PHYSICAL REVIEW VOLUME 173, NUMBER 2 10 SEPTEMBER 1968

Fermi-Liquid Effects on Plasma Wave Propagation in Metals*

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(Received 2 February 1968)

The dispersion relation for plasma waves propagating perpendicular to a dc magnetic field in a metal with a spherical Fermi surface is studied. It is found that the dispersion relation differs only slightly from the prediction of the free-electron model. The experimental data can be used to estimate the value of the interaction coefficient A_2, but they should be independent of A_0 and A_1.

THE propagation of plasma waves near the fundamental cyclotron resonance in potassium has been studied recently by Walsh and Platzman.[1,2] The experiments are carried out in the Azbel-Kaner geometry, with samples which are thin, plane-parallel slabs. The plasma waves propagate perpendicular to the dc magnetic field $\mathbf{B}$, and may be polarized either parallel or perpendicular to $\mathbf{B}$. In the former case,[1] the experimental data differs slightly from the prediction of the free-electron model, while in the latter,[2] there appears to be considerable deviation. Platzman and Walsh attributed the deviation to Fermi-liquid effects, and presented an analysis which involved only the Fermi-liquid interaction coefficients A_0 and A_1. This analysis contained some numerical errors,[3,4] and the agreement of theory and experiment was fortuitous. The object of this paper is to present the results of a study of plasma wave propagation in a situation in which the interaction coefficients A_n can be neglected for $n>2$.

We begin with the kinetic equation for a collisionless electron liquid[5]:

$$[-i\omega+i\mathbf{q}\cdot\mathbf{v}+\omega_c(\partial/\partial\phi)]f(\theta,\phi) \\ +[i\mathbf{q}\cdot\mathbf{v}+\omega_c(\partial/\partial\phi)]\delta\epsilon_1(\theta,\phi)=-e\mathbf{E}\cdot\mathbf{v}. \quad (1)$$

Here θ and ϕ are angles denoting a direction in $\mathbf{k}$ space and $f(\theta,\phi)$ is defined by

$$\delta f(\mathbf{k})=(-\partial f_0/\partial\epsilon)f(\theta,\phi), \quad (2)$$

where $\delta f(\mathbf{k})$ is half the trace with respect to spin of the deviation from thermal equilibrium of the density matrix caused by the electric field $\mathbf{E}$. We have assumed space-time dependence of the form $\exp(-i\omega t+i\mathbf{q}\cdot\mathbf{r})$, and taken the dc magnetic field to lie in the z direction $(\theta=0)$. Without loss of generality we can choose $\mathbf{q}$ to lie in the x-z plane. The electron cyclotron frequency is denoted by ω_c, and the function $\delta\epsilon_1$ is given by

$$\delta\epsilon_1(\mathbf{k})=\frac{2}{(2\pi)^3}\int d^3k'\Phi(\mathbf{k},\mathbf{k}')\delta f(\mathbf{k}'), \quad (3)$$

where $\Phi(\mathbf{k},\mathbf{k}')$ is the spin-independent part of the interaction function.[6] We introduce $\mathbf{R}(\theta,\phi)$, the periodic part of the position vector in real space of an electron on the Fermi surface, and note that

$$\exp[i\mathbf{q}\cdot\mathbf{R}(\theta,\phi)]=\sum_{m=-\infty}^{\infty}J_m(X\sin\theta)e^{im\phi}. \quad (4)$$

Here $X=q_xv_F/\omega_c$ and J_m is the mth-order Bessel function. We define the Fourier coefficients $f_m(\theta)$, $\mathbf{v}_m(\theta)$, and $\Phi_m(\theta,\theta')$ by

$$f(\theta,\phi)\exp[+i\mathbf{q}\cdot\mathbf{R}(\theta,\phi)]=\sum_m f_m(\theta)e^{im\phi},$$

$$\mathbf{v}(\theta,\phi)\exp[+i\mathbf{q}\cdot\mathbf{R}(\theta,\phi)]=\sum_m \mathbf{v}_m(\theta)e^{im\phi},$$

and

$$\Phi(\mathbf{k},\mathbf{k}')=\sum_m\Phi_m(\theta,\theta')\exp[im(\phi-\phi')]. \quad (5)$$

