

the manuscript. I also wish to thank Phillip M. Yager for fitting these data to an algebraic equa-

tion and Dr. E. Dwight Adams for use of his data on the melting curve.

*This research was supported by the National Science Foundation and submitted in partial fulfillment of the requirements for the Ph.D. degree at the University of California, San Diego, California.

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found to be negligible (that is, less than the noise of ± 0.02 atm) by performing a demagnetization with no He^3 in the container. Hysteresis in the nylon was not present at the melting curve minimum or at calibration temperatures, but unfortunately could not be measured directly at low temperatures because of a rapid warm-up time which prevented the sample from achieving thermal equilibrium while warming.

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Phonon-Quasiparticle Interactions in Dilute Solutions of He^3 in Superfluid He^4 : III. Attenuation of First Sound Above 0.2°K *

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(Received 5 April 1968)

Utilizing a previously developed phonon Boltzmann equation we extend to higher temperatures an earlier calculation of first sound attenuation in dilute solutions of He^3 in superfluid He^4 . The present calculation takes into account thermal phonon- He^3 scattering and is valid up to about 0.7°K . The results are in good overall agreement with experiment.

I. INTRODUCTION

In this paper we extend to higher temperatures the theory¹ of ultrasonic attenuation in dilute solutions of He^3 in superfluid He^4 . For temperatures below about 0.3°K for He^3 molar concentrations x of a few percent the primary mechanism responsible for sound attenuation is absorption by the He^3 impurities, a process which is kinematically possible because of He^3 - He^3 interactions. In the temperature range from 0.3°K to about 0.6 – 0.7°K ,

attenuation due to thermal phonons becomes important. This latter process, to which we address ourselves in this work, depends critically on the form of the scattering of thermal phonons by the He^3 . For temperatures above 0.7°K rotons and phonon-phonon scattering, neglected here, also become important.

Because He^3 - He^3 scattering rates are so relatively rapid, the phonon- He^3 collision integral appearing in the phonon Boltzmann equation may be evaluated under the assumption that the He^3

quasiparticles maintain local equilibrium among themselves. This simplification makes it possible to reduce the collision integral to a form which may be easily handled; the details have been presented in a previous paper.² The phonon-He³ scattering matrix essentially contains only elastic s , p , and d wave parts; for this reason the phonon Boltzmann equation is exactly soluble. We are

thus able to obtain the sound attenuation for arbitrary frequency ω subject only to the requirement that for temperatures where He³-phonon scattering is important we must have $\omega\tau_\eta \ll 1$, where τ_η is the He³-He³ scattering time appropriate to viscosity. The hydrodynamic limit of our result is of interest since measurements in this regime would enable us to refine further our estimates of τ_η .

II. EQUATIONS OF MOTION

In this section we present the set of equations from which the sound attenuation will be derived. Linearized equations will be used throughout. We start with the phonon and He³ Boltzmann equations

$$\partial n_{\vec{q}}/\partial t + \vec{\nabla}_{\vec{q}} \omega_{\vec{q}} \cdot \vec{\nabla}_{\vec{r}} n_{\vec{q}} - \vec{\nabla}_{\vec{r}} \omega_{\vec{q}} \cdot \vec{\nabla}_{\vec{q}} n_{\vec{q}} = I_{3\text{-ph}} + I_{\text{ph-ph}} \quad (1)$$

$$\text{and } \partial f_{\vec{p}}/\partial t + \vec{\nabla}_{\vec{p}} \epsilon_{\vec{p}} \cdot \vec{\nabla}_{\vec{r}} f_{\vec{p}} - \vec{\nabla}_{\vec{r}} \epsilon_{\vec{p}} \cdot \vec{\nabla}_{\vec{p}} f_{\vec{p}} = I_{3\text{-3}} + I_{\text{ph-3}} \quad (2)$$

together with the equation of motion for superfluid flow

$$m_4 n_4 \partial \vec{v}_s/\partial t + n_4 \vec{\nabla} \mu_4 = 0. \quad (3)$$

$n_{\vec{q}}$ and $f_{\vec{p}}$ are the phonon and He³ distribution functions, $\epsilon_{\vec{p}}$ and $\omega_{\vec{q}}$ are the local He³ quasiparticle and phonon energies, m_4 is the He⁴ atomic mass, n_4 is the He⁴ number density, and μ_4 is the He⁴ chemical potential. $I_{3\text{-ph}}$, $I_{3\text{-3}}$, $I_{\text{ph-3}}$, and $I_{\text{ph-ph}}$ are the collision integrals for the various He³ and phonon collision processes. In the presence of superfluid flow we have (in linear order)¹

$$\epsilon_{\vec{p}} = \epsilon_0 + p^2/2m + (\delta m/m) \vec{p} \cdot \vec{v}_s \quad (4)$$

$$\text{and } \omega_{\vec{q}} = sq + \vec{q} \cdot \vec{v}_s, \quad (5)$$

where ϵ_0 is the He³ chemical potential at zero concentration, $\delta m = m - m_3$ is the difference between the He³ quasiparticle mass m and the He³ atomic mass m_3 , and s is the first sound velocity in pure He⁴ at zero temperature; ϵ_0 , m , and s depend on position and time through their dependence on density.

