, which is completely determined by the components along the two principal directions, namely, the components parallel to and perpendicular to the static magnetic field. (This is, in general, only true if we neglect crystalline anisotropy.) Therefore it is only necessary to evaluate $s=s_1-is_2$ for the two cases $\beta=0$ and $\frac{1}{2}\pi$. The values of

$$(Z/Z_n)=[(R/R_n)+i\sqrt{3}(X/X_n)]/(1+i\sqrt{3})$$
$$=\tfrac{1}{2}\exp(-i\pi/3)[r+i\sqrt{3}x]$$

are then computed in the extreme anomalous limit from Eq. (3). The quantities r and x are called the surface-resistance ratio and the surface-reactance ratio, respectively, and are defined by the last equation.

A. High-Frequency Surface Impedance at $t=0$

Since there are no thermally excited quasiparticles at $t=0$, and since we can limit our region of interest to $pv\leq\Delta$ (for type-1 superconductors $pv>\Delta$ corresponds to $H>H_c$), Eq. (4′) becomes

$$s_1(\omega,0,h,\beta)=(2/\pi)(1/\hbar\omega)\times\int_0^{\theta_0}\cos^2(\beta+\theta)\int_{\Delta-\hbar\omega-pv\cos\theta}^{-\Delta}(1+\Delta^2/EE')\times[\rho(E)\rho(E')]dEd\theta,\quad(6)$$

where θ_0 is defined by the condition that $\hbar\omega+pv\cos\theta>2\Delta$ for all allowed θ. This can then be rewritten as

$$s_1(\omega,0,h,\beta)=(2/\pi)\int_0^{\pi}\cos^2(\theta+\beta)\times(1+(h\Delta/\hbar\omega)\cos\theta)s_1(\omega',0,0,0)d\theta,\quad(6')$$

where $\hbar\omega'=\hbar\omega+pv\cos\theta$, $s_1(\omega',0,0,0)$ is just the value of s_1 at the frequency ω' in the absence of a static magnetic field, and $h=pv/\Delta$ (for nonzero temperature, h is more complicated; see Sec. III B below for discussion). Since $s_1(\omega,0,0,0)$ can be written as[33]

$$s_1(\omega,0,0,0)=[1+(2\Delta/\hbar\omega)]\mathcal{E}(k)-2(2\Delta/\hbar\omega)\mathcal{K}(k),\quad\text{for }(\hbar\omega/2\Delta)>1$$
$$=0\qquad\text{otherwise,}\quad(7)$$

where $k=(2\Delta-\hbar\omega)/(2\Delta+\hbar\omega)$, and $\mathcal{E}$ and $\mathcal{K}$ are complete elliptic integrals of the first and second kind; Eq. (6′) can be evaluated as an integration over θ.

The reactive part of the conductivity is treated similarly to give

$$s_2(\omega,0,h,\beta)=(2/\pi)\int_0^{\pi}\cos^2(\beta+\theta)\times(1+(h\Delta/\hbar\omega)\cos\theta)s_2(\omega',0,0,0)d\theta,\quad(8)$$

where

$$s_2(\omega,0,0,0)=\tfrac{1}{2}\{[(2\Delta/\hbar\omega)+1]\mathcal{E}(k')+[(2\Delta/\hbar\omega)-1]\mathcal{K}(k')\}\quad(9)$$

and $k'\equiv(1-k^2)^{1/2}$.

B. Temperature Dependence of the Low-Frequency Surface Impedance

By limiting our range of frequency and magnetic field to $(\hbar\omega+pv)<2\Delta$, we eliminate the second integral over E in Eq. (4′) and are left with absorption only by thermally excited quasiparticles. The real part of the conductivity thus must vanish at $t=0$. At higher temperatures we must use the temperature dependences of Δ and v. Since the drift velocity at the surface v_0 is given by

$$v_0=(e/mc)A$$
$$=(e/mc)\lambda(t)H$$
$$=(e/mc)\lambda(t)H_c(t)[H/H_c(t)],$$

we can get the temperature dependence of v_0 from the temperature dependence of λ and H_c. We have obtained the temperature dependence of Δ, λ, and H_c from the tables derived by Mühlschlegel.[37]