In terms of these functions the kinetic equation can be written

$$f_n(\theta)=\frac{-ie\mathbf{E}\cdot\mathbf{v}_n}{\omega-q_zv_z-n\omega_c}+\left(-1+\frac{\omega}{\omega-q_zv_z-n\omega_c}\right)Y_n(\theta), \quad (6)$$

where the function $Y_n(\theta)$ is given by

$$Y_n(\theta)=\frac{m^*k_F}{2\pi^3}\sum_{ml}J_{n-l}(X\sin\theta)\int d(\cos\theta')\Phi_l(\theta,\theta') \\ \times J_{m-l}(X\sin\theta')f_m(\theta'). \quad (7)$$

Here m^* is the effective mass and k_F the Fermi momentum (we set $\hbar=1$ throughout). The function $\Phi_l(\theta,\theta')$ can easily be expressed in terms of the usual Fermi-liquid interaction coefficients A_n:

$$\Phi_l(\theta,\theta')=(4\pi^3/m^*k_F)\sum_{n\geq|l|}A_n(L_n^l)^2P_n^{|l|}(\theta)P_n^{|l|}(\theta'), \quad (8)$$

* Supported in part by the Advanced Research Projects Agency, and the National Science Foundation under Grant No. GP 4815.
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[2] P. M. Platzman and W. M. Walsh, Jr., Phys. Rev. Letters **19**, 514 (1967).
[3] A. L. McWhorter and D. Hamilton (private communication).
[4] S. C. Ying and J. J. Quinn, Phys. Rev. Letters **20**, 1007 (1968).
[5] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 495 (1957) [English transl.: Soviet Phys.—JETP **6**, 387 (1958)].
[6] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956) [English transl.: Soviet Phys.—JETP **3**, 920 (1956)].

where $L_n^l=[(2n+1)(n-|l|)!/4\pi(n+|l|)!]^{1/2}$ and $P_n^l(\theta)$ is an associated Legendre polynomial. By substituting (8) into (7) we obtain

$$Y_n(\theta)=\sum_{s\geq|l|}\sum_l \Gamma_{ls}A_s(L_s^l)^2P_s^{|l|}(\theta)J_{n-l}(X\sin\theta), \quad (9)$$

where

$$\Gamma_{ls}=2\sum_m\int d(\cos\theta)J_{m-l}(X\sin\theta)P_s^{|l|}(\theta)f_m(\theta). \quad (10)$$

If $A_n=0$ for all n greater than some value N, there are $(N+1)^2$ different Γ_{ls}. In this paper we take $N=2$; thus there are nine coefficients Γ_{ls}. It is apparent that because of the form of $\Phi_l(\theta,\theta')$, one can solve the integral equation (6) by multiplying by $P_{s'}^{|l'|}(\theta)J_{n-l'}(X\sin\theta)$, integrating over $\cos\theta$, and summing over n. The integral equation is then reduced to $(N+1)^2$ simultaneous equations for the coefficients $\Gamma_{l's'}$.

For propagation perpendicular to the dc magnetic field, the 9×9 matrix equation (for the case $A_n=0$ if $n>2$) for the coefficients Γ_{ls} separates into a 3×3 and a 6×6. The solution of the 3×3 matrix equation gives σ_{zz}, and hence determines the dispersion relation for plasma waves polarized parallel to **B**. The solution of the 6×6 gives σ_{xx}, σ_{yy}, σ_{xy}, and σ_{yx}, and determines the dispersion relation for the other polarization. We have studied the solutions of both of these equations by the same approximate method. Theoretical estimates[7] of the interaction coefficients indicate that $|A|_0$ is of the order of a few tenths, while $|A_1|$ and $|A_2|$ are smaller by roughly one order of magnitude. We therefore consider A_0 to be of first order, A_1 and A_2 of second order, A_0A_1 and A_0A_2 of third order, etc. In solving the 3×3 matrix equation, we retain terms up to third order in *both the numerator and denominator* of the correction to σ_{ij} caused by Fermi-liquid effects.[8] In other words we write

$$\sigma_{zz}=\sigma_{zz}^0+N(A_n)/D(A_n), \quad (11)$$

where σ_{zz}^0 is the value of σ_{zz} in the absence of Fermi-liquid effects, and $N(A_n)$ and $D(A_n)$ are functions of A_0, A_1, A_2, which are accurate up to terms of third-order. When this is done, the dispersion relation[9] for waves polarized parallel to **B**, viz., $\sigma_{zz}\approx0$, becomes