We now use equations (1)–(5) to obtain expressions for He³ number conservation and overall mass, energy, and momentum conservation.³ Summing (2) over all \vec{p} , using (4), and noting that collisions conserve He³ atoms we have

$$\partial n_3/\partial t + n_3 \vec{\nabla} \cdot \vec{v}_3 = 0; \quad (6)$$

here n_3 is the He³ number density and the He³ velocity \vec{v}_3 is defined by

$$\sum_{\vec{p}} \vec{p} f_{\vec{p}} = n_3 (m \vec{v}_3 - \delta m \vec{v}_s). \quad (7)$$

The equation expressing conservation of mass is

$$m_4 \partial n_4/\partial t + m_3 \partial n_3/\partial t + \vec{\nabla} \cdot \vec{g} = 0, \quad (8)$$

where the total momentum density is given by

$$\vec{g} = \sum_{\vec{p}} \vec{p} f_{\vec{p}} + \sum_{\vec{q}} \vec{q} n_{\vec{q}} + m_4 n_4 \vec{v}_s. \quad (9)$$

In linear order

$$\sum_{\vec{q}} \vec{q} n_{\vec{q}} = \rho_{\text{ph}} (\vec{v}_{\text{ph}} - \vec{v}_s), \quad (10)$$

thus defining the phonon normal velocity \vec{v}_{ph} ; the phonon normal mass density ρ_{ph} is given by the Landau result (working in unit volume)

$$\rho_{\text{ph}} = \sum_{\vec{q}} (q^2/3) \partial n_q^0/\partial \omega_q = (2\pi^2/45)(KT)^4/s^3 \hbar^3. \quad (11)$$

$n_q^0 = [\exp(\beta\omega_q) - 1]^{-1}$ is the equilibrium phonon distribution function and $\beta = 1/KT$. Using (7), (9), and (10) we may rewrite (8) in the form

$$m_4 \partial n_4 / \partial t + m_3 \partial n_3 / \partial t + \rho_s \vec{\nabla} \cdot \vec{v}_s + m n_3 \vec{\nabla} \cdot \vec{v}_3 + \rho_{ph} \vec{\nabla} \cdot \vec{v}_{ph} = 0, \quad (12)$$

in which $\rho_s = m_4 n_4 - \delta m n_3 - \rho_{ph}$ is the superfluid mass density.

The equation for conservation of momentum may be derived by multiplying (2) by \vec{p} and summing over \vec{p} , multiplying (1) by \vec{q} and summing over \vec{q} , and noting that total momentum is conserved in all collisions; we find that

$$\begin{aligned} \partial \vec{g} / \partial t &= (\partial / \partial t) \left(\sum_{\vec{p}} \vec{p} \delta f_{\vec{p}} + n_3 \delta m \vec{v}_s + \sum_{\vec{q}} \vec{q} \delta n_{\vec{q}} + \rho_{ph} \vec{v}_s + \rho_s \vec{v}_s \right) \\ &= -n_4 \vec{\nabla} \mu_4 - n_3 \vec{\nabla} \epsilon_0 - \rho_{ph} s \vec{\nabla} s + \frac{5}{2} P_f(T) (1/m) \vec{\nabla} m - \vec{\nabla} \cdot \left[\sum_{\vec{p}} (\vec{p} \vec{p} / m) \delta f_{\vec{p}} + \sum_{\vec{q}} (s \vec{q} \vec{q} / q) \delta n_{\vec{q}} \right]. \end{aligned} \quad (13)$$

$\delta f_{\vec{p}}$ and $\delta n_{\vec{q}}$ are the deviations of the distribution functions from global equilibrium and $P_f(T)$ is the pressure of a free Fermi gas composed of particles of mass m and having number density n_3 .

We may rewrite $\vec{\nabla} \cdot \sum_{\vec{p}} (\vec{p} \vec{p} / m) \delta f_{\vec{p}}$ in the hydrodynamic limit¹ as

$$\begin{aligned} \vec{\nabla} \cdot \sum_{\vec{p}} (\vec{p} \vec{p} / m) \delta f_{\vec{p}} &= \sum_{\vec{p}} (\vec{p} \cdot \vec{p} / 3m) \vec{\nabla} f_{\vec{p}} + \sum_{\vec{p}} [\vec{p} \vec{p} \cdot \vec{\nabla} / m - (\vec{p} \cdot \vec{p}) \vec{\nabla} / 3m] f_{\vec{p}} \\ &= \sum_{\vec{p}} (p^2 / 3m) \vec{\nabla} f_{\vec{p}} - \eta_3 [\nabla^2 \vec{v}_3 + \frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{v}_3)] \end{aligned} \quad (14)$$

where $\eta_3 = P_f(T) \tau_\eta$ is the He³ first viscosity and τ_η is the He³-He³ quasiparticle collision time appropriate to viscosity. More generally, whenever $v k \ll |\omega + i / \tau_\eta|$, where v is a typical He³ velocity, η_3 in (14) is replaced by $\eta_3 (1 - i \tau_\eta \omega)^{-1}$.