Converting the values of static magnetic field H into the energy term pv requires both a knowledge of the Fermi momentum p and the relationship between v and H. As discussed in Sec. II, the drift velocity is a strong function of the trajectory of the particle in question. Since in our simple model we wish to use a single, average velocity v for all particles that contribute to the surface impedance (i.e., those that are traveling almost parallel to the surface), we give up the knowledge of the exact relationship between v and H and carry out the calculation in terms of a critical value of field H_0 defined by $H_0(t)=H_0(0)[H_c(t)/H_c(0)]$, where $H_c(t)$ is the actual bulk-critical magnetic field, and $H_0(0)$ is defined at $t=0$ by $H_0(0)=(\Delta(0)H)/pv(0)$. Then $h\equiv H/H_0(t)$ gives us

$$h=[H_c(0)pv(0)]/[H_c(t)\Delta(0)].$$

This is not completely satisfactory, since we would prefer that h be defined in terms of the thermodynamic critical field $H_c(0)$. However, H_0 and H_c are of the same order of magnitude. Unfortunately, this is the best we can do without a complete theory.

The calculation using the BCS density of states $\rho(E)=E(E^2-\Delta^2)^{-1/2}$ has certain difficulties which can

[37] B. Mühlschlegel, Z. Physik **155**, 313 (1959).

be avoided by eliminating the divergence at $E=\Delta$. We do this by cutting off the density-of-states function as shown in Fig. 1. This procedure conserves the number of states. The calculation was carried out for the values of δ in the range given by $\delta/\Delta=0.00001$ to 0.015, as well as for the BCS case, to check the sensitivity of the results to this cutoff.

With the above values for the parameters and the approximations cited, Eqs. (4′) and (5′) have been evaluated by quadrature, using a Gaussian integration formula for the integrals over both E and θ. The results of the computation, which was carried out on the Cambridge University Titan, are presented in Sec. IV.

IV. RESULTS AND DISCUSSION

A. High-Frequency Surface Impedance at $t=0$

The calculation described in Sec. III A was carried out for two values of the magnetic field, $h=1.0$ and 0.5;

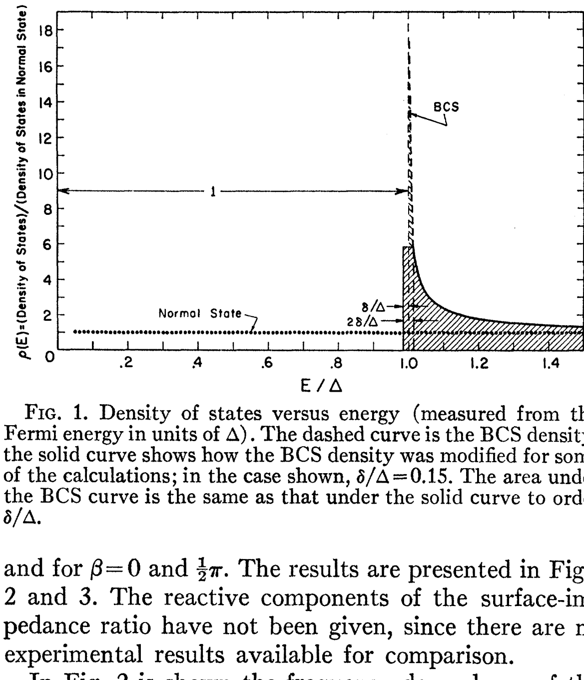

FIG. 1. Density of states versus energy (measured from the Fermi energy in units of Δ). The dashed curve is the BCS density; the solid curve shows how the BCS density was modified for some of the calculations; in the case shown, $\delta/\Delta=0.15$. The area under the BCS curve is the same as that under the solid curve to order δ/Δ.

and for $\beta=0$ and $\frac{1}{2}\pi$. The results are presented in Figs. 2 and 3. The reactive components of the surface-impedance ratio have not been given, since there are no experimental results available for comparison.