$$X^2L[1+2A_2-\tfrac{15}{2}A_2a^2M]+\tfrac{5}{3}A_2[(3a^2L-1)^2+(\tfrac{3}{2}aXL')^2]=0, \quad (12)$$

where $a=\omega/\omega_c$, $X=q_xv_F/\omega_c$, $L'=\partial L/\partial X$, and

$$L=\sum_{m=0}^{\infty}(1+\delta_{m0})^{-1}S_m(a^2-m^2)^{-1}, \quad (13)$$

and

$$M=\sum_{m=0}^{\infty}(1+\delta_{m0})^{-1}(T_{m+1}+T_{m-1})(a^2-m^2)^{-1}. \quad (14)$$

In (13) and (14)

$$S_m=\int d(\cos\theta)\cos^2\theta\, J_m^2(X\sin\theta),$$

$$T_m=\int d(\cos\theta)\sin^2\theta\cos^2\theta\, J_m^2(X\sin\theta). \quad (15)$$

Notice that the dispersion relation does not depend on A_0 or A_1 at all.[3] The solution of (12) has been obtained numerically for several values of the parameter A_2. In Fig. 1 we display the experimental data of Walsh and Platzman[1] together with the solutions of (12) for $A_2=-0.026$, -0.036, and 0. We see that the experimental data favor a value of $A_2\approx-0.026$. The long-wavelength limit[10] (i.e., $X^2\ll|A_2|$) of the dispersion relation can be written[11]

$$a\approx1+A_2-\tfrac{1}{10}X^2[1+(20/7)A_2]. \quad (16)$$

Thus, we see, the main effect of the Fermi-liquid interaction is the shift of a from the value of unity of $X=0$; the change in the coefficient of X^2 is of the order of only 10%.

We have used the same approximation to study the other polarization. It must be mentioned that con-

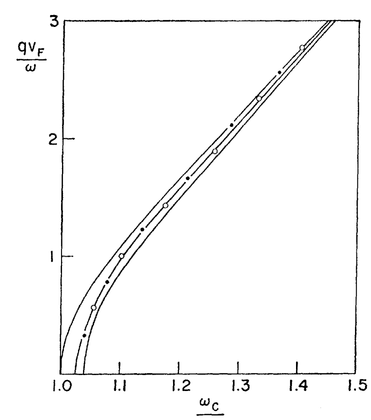

FIG. 1. Plot of ω_c/ω versus qv_F/ω for plasma waves polarized parallel to the dc magnetic field. The open and solid circles represent the experimental minima and maxima, respectively, of the derivative with respect to B of the power absorbed (Ref. 2). The three curves are results of the present calculation for values of A_2 of 0, -0.026, and -0.036, going from left to right.

[7] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **34**, 781 (1958) [English transl.: Soviet Phys.—JETP **3**, 538 (1958)].

[8] It should be pointed out the calculation which attempts to represent the change in $f(\theta,\phi)$ or σ by a power series in A_n (or in qR) cannot very easily reproduce results of the type obtained here.

[9] We have been informed that a similar dispersion relation has been obtained independently by P. M. Platzman, W. M. Walsh, Jr., and E.-Ni Foo, Phys. Rev. **172**, 689 (1968).

[10] The functions Γ_{ls} defined by Eq. (10) can be shown to be proportional to the F_{sl} used (V. P. Silin, Zh. Eksperim. i Teor. Fiz. **35**, 1243 (1958) [English transl.: Soviet Phys—JETP **8**, 870 (1959)]) in the expansion of $f=\sum_{sl}F_{sl}Y_s^l$ in spherical harmonics. For $q=0$, the different spherical harmonics are independent of one another. The mode displayed in Fig. 1 corresponds at $q=0$ to the $s=2$, $l=1$ mode.

[11] The result $a=1+A_2$ for $X=0$ was obtained by a different method by P. M. Platzman and W. M. Walsh (private communication).

siderable care must be used in treating the dispersion relation[2] $\sigma_{xx}\sigma_{yy}+\sigma_{xy}^2\approx 0$, because of the cancellation of a factor $D(A_n)$ between numerator and denominator. We obtain the correct dispersion relation using our approximate solution of the 6×6 matrix equation after demonstrating that this cancellation must occur. The full dispersion relation is too complicated to display here, and its long-wavelength limit is of no value because nonlinear terms in A_2 become important for very small X. We can however, make the following statements about the dispersion relation: first, it contains no Fermi-liquid terms lower than fourth order except the term linear in A_2, and second, in contrast to the other polarization, there is no shift of ω away from the value ω_c for $X=0$. In this respect the dispersion relation is much closer to the prediction of the free-electron model than the experimental data.[12] The exact numerical results for both polarizations will be presented in a more detailed later publication.