The equation for energy conservation can be derived if we multiply (1) by $s q$ and sum over \vec{q} , multiply (2) by $p^2 / 2m$ and sum over \vec{p} , and add the results taking into account overall energy conservation in collisions; we obtain

$$(\partial / \partial t) \left(\sum_{\vec{p}} (p^2 / 2m) \delta f_{\vec{p}} + \sum_{\vec{q}} s q \delta n_{\vec{q}} \right) + \frac{5}{2} P_f(T) \vec{\nabla} \cdot \vec{v}_3 + s^2 \rho_{ph} \vec{\nabla} \cdot \vec{v}_{ph} = 0. \quad (15)$$

In deriving (15) the thermal conductivity of the He³ has been assumed negligible; this assumption is valid for temperatures greater than about 20-°0 m°K^{1,4} for the He³ concentrations considered.

For convenience we reproduce at this point the expression given in II (Eq. 44) for variations in μ_4 ,

$$\delta \mu_4 = \left(\frac{\partial \mu_4}{\partial n_4} \right)_{n_{\vec{q}}, f_{\vec{p}}} \delta n_4 + \sum_{\vec{p}} \left(\frac{\partial \epsilon_{\vec{p}}}{\partial n_4} \right)_{n_{\vec{q}}, f_{\vec{p}}} \delta f_{\vec{p}} + \sum_{\vec{q}} \left(\frac{\partial \omega_{\vec{q}}}{\partial n_4} \right)_{n_{\vec{q}}, f_{\vec{p}}} \delta n_{\vec{q}}. \quad (16)$$

When working in the linear approximation, the derivatives at constant $n_{\vec{q}}$ and $f_{\vec{p}}$ in (16) may be replaced by equilibrium derivatives at constant n_3 and total entropy S ; hence,

$$\delta \mu_4 = \left(\frac{\partial \mu_4}{\partial n_4} \right)_{n_3, S} \delta n_4 + \left(\frac{\partial \epsilon_0}{\partial n_4} \right)_{n_3, S} \delta n_3 + \frac{s u_1}{n_4} \sum_{\vec{q}} q \delta n_{\vec{q}} - \frac{u_2}{n_4} \sum_{\vec{p}} \frac{p^2}{2m} \delta f_{\vec{p}}, \quad (17)$$

where⁵ $u_1 = (n_4 / s) (\partial s / \partial n_4)_{n_3, S} \approx 2.7$, and⁶ $u_2 = (n_4 / m) (\partial m / \partial n_4)_{n_3, S} \approx 1.24$.

III. FIRST SOUND ATTENUATION

Our first step in this section is to derive an expression for the attenuation of first sound in terms of the response of the He³ and the phonons to a local variation δn_4 in the He⁴ density. Only longitudinal variations are considered here. We work with the Fourier transformed forms of the equations given in the previous section. Combining (17) with equations (3), (6), (12), and (15) gives

$$\begin{aligned} \frac{\omega^2}{k^2} - \frac{\rho_s}{m_4^2} \left(\frac{\partial \mu_4}{\partial n_4} \right)_{n_3, S} &= \frac{\rho_s}{\rho_4} \left[\frac{n_4}{m_4} \left(\frac{\partial \epsilon_0}{\partial n_4} \right)_{n_3, S} + \frac{\delta m}{m_4} \frac{\rho_4}{\rho_s} \frac{\omega^2}{k^2} - \frac{5}{2} \frac{u_2 P_f(T)}{m_4 n_3} \right] \frac{\delta n_3}{\delta n_4} \\ &+ \frac{s \rho_{ph} \rho_s}{m_4 \rho_4} \frac{\delta}{\delta n_4} \left[\frac{\omega}{s k} \left(\frac{\rho_4}{\rho_s} - u_2 \frac{s^2 k^2}{\omega^2} \right) v_{ph} - 3(u_1 + u_2) \langle \Phi_0 \rangle \right]. \end{aligned} \quad (18)$$

Here $\rho_4 = m_4 n_4$ and we have introduced the notation

$$\delta n_{\vec{q}} = q \frac{\partial n_q^0}{\partial \omega_q} \Phi(\vec{q}) = q \frac{\partial n_q^0}{\partial \omega_q} \sum_{l=0}^{\infty} P_l(\hat{k} \cdot \hat{q}) \Phi_l(q);$$

the average value of a quantity $A(q)$ is defined as $\langle A \rangle = -(1/\rho_{\text{ph}} \sum_{\vec{q}} (q^2/3) (\partial n_q^0 / \partial \omega_q) A(q)$.