In Fig. 2 is shown the frequency dependence of the surface resistance for the case of alternating and static magnetic fields parallel at values of $h=1.0$ and 0.5. The curves are similar to those observed by Budzinski and Garfunkel[13,15] for superconducting aluminum. The changes which they observed in surface-resistance ratio for frequencies below $\omega=2.5\Delta/\hbar$ are of about the same magnitude as those in Fig. 2 (recall that we do not know how accurately h represents H/H_c). Budzinski and Garfunkel have not made measurements at frequencies above $\omega=2.5\Delta/\hbar$ on aluminum to compare with the calculated *decrease* in surface resistance with static field that appears in the figure. However, experiments by Budzinski *et al.*[38] on superconducting zinc up to frequencies of $\omega=4\Delta/\hbar$ do not show the predicted decrease. Possibly the absence of a decrease in zinc is somehow related to the large crystalline anisotropy of the energy gap in zinc.[38] Alternatively, it may be a consequence of the effect of a static magnetic field on the energy-gap parameter Δ. The field will, in general, reduce Δ, causing an increase in the surface-resistance ratio, which may, in the case of zinc, compensate the calculated decrease shown in Fig. 2. Although this is a possible explanation of the absence of a decrease, it is not very likely, because the fractional change in Δ expected at the critical magnetic field is only[36] $[\Delta(0)-\Delta(H_c)]/\Delta(0)=\lambda/[(\sqrt{8})\xi]$, which should be of the order of 1% in either zinc or aluminum.

A decrease in absorptivity with field which has been observed by Fischer and Klein[17] in the microwave absorptivity of normal-superconductor sandwiches has been attributed by them to the same source as that responsible for the decrease in Fig. 2. Possibly the Fischer-Klein experiment may be a more sensitive way to look for this feature of the results.

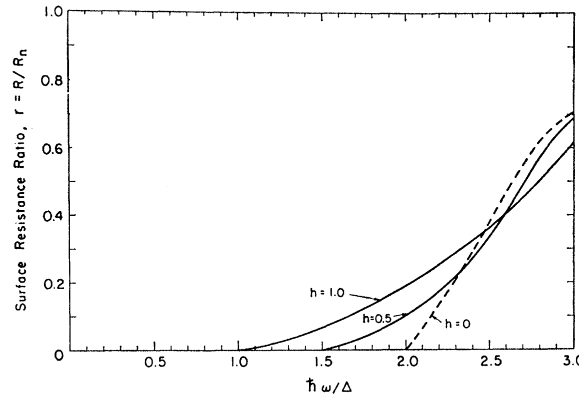

FIG. 2. High-frequency surface-resistance ratio versus photon energy with static magnetic field parallel to alternating magnetic field. The onset of absorption is shifted from the $h=0$ curve (dashed line) by an amount proportional to h, but because the energy gap is anisotropic in a field, the onset is less sharp and thus more difficult to define accurately experimentally.

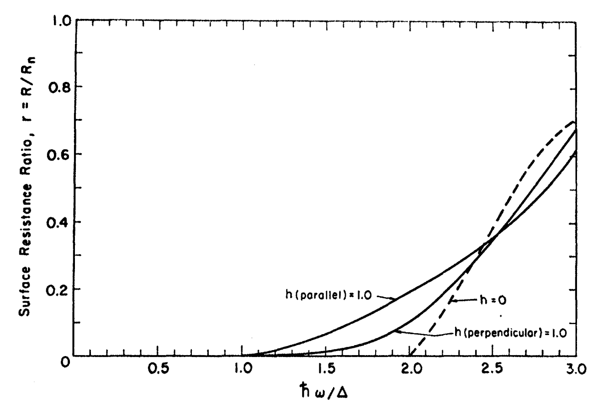

FIG. 3. Comparison of the high-frequency surface-resistance ratio versus photon energy for the cases of static magnetic field parallel to and perpendicular to the alternating magnetic field. The solid curves are both at $h=1.0$ and the dashed curve is for $h=0$.

[38] W. V. Budzinski, J. B. Evans, M. P. Garfunkel, and D. Hays (unpublished).