The authors would like to acknowledge several discussions with P. M. Platzman and W. M. Walsh, Jr.

[12] W. M. Walsh, Jr., has informed us that further experimental studies have shown that the original assignment of the wavelengths of the plasma waves in Ref. 2 was incorrect, and that the correct experimental data does lie much closer to the prediction of the free-electron model.

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Free Energy of the Classical Heisenberg Model

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High-temperature power-series expansions for the free energy of a classical Heisenberg ferromagnet in an applied field are given in the form

$$-F/NkT=\sum_{n,l} a_{2n,l}h^{2n}x^l,$$

where $h=g\beta H/kT$ and $x=J/kT$. The coefficients are given for $l\leq 7$ and $4\leq 2n\leq 10$. Estimates of the critical exponents for the fourth to tenth field derivatives of F are given.

I. INTRODUCTION

In recent years much work has been done on obtaining high-temperature series expansions of thermodynamic functions for the Heisenberg model and the estimation, from these series, of various critical parameters. For general spin only six terms of the susceptibility expansion and five terms of the specific-heat expansion are known.[1] For the special cases of $S=\frac{1}{2}$ and $S=\infty$ further terms are known. Baker *et al.*[2] have given the free energy for the $S=\frac{1}{2}$ Heisenberg model to order $1/T^{10}$ [for the bcc and simple cubic (sc) lattices] and $1/T^9$ (for the fcc lattice) and also the field dependence in powers of the applied field H up to $(H/T)^8$.

It is the purpose of the present paper to present the results of a similar calculation for the $S=\infty$ Heisenberg model. Currently, high-temperature expansions for the zero-field susceptibility and specific heat are known to 8 and 9 terms for close-packed lattices and 9 and 10 terms for open lattices, respectively, using this model.[3] We shall give the temperature and field dependence to orders $1/T^7$ and $(H/T)^{10}$.

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[2] G. A. Baker, H. E. Gilbert, J. Eve, and G. S. Rushbrooke, Phys. Letters **20**, 146 (1966); **22**, 269 (1966); Phys. Rev. **164**, 800 (1967).
[3] H. A. Brown and J. M. Luttinger, Phys. Rev. **100**, 685 (1955); H. E. Stanley and T. A. Kaplan, Phys. Rev. Letters **16**, 981 (1966); P. J. Wood and G. S. Rushbrooke, *ibid.* **17**, 307 (1966); G. S. Joyce and R. G. Bowers, Proc. Phys. Soc. (London) **88**, 1053 (1966).

In Sec. II we shall describe how the calculation was performed. Section III will be concerned with the susceptibility series, Sec. IV with the high-temperature series proportional to powers of the applied field greater than 2. Finally, in Sec. V we consider some two-dimensional lattices.

II. CALCULATION OF THE FREE ENERGY

We start with the Hamiltonian

$$\mathcal{H}=-2J\sum_{\langle ij\rangle}\mathbf{S}^{(i)}\cdot\mathbf{S}^{(j)}-g\beta H\sum_i S_z^{(i)}=\mathcal{H}_1+\mathcal{H}_0, \quad (1)$$

where J is the exchange-energy constant for nearest-neighbor interactions, $\mathbf{S}^{(i)}$ the spin vector on lattice site i, g the gyromagnetic ratio, β the Bohr magneton, and H the applied field (taken to be in the z direction). We use the abbreviation $S(S+1)=X$.

We introduce a new vector $\mathbf{T}^{(i)}=\mathbf{S}^{(i)}/\sqrt{X}$. It is then easily seen that in the limit $S\rightarrow\infty$, $\mathbf{T}^{(i)}$ becomes a unit classical vector. In terms of the vectors $\mathbf{T}^{(i)}$ the Hamiltonian becomes

$$\mathcal{H}=-2JX\sum_{\langle ij\rangle}\mathbf{T}^{(i)}\cdot\mathbf{T}^{(j)}-g\beta H\sqrt{X}\sum_{i=1}^{N}\mathbf{T}_z^{(i)}. \quad (2)$$

In the following we shall write J for JX and H for