$\delta n_3 / \delta n_4$ and $\delta \Phi / \delta n_4$ are the total variations in n_3 and Φ due to variations in n_4 . For the low temperatures and low concentrations of interest we may set

$$\left(\frac{\partial \epsilon_0}{\partial n_4} \right) n_3, S \approx \left(\frac{\partial \mu_3}{\partial n_4} \right) n_3, S = (1 + \alpha) \left(\frac{\partial \mu_4}{\partial n_4} \right) n_3, S = (1 + \alpha) \frac{m_4 s^2}{n_4}, \quad (19)$$

the zero concentration result, where $\alpha \approx 0.28$ is the fractional excess molar volume of He³ in He⁴. The attenuation amplitude α_I of first sound associated with the solution of (18) for ω near sk is easily shown to be

$$\alpha_I = (\omega/2s)(1 + \alpha + \delta m/m_4) \text{Im}(\delta n_3 / \delta n_4) + (\omega \rho_{\text{ph}} / 2s^2 m_4) \text{Im}(\delta / \delta n_4) [3(u_1 + u_2) \langle \Phi_0 \rangle - (1 - u_2) v_{\text{ph}}] \quad (20)$$

to lowest order in x and ρ_{ph}/ρ_4 . Note that $v_{\text{ph}} = v_S - \langle \Phi_1 \rangle$ (see Eq. 10).

We must now calculate the He³ response $\delta n_3 / \delta n_4$ and the phonon response $\delta \langle \Phi(\vec{q}) \rangle / \delta n_4$ to He⁴ density variations. It is clear that it will be necessary to include in $\delta n_3 / \delta n_4$ terms of order ρ_{ph}/ρ_4 corresponding to phonon drag on the motion of the He³ quasiparticles. To zero order in ρ_{ph}/ρ_4 and x the quantity $\delta \langle \Phi(\vec{q}) \rangle / \delta n_4$ is determined solely by He³-phonon scattering and absorption processes.

The calculation of $\text{Im}(\delta n_3 / \delta n_4)$ is straightforward. We use Eq. (15) to eliminate $\sum_{\vec{p}} p^2 \delta f_{\vec{p}}$ in (14) [with the factor $(1 - i\omega\tau_{\eta})^{-1}$ included] and substitute the resulting expression for the He³ stress tensor in (13). We then use (3) to eliminate μ_4 in (13), (12) to eliminate v_S and finally (6) to eliminate v_3 . Assuming $\rho_{\text{ph}} \ll mn_3$ we find, to first order in x and ρ_{ph}/ρ_4 , and for $\omega = sk$, that

$$\text{Im} \delta n_3 / \delta n_4 = - (4\lambda^2 \omega \eta_3 / 3mn_4 s^2) [1 / (1 + \omega^2 \tau_{\eta}^2)] + (p_{\text{ph}} / ms) \text{Im}(\delta / \delta n_4) (\langle \Phi_0 \rangle - \frac{1}{3} v_{\text{ph}} - \frac{2}{5} \langle \Phi_2 \rangle), \quad (21)$$

where $\lambda = (m_4/m)[1 + \alpha + (\delta m/m_4)] \approx 1.30$. With this result, (20) may be rewritten as

$$\alpha_I = (2\omega^2 \lambda^2 / 3s^3 \rho_4) [\eta_3 / (1 + \omega^2 \tau_{\eta}^2)] + (\omega \rho_{\text{ph}} / 2s^2 m_4) \text{Im}(\delta / \delta n_4) [(3u_1 + 3u_2 - \lambda) \langle \Phi_0 \rangle - (1 - u_2 - \lambda/3) v_{\text{ph}} + \frac{2}{5} \lambda \langle \Phi_2 \rangle]. \quad (22)$$

The first term in (22) is the attenuation due to the He³ viscosity.

To calculate the response of $\Phi(\vec{q})$ to variations in n_4 we turn now to the phonon Boltzmann equation, given in II [Eq. (27)] as

$$\frac{\partial n_{\vec{q}}}{\partial t} + \vec{\nabla}_{\vec{q}} \omega_{\vec{q}} \cdot \vec{\nabla}_{\vec{r}} n_{\vec{q}} - s \hat{q} \cdot \vec{\nabla}_{\vec{r}} (s q + \vec{q} \cdot \vec{v}_S) \frac{\partial n_q^0}{\partial \omega_q} = - q \frac{\partial n_q^0}{\partial \omega_q} \left[\frac{\Phi(\vec{q}) + \hat{q} \cdot (\vec{v}_3 - \vec{v}_S) - \Phi_0^{\text{le}}}{\tau_a(q)} + \frac{\hat{q} \cdot (\vec{v}_3 - \vec{v}_S)}{\tau_1(q)} + \sum_{l=1}^{\infty} \frac{\Phi_l(q)}{\tau_l(q)} P_l(\cos \theta_q) \right] + I_{\text{ph-ph}}. \quad (23)$$

$\Phi_0^{\text{le}} = \beta^{-1} \delta(\beta_3 s)$ is the local equilibrium value of $\Phi_0(q)$. In deriving the right side of (23) it was assumed that the He³ quasiparticles were in local thermodynamic equilibrium at temperature $K/\beta_3 = T_3 = T + \delta T_3$ and velocity v_3 . The phonon-He³ scattering lifetimes $\tau_l(q)$ are given by

$$\tau_1(q) = 4.35 n_4 / x s q^4, \quad \tau_2(q) = 5.96 n_4 / x s q^4, \quad \tau_l(q) = 5.48 n_4 / x s q^4 \equiv \tau_0(q) \text{ for } l \geq 3.$$