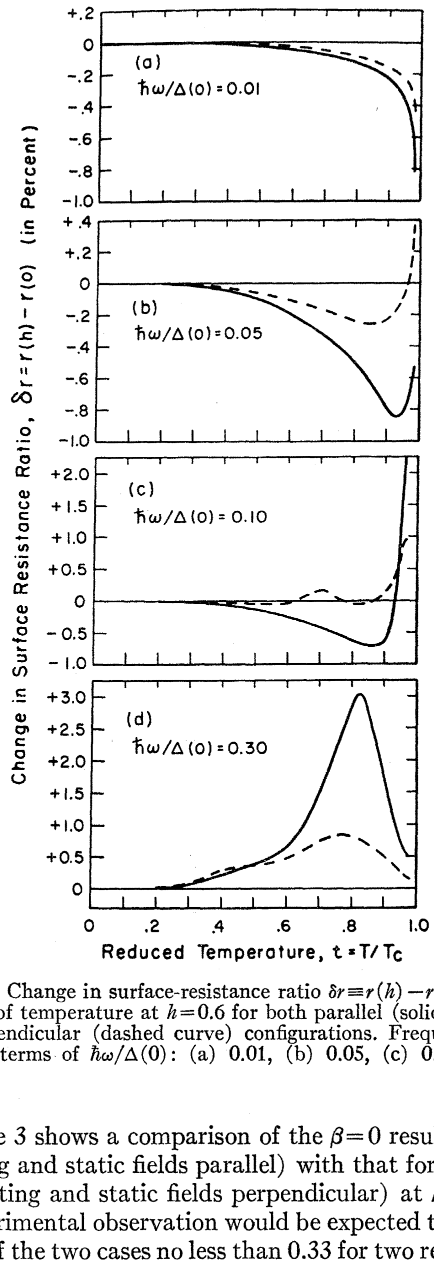

FIG. 4. Change in surface-resistance ratio $\delta r \equiv r(h) - r(0)$ as a function of temperature at $h = 0.6$ for both parallel (solid curve) and perpendicular (dashed curve) configurations. Frequency is given in terms of $\hbar\omega/\Delta(0)$: (a) 0.01, (b) 0.05, (c) 0.10, and (d) 0.30.

Figure 3 shows a comparison of the $\beta = 0$ results (alternating and static fields parallel) with that for $\beta = \frac{1}{2}\pi$ (alternating and static fields perpendicular) at $h = 1.0$. An experimental observation would be expected to have a ratio of the two cases no less than 0.33 for two reasons: (1) The microwave polarization cannot be rigorously maintained, so that it is experimentally impossible to have pure cases; and (2) the effect of a static magnetic field on Δ *adds* a resistance to both parallel and perpendicular cases which should decrease the ratio of the two (below $\omega = 2.0\ \Delta/\hbar$) to no less than 0.33. Thus it is surprising that the experimental results of Budzinski and Garfunkel[15] exhibit a smaller ratio than the theoretical lower limit of 0.33. This might result from the effect of crystalline anisotropy, which has been neglected in the present model. It will require further experiments to determine whether this is indeed the source of the disagreement.

One known effect that has been omitted from our discussion is the correction necessary to the surface impedance for nonzero λ/ξ, which in the case of aluminum is 0.031. Miller[32] has calculated this correction for aluminum and tin. In the case of aluminum, the values of r in Figs. 2 and 3 would be increased by an amount which varies from about 30% at $\omega = \Delta/\hbar$ to less than 10% at $\omega = 3\Delta/\hbar$. These changes would not in any way alter the general features of the results.

Finally, we must mention the effect of the small fraction of particles that are in trajectories almost parallel to the surface both before and after scattering from the surface (or we may alternatively say, those pairs created with both quasiparticles travelling in the plane of the surface). These particles have been omitted from the calculation. They would cause absorption at much lower frequencies, but since the probability of such trajectories is so small, we expect only a small absorption at these frequencies, presumably too small to have a large effect in the experiments of Budzinski and Garfunkel.[13,15]

B. Temperature Dependence of the Low-Frequency Surface Impedance

The calculation described in Sec. III B was carried out as a function of frequency, temperature, and mag-

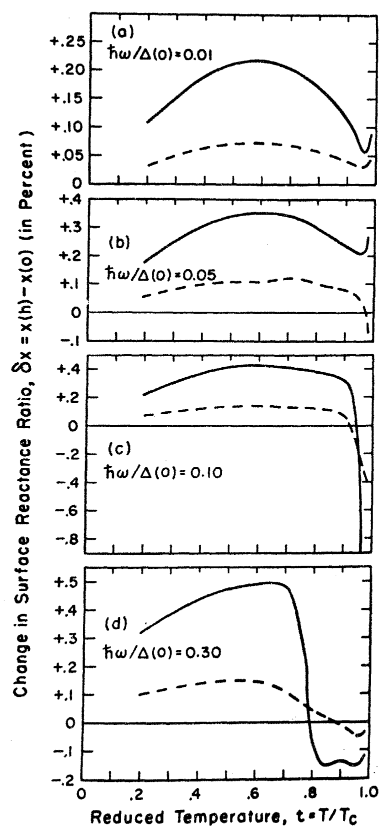

FIG. 5. Change in surface-reactance ratio $\delta x \equiv x(h) - x(0)$ as a function of temperature at $h = 0.6$ for both parallel (solid line) and perpendicular (dashed line) configurations. Frequency is given in terms of $\hbar\omega/\Delta(0)$: (a) 0.01, (b) 0.05, (c) 0.10, and (d) 0.30.

netic field, and also as a function of the density-of-states cutoff parameter δ shown in Fig. 1. As might have been expected, the general features of the results were unaffected when $\hbar\omega \gg \delta$, although the magnitude of the surface impedance does depend somewhat on the value of δ used. We present results for the BCS case, i.e., $\delta/\Delta = 0$.

In discussing these results, we further subdivide our calculation into two frequency ranges given by (1) $0.01 \leq \hbar\omega/\Delta(0) \leq 0.3$ and (2) $\hbar\omega/\Delta(0) < 0.01$. Experimental results in the range (1) have been obtained on both surface resistance and surface reactance, while for the frequency range (2), only the reactance has been measured, since the resistance is too small for measurement. We shall only present results of the calculations for those cases where there are experimental results.

1. Frequency Range $0.01 \leq \hbar\omega/\Delta(0) \leq 0.3$

The results of the calculation are given in Figs. 4–7. Figures 4 and 5 show the temperature dependence of

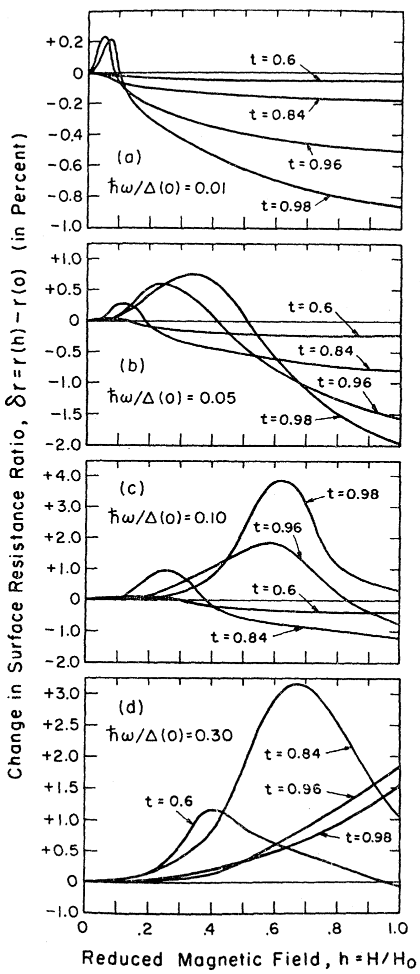

FIG. 6. Change in surface-resistance ratio as a function of magnetic field for the parallel configuration at various temperatures. Frequency is given in terms of $\hbar\omega/\Delta(0)$: (a) 0.01, (b) 0.05, (c) 0.10, and (d) 0.30.