The time τ_a characterizing phonon absorption and emission by the He³ quasiparticles is given by

$$\tau_a^{-1}(q) = \frac{4}{3} [P_f(T) / \rho_4] \frac{\lambda^2 q^2 \tau_{\eta} [1 + (\hbar s q / 2\pi K T)^2]}{[1 + (\hbar s q / 2\pi K T)^2] + s^2 q^2 \tau_{\eta}^2}. \quad (24)$$

Since all of the $\tau_l(q)$ are equal for $l \geq 3$, the phonon Boltzmann equation is exactly soluble for temperatures below roughly 0.6–0.7°K where the phonon-phonon collision integral $I_{\text{ph-ph}}$ is negligible.

Upon Fourier transformation, (23) becomes

$$(\omega - sk\xi + i/\tau_z) \Phi(\vec{q}) + sk\xi(\delta s + \zeta v_s) = i\Phi_0^{le}/\tau_a + i\Phi_0/\tau_0 + i\zeta\Phi_1/t_1 + i\zeta(v_s - v_3)/\tau_K + iP_2(\xi)\Phi_2/t_2, \quad (25)$$

where $\xi = \hat{k} \cdot \hat{q}$ and for simplicity we have used the notation $1/t_1 = 1/\tau_0 - 1/\tau_1$, $1/t_2 = 1/\tau_0 - 1/\tau_2$, $1/\tau_K = 1/\tau_a + 1/\tau_1$, and $1/\tau_z = 1/\tau_0 + 1/\tau_a$. We may set $\omega = sk$ in (25) since the solution for $\Phi(\vec{q})$ will be used only in (22). Two equations relating Φ_0 , Φ_1 , and Φ_2 may be obtained by taking the $l=0$ and $l=1$ angular moments of (25). A third relation is obtained by solving (25) for $\Phi(\vec{q})$ and taking the $l=1$ moment of the result. Solution of this system of equations yields for Φ_1

$$\begin{aligned} \Phi_1(q)D(q) = & \left(I_2 + \frac{iI_3}{4\omega t_2} \right) \delta s + \frac{i}{\omega\tau_K} \left(I_2 + \frac{iI_3}{4\omega t_2} \right) v_3 + \left[\frac{I_1 - I_3/5}{3} - \frac{iI_2}{\omega\tau_K} + \frac{iI_1}{3\omega\tau_0(1+i/\omega\tau_a)} \right. \\ & \left. + \frac{I_3}{4\omega^2 t_2 \tau_K} \left(1 + \frac{i\omega\tau_K}{3(1+i/\omega\tau_a)} \right) \right] v_s - \frac{i}{\omega\tau_a} \left[I_1 \left(1 + \frac{i}{\omega\tau_0(1+i/\omega\tau_a)} \right) + \frac{iI_3}{4\omega\tau_2(1+i/\omega\tau_a)} \right] \Phi_0^{le} \end{aligned} \quad (26)$$

$$\text{where } D(q) = 1 + \frac{iI_2}{\omega t_1} + \frac{iI_1}{3\omega\tau_0(1+i/\omega\tau_a)} - \frac{iI_3}{4\omega\tau_2} \left((1+i/\omega\tau_K) - \frac{1}{3(1+i/\omega\tau_a)} \right), \quad (27)$$

$$\text{and } I_1 = 3 \left[1 + \frac{z(q)}{2} \ln \left(\frac{z(q)-1}{z(q)+1} \right) \right], \quad I_2 = z(q)I_1, \quad I_3 = 5[I_1 - 3 - 3z(q)I_2], \quad (28)$$

with $z(q) = 1 + i/\omega\tau_z(q)$. Φ_0 and Φ_2 are given by

$$\Phi_0 = [(\Phi_1 - v_s)/3 + i\Phi_0^{le}/\omega\tau_a] / (1 + i/\omega\tau_a) \quad (29)$$

$$\text{and } 2\Phi_2/5 = \delta s - \Phi_0 - i(v_s - v_3)/\omega\tau_K + \Phi_1(1 + i/\omega\tau_K). \quad (30)$$

For use in (22), the quantities $\delta s/\delta n_4$, $\delta v_s/\delta n_4$, $\delta v_3/\delta n_4$, and $\delta\Phi_0^{le}/\delta n_4$ may be replaced by their values for $x=0$, $\rho_{ph}/\rho_4=0$, and $\omega = sk$. In this case we have $\delta s/\delta n_4 = su_1/n_4$ and $\delta v_s/\delta n_4 = s/n_4$. $\delta v_3/\delta n_4$ is related to $\delta n_3/\delta n_4$ by He³ number conservation; a short calculation along the lines of that given for $\text{Im}(\delta n_3/\delta n_4)$ yields $\delta v_3/\delta n_4 = s\lambda/n_4$. Φ_0^{le} is given by

$$\Phi_0^{le} = \delta s - s\delta T_3/T = s(u_1\delta n_4/n_4 - \delta T_3/T); \quad (31)$$