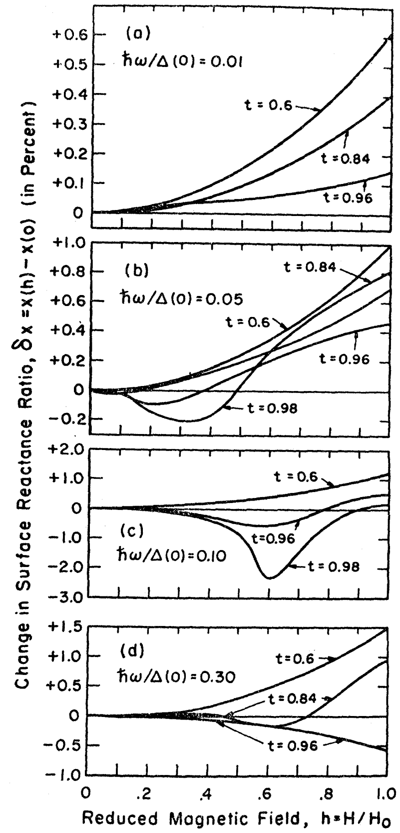

FIG. 7. Change in surface-reactance ratio as a function of magnetic field for the parallel configuration at various temperatures. Frequency is given in terms of $\hbar\omega/\Delta(0)$: (a) 0.01, (b) 0.05, (c) 0.10, and (d) 0.30.

the changes in the surface-resistance ratio $\delta r \equiv r(h) - r(0)$ and the surface-reactance ratio $\delta x \equiv x(h) - x(0)$ at a value of $h = 0.6$ for $\beta = 0$ and $\frac{1}{2}\pi$ (alternating and static magnetic fields parallel and perpendicular, respectively). As we go from (a) to (d) in both figures, the frequency varies through the values $\hbar\omega/\Delta(0) = 0.01$, 0.05, 0.10, and 0.30.

In the surface resistance (Fig. 4), the changes vary from a characteristic decrease at frequencies below $\hbar\omega/\Delta(0) = 0.1$ to an increase above. At $\hbar\omega/\Delta(0) = 0.1$ the resistance change for $\beta = 0$ is negative at low temperatures, while for $\beta = \frac{1}{2}\pi$ it is positive over much of the temperature range. At the other frequencies shown, the magnitude of the resistance for $\beta = 0$ is generally larger than for $\beta = \frac{1}{2}\pi$. We also note that at $\hbar\omega/\Delta(0) =$ 0.05 and 0.10 there is a sign change from negative to positive as the temperature approaches T_c. In general, these curves resemble in many ways the various experimental results.[1–3,6,8,11,16] The similarity could be improved by making three corrections that are applicable. These are the following: (i) the correction for the magnetic field dependence of Δ, which would increase the resistance at all temperatures, possibly making the $\beta = \frac{1}{2}\pi$ curve positive for curves (a) and (b) in mate-

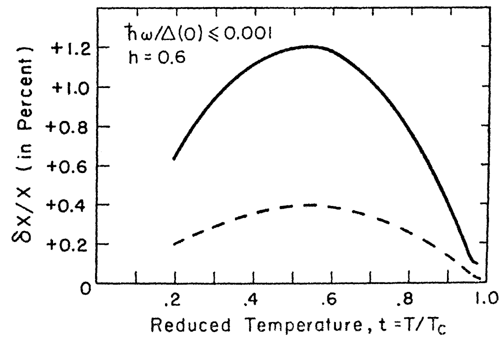

FIG. 8. Low-frequency limit of the fractional change in the surface reactance $\delta x/x$ as a function of temperature $h=0.6$ for both parallel (solid line) and perpendicular (dashed line) configurations. These curves are for $\hbar\omega/\Delta(0)\lesssim 0.001$ and are independent of frequency in this range.

rials in which λ/ξ is not very small; (ii) the correction to the surface resistance, which, because we are not in the extreme anomalous limit,[32] would increase all the magnitudes without affecting the sign (in materials such as tin this would amount to a factor of the order of 5); and (iii) the inclusion in the calculation of those quasiparticles which are in trajectories that are parallel to the surface both before and after scattering from the surface, which would make the low-temperature changes always positive. At the lowest temperatures the absorption is actually dominated by those quasiparticles that are specularly reflected from the surface; these are just the ones calculated in the low-temperature region treated by Maki,[21] but omitted from our model.