as noted above δT_3 is the variation of the local He³ temperature. δT_3 may be simply calculated when $\rho_{ph}=0$ by noting that when the thermal conductivity is neglected the entropy per quasiparticle, which depends only on $T_3/T_f \sim mT_3/n_3^{2/3}$, is a constant; the result is

$$\delta T_3/T = \frac{2}{3} \delta n_3/n_3 - u_2 \delta n_4/n_4. \quad (32)$$

Since to lowest order $\delta n_3/n_3 = \lambda \delta n_4/n_4$, we have

$$(n_4/T)\delta T/\delta n_4 = \frac{2}{3} \lambda - u_2. \quad (33)$$

Putting together all the pieces, we arrive at the final formula for the first sound attenuation

$$\begin{aligned} \alpha_I = & \frac{\omega^2}{s^3\rho_4} \frac{2\lambda^2}{3} \frac{\eta_3}{1+\omega^2\tau_\eta^2} + \frac{\omega\rho_{ph}}{2s^2m_4} \left\{ \left\langle \left(\frac{u_1+u_2-2\lambda/3}{1+1/\omega^2\tau_a^2} + (1-u_2 + \frac{2\lambda}{3}) \right) \text{Im} \frac{\delta}{\delta n_4} \Phi_1 \right\rangle \right. \\ & + \left\langle \left(\frac{\lambda}{\omega\tau_K} - \frac{(u_1+u_2-2\lambda/3)}{\omega\tau_a(1+1/\omega^2\tau_a^2)} \right) \text{Re} \frac{\delta}{\delta n_4} \Phi_1 \right\rangle + \lambda(\lambda-1) \left\langle \frac{1}{\omega\tau_K} \right\rangle \\ & \left. + (u_1+u_2-2\lambda/3)(3u_1+3u_2-2\lambda+1) \left\langle [\omega\tau_a(1+1/\omega^2\tau_a^2)]^{-1} \right\rangle \right\}, \end{aligned} \quad (34)$$

with $\delta\Phi_1(q)/\delta n_4$ now given by

$$\begin{aligned} \frac{n_4}{s} \frac{\delta\Phi_1(q)}{\delta n_4} \Phi_1(q)D(q) = & \left(u_1 + \frac{i\lambda}{\omega\tau_K} \right) \left(I_2 + \frac{iI_3}{4\omega\tau_2} \right) + \left[\frac{I_1}{3} - \frac{I_3}{15} - \frac{iI_2}{\omega\tau_K} + \frac{iI_1}{3\omega\tau_0(1+i/\omega\tau_a)} + \frac{I_3}{4\omega^2 t_2 \tau_K} \left(1 + \frac{i\omega\tau_K}{3(1+i/\omega\tau_a)} \right) \right] \\ & - \frac{i}{\omega\tau_a} \left(u_1 + u_2 - \frac{2\lambda}{3} \right) \left[I_1 \left(1 + \frac{i}{\omega\tau_0(1+i/\omega\tau_a)} \right) + \frac{iI_3}{4\omega t_2(1+i/\omega\tau_a)} \right]. \end{aligned} \quad (35)$$

The calculation of the He³-phonon collision integral given in II used the assumption that the He³ quasi-particles were in local equilibrium among themselves. This requires $\omega\tau_\eta \ll 1$ for temperatures where the effects of I_{3-ph} are important; this criterion is satisfied, for He³ concentrations of a few percent, for frequencies less than about 100 MHz. Subject to this limitation our derivation of first sound attenuation is exact for temperatures below 0.6–0.7°K where rotons and phonon-phonon interactions may be neglected and above about 0.01°K where boundary scattering becomes important.

We now present results of calculations of ultrasonic attenuation, using Eqs. (34) and (35). It must be pointed out that for the frequency region in which experimental results are presently available it is necessary to use these general formulas rather than their collisionless or hydrodynamic limits (given in the following section). The integrations involved in averaging over the phonon momenta have been performed numerically.

Shown in Fig. 1 for a frequency of 20 MHz are the theoretical values of the attenuation in 1.3% solutions and both the experimental⁷ and theoretical values of the attenuation in 5% solutions. Figure 2 provides the same information for a frequency of 60 MHz, the experimental data also coming from Ref. 7. The attenuation in 5.5% solutions for frequencies of 15, 45, and 75 MHz⁸ is given in Fig. 3; except at very low T , the experimental data for 15 MHz, are, on the average, approximately 1 dB above the calculated values. The He³ viscosity lifetimes τ_η used for the 1.3% and 5.0% solutions have been constructed in I. τ_η for the 5.5% solutions shown in Fig. 4 has been constructed in the following similar fashion. At low temperature we write τ_η in the form

$$\tau_\eta = (A/T^2)[1 + B(T/T_F)^2].$$

The values $A = 1.99 \times 10^{-11} \text{ sec} \cdot (\text{°K})^2$ and $B = 2.0$ are determined, as in Ref. 8, by requiring that for each of the frequencies 15 and 45 MHz the theoretical and experimental low-temperature peaks occur at the same temperatures. At high temperatures τ_η has been related to its value in 5% solutions by noting that it scales inversely with concentration. For intermediate temperatures, a smooth interpolation following τ_η in the 5% solutions has been used.

In all cases, the overall agreement between theory and experiment is quite good between 0.04°K and about 0.7°K, the highest temperature at which the theory may be expected to be valid. The disagreement near the minima in the curves for the 5% and 5.5% solutions at the higher frequencies is probably attributable to the neglect of the exclusion principle in the calculation of the phonon lifetimes used in the phonon Boltzmann equation. We see no theoretical reason for the disagreement below 0.04°K. Deviation from a T^2 dependence would be in contradiction with the Landau-Fermi liquid theory.⁹

The disagreement between theory and experiment above about 0.7°K is due to the neglect of phonon-phonon interactions and rotons. The high-temperature peak in the attenuation occurs in a regime midway between the collisionless and hydrodynamic limits. In the peak region, the attenuation is limited by roton-phonon scattering, as in pure He⁴,¹⁰ augmented by the phonon-He³ interactions; preliminary calculations

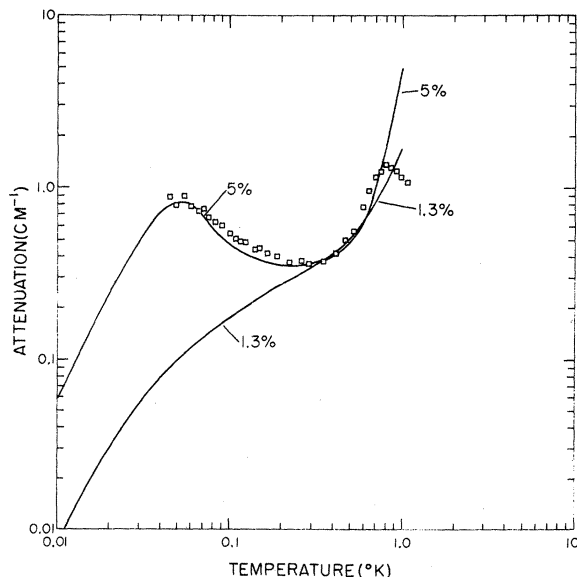


FIG. 1. Attenuation of first sound of frequency 20 MHz in 1.3% and 5% solutions of He³ in He⁴. The solid lines are the calculated attenuation; the data points are those of Abraham *et al.* (Ref. 7).

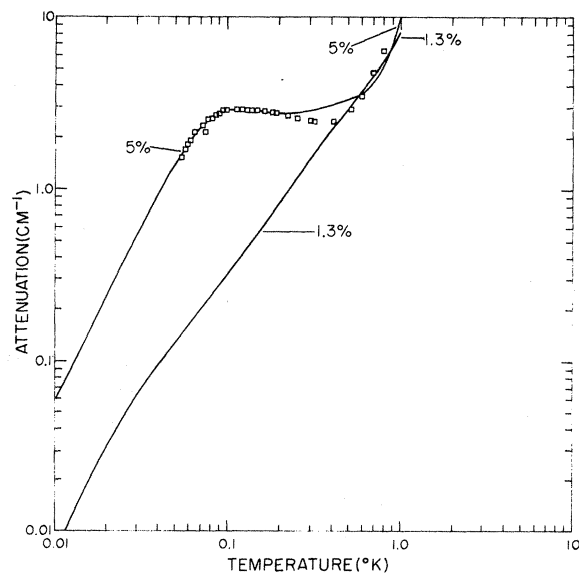


FIG. 2. Attenuation of first sound of frequency 60 MHz in 1.3% and 5% solutions of He³ in He⁴. The solid lines are the calculated attenuation; the data points are those of Abraham *et al.* (Ref. 7).

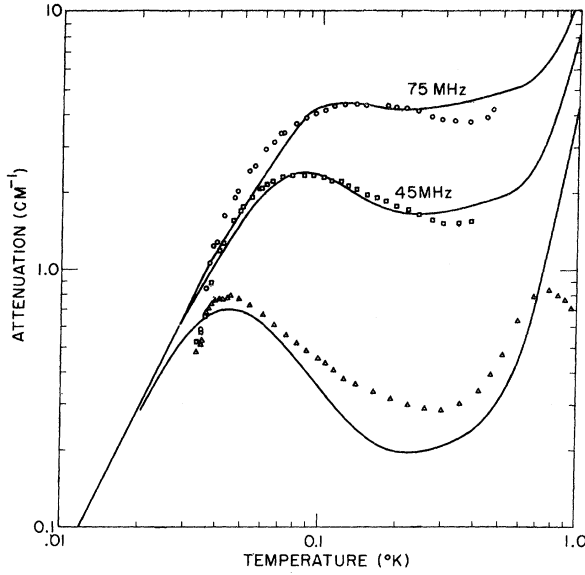


FIG. 3. Attenuation of first sound in a 5.5% solution of He^3 in He^4 . The solid lines are the calculated attenuation at 15, 45, and 75 MHz. The data points are those of Abraham *et al.* (Ref. 8).