The surface-reactance curves of Fig. 5 also bear a similarity to the experimental results, although the similarity is not quite so striking. In fact, we must note that the "anomalous" negative change in the surface-reactance ratio at high temperatures does not occur at the lowest frequencies, but only above $\hbar\omega/\Delta(0)=0.05$. This seems at variance with some of the experiments[6,8,9,24,25] which exhibit the negative behavior in a somewhat lower-frequency range. It is not clear whether the corrections above could improve the agreement in any way. Recent calculations for the $t=0$ case show that δx becomes temperature-independent at a small, positive value as t goes to zero.

In Figs. 6 and 7, the data are plotted for the same values of frequency as above, but as a function of static field h for the case $\beta=0$. The most striking feature of the results is the complexity of the curves. At the lowest frequencies there is more or less a monatonic decrease in the surface resistance (Fig. 6) with field (above $h=0.1$), but as the frequency is increased, the curves become increasingly complex, in no way resembling the simple quadratic dependence on field that earlier theories have predicted. The curves for the change in the surface reactance (Fig. 7) show a similar pathological behavior. In fact, it is evident from Figs. 6 and 7 that the particular form of the curves of Figs. 4 and 5 depends to a large extent on the value of the static magnetic field h.

In order to examine the theory properly, it is necessary to compare its predictions with experiments on a metal which has $\lambda/\xi\ll 1$. In this case, the corrections mentioned above would be small and the general features of the experiment should resemble those presented here, unless there were a strong dependence on crystalline anisotropy. Unfortunately, the only experiment in the frequency region $0.01\lesssim\hbar\omega/\Delta(0)\lesssim 0.30$ which satisfies the condition $\lambda/\xi\ll 1$ is the superficial study of aluminum by Glosser and Douglas.[11] It is desirable to make a more thorough study than they have done either on aluminum or on some other superconductor with $\lambda/\xi\ll 1$.

2. *Frequency Range* $\hbar\omega/\Delta(0)<0.01$

In this frequency range the results have become quite simple. In fact, if we plot the fractional change in surface reactance $\delta x/x$, there seems to be no frequency dependence. As $\hbar\omega\to 0$, this ratio becomes $\delta\lambda/\lambda$, the fractional change in penetration depth. In Fig. 8 is shown the temperature dependence of $\delta x/x$ for $\beta=0$ and $\frac{1}{2}\pi$ at $h=0.6$. The change is always positive, decreasing near T_c. Recent calculations at $t=0$ show that $\delta x/x$ becomes temperature-independent at low temperature and at $t=0$, $\delta x/x\approx 0.5\%$ for the case $\beta=0$. Comparison of the $\beta=0$ and $\beta=\frac{1}{2}\pi$ curves shows that over most of the range there is about a factor of 3 difference in the two, just the ratio given by Ginzburg-Landau (GL) theory.[29,4] Figure 9 shows the field dependence at several temperatures. The dependence appears to be initially quadratic, as one would expect from the GL theory,[29] and as was found experimentally by Sharvin and Gantmaker[4] and by Connell[7] at $\hbar\omega/\Delta(0)\approx 10^{-5}$. In form, if not in magnitude, the experiments, the predictions of the GL theory, and the present theory agree. To distinguish between the two theories at low frequencies requires that experiments be con-

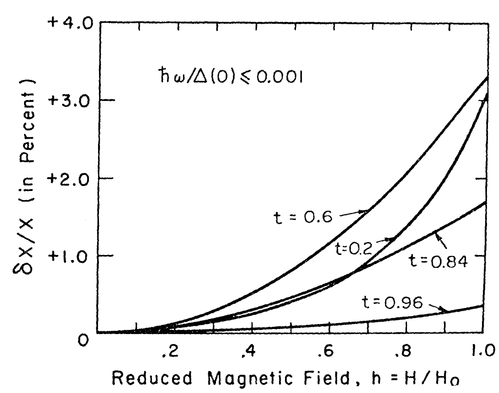

FIG. 9. Low-frequency limit of the fractional change in the surface reactance as a function of static magnetic field for the parallel configuration at various temperatures.

ducted on a superconductor with $\lambda/\xi \ll 1$ to reduce the magnitude of the changes in surface reactance from the GL theory.

Finally, we must comment on the results of the experiment by Josephson,[9] in which he found that the change in the surface reactance with field at $\hbar\omega/\Delta(0) \approx 10^{-3}$ was negative for the $\beta=0$ case at the lowest and highest temperatures and positive in between, i.e., from $0.5 \lesssim t \lesssim 0.92$. This is not a feature of the present results. However, when calculations were carried out for $\delta/\Delta \gtrsim 0.001$ ($=\hbar\omega/\Delta$ for Josephson's experiment), a curious negative change resulted. It is not clear whether this is the origin of Josephson's result or simply an extraneous result of the calculation. The case $\delta \gtrsim \hbar\omega$ must be treated much more carefully than has been done in this calculation before trusting the results. Another observation made by Josephson[9] was that the variation of surface reactance with field at high temperatures ($t \approx 0.97$) is linear to very small fields, and is therefore not analytic about $h=0$. This is not a feature of the present theory for the BCS density of states, but does again appear for nonzero δ. To establish whether this nonanalytic behavior is a consequence of the broadening and reduction in height of the BCS density of states requires a more thorough treatment.

V. CONCLUSION

The model, explaining the effect of a static magnetic field on the microwave surface impedance of superconductors proposed by Budzinski and Garfunkel,[14] has been shown to be successful in predicting many of the features of the experiments. In particular, the "anomalous" decrease in the resistance which occurs at low frequency [$\hbar\omega/\Delta(0) \lesssim 0.1$] was shown to result from depressing the large peak (which appears in the BCS density of states at the edge of the energy gap) by a static magnetic field. This reduction comes about by the addition of a term to the one-particle energy of the form $\mathbf{p}\cdot\mathbf{v}$, where $\mathbf{p}$ is the electron momentum near the Fermi surface and $\mathbf{v}$ is the drift velocity associated with the Meissner current. In addition to the success in predicting the "anomalous" decrease at low frequency, the model succeeds in showing the origin of the absorption at $T=0$ at photon energies near the energy gap. The change from a decrease in the absorption at low frequency to an increase at higher frequency also comes directly from the model. Finally, the effect of impurities is easily seen, since the correction to the quasiparticle energies $\mathbf{p}\cdot\mathbf{v}$ becomes negligible in the limit of short electron free paths. This follows from the Anderson theory of dirty superconductors.[39]

The theory presented here cannot be directly compared with the models that were proposed by Pincus[22] and by Maki.[21] The Pincus[22] theory finds that absorption can occur at photon energies below the gap, but does not give the correct magnitude of the absorption as observed in the experiments of Budzinski and Garfunkel.[12–14] However, his theory, using specular reflection of the electrons at the metal surface, obtains some of the features which are thought of as properties of diffuse reflection. This may make unnecessary the dependence on diffuse reflection which is characteristic of the present work. The model of Maki[21] is somewhat more difficult to examine. He seems to use expressions which are characteristic of specular reflection, but claims to be considering the case of diffuse reflection. At any rate, he does not justify any of his procedure and would not get results that in any way resemble the high-frequency experiments.[12–14] Furthermore, he has not worked out the details of the low-frequency predictions of his model.

For a better comparison with experiment, it is desirable to examine superconductors in which the $\mathbf{p}\cdot\mathbf{v}$ correction dominates the field dependence, i.e., superconductors with a small value of λ/ξ rather than those such as tin ($\lambda/\xi \approx 0.2$) in which the effect of the $\mathbf{p}\cdot\mathbf{v}$ term is diluted by the presence of other sources of the field dependence of the surface impedance. There is still one prediction to be examined, that is, the decrease in absorptivity with field at frequencies given by $\hbar\omega/\Delta(0) > 2.5$.

ACKNOWLEDGMENTS

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[39] P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).