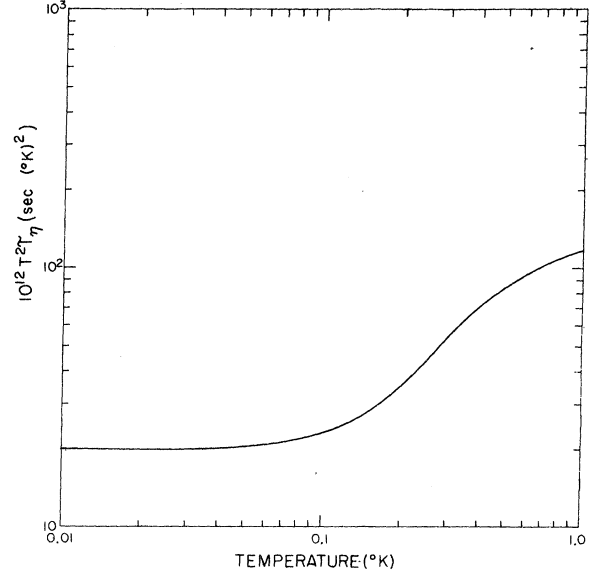


FIG. 4. Lifetime for He^3 viscosity in a 5.5% solution of He^3 in He^4 .

based on this picture appear to explain the concentration and frequency dependence of the peak.

It is important to understand how sensitive our results are to changes in the parameters τ_η and τ_l . Up to about 0.4°K the attenuation is dominated by the He^3 viscosity and is insensitive to changes in τ_l but quite sensitive to any change in τ_η . Thus, attenuation measurements pin down τ_η rather well up to 0.4°K. Above 0.4°K our results are only mildly sensitive to changes in τ_η and τ_l . For example, a 30% reduction in τ_η changes the attenuation by less than 10% here; the change is negative at temperatures near 0.4°K and becomes positive at higher temperatures. Above 0.4°K, a 50% reduction in τ_l reduces the attenuation by less than 15%; in general, the change is an increasing function of temperature. We conclude that ultrasonic attenuation data do not provide a very sensitive measure of the lifetimes τ_η and τ_l for temperature between 0.4°K and 0.7°K, a not unexpected result in the regime on the collisionless side of the high-temperature peak.

IV. COLLISIONLESS AND HYDRODYNAMIC LIMITS

In this section, we give the limits of our expression (34) for the sound attenuation when $\omega\tau \ll 1$ where τ is a typical phonon lifetime; $\omega\tau_\eta$ will still be assumed small for temperatures where $I_{3\text{-ph}}$ is important.

The $\omega\tau \gg 1$ or collisionless limit is easily found to be

$$\alpha_I^{\text{n.c.}} = \frac{2}{3}\lambda^2(\omega^2/s^3\rho_4)[\eta_3/(1+\omega^2\tau_\eta^2)] + (3\pi\omega\rho_{\text{ph}}/8s\rho_4)(1+u_1)^2. \quad (36)$$

The second term in this expression is the usual^{10,11} result for pure He^4 . For He^3 concentrations of a few percent, this result holds for frequencies in the 10–100 MHz range for temperatures below 0.1–0.2°K; the second term is of no practical importance here.

The hydrodynamic limit is valid when $\omega\tau \ll 1$ and $\omega\tau_\eta \ll 1$; it is given by

$$\alpha_I^{\text{hydro}} = (2\lambda^2\omega^2/3s^3\rho_4)\eta_3 + (\omega^2\rho_{\text{ph}}/2s\rho_4)[3(u_1+u_2-\lambda/3)^2\langle\tau_\alpha\rangle + (1-u_2-\lambda/3)^2\langle\tau_\kappa\rangle + \frac{4}{15}\lambda^2\langle\tau_\nu\rangle] \quad (37)$$

where $\tau_\nu^{-1} = \tau_\alpha^{-1} + \tau_2^{-1}$. In (37) the term proportional to $\langle\tau_\alpha\rangle$ gives the contribution from second viscosity; this arises from energy exchange between the phonons and He^3 . The term proportional to $\langle\tau_\kappa\rangle$ arises from both phonon thermal conductivity and He^3 quasiparticle diffusion. Lastly, the term containing $\langle\tau_\nu\rangle$ is the damping due to the phonon viscosity. Because of the relatively slow rate at which energy is exchanged between the phonons and He^3 , the second viscosity dominates these three terms; thus for purposes of numerical calculation we have

$$\alpha_I^{\text{hydro}} = (2\lambda^2\omega^2/3s^3\rho_4)\eta_3 + (3\omega^2\rho_{\text{ph}}/2s\rho_4)(u_1+u_2-\lambda/3)^2\langle\tau_a\rangle, \quad (38)$$

which is plotted for a 5% solution in Fig. 5. From this expression and that given in (24) for τ_a we see that measurements of the attenuation of hydrodynamic sound would provide an excellent means of inferring τ_η over the temperature range from 0.01 to about 0.7°K. Measurements in the required frequency range ($\omega \lesssim 100$ KHz) are not, however, available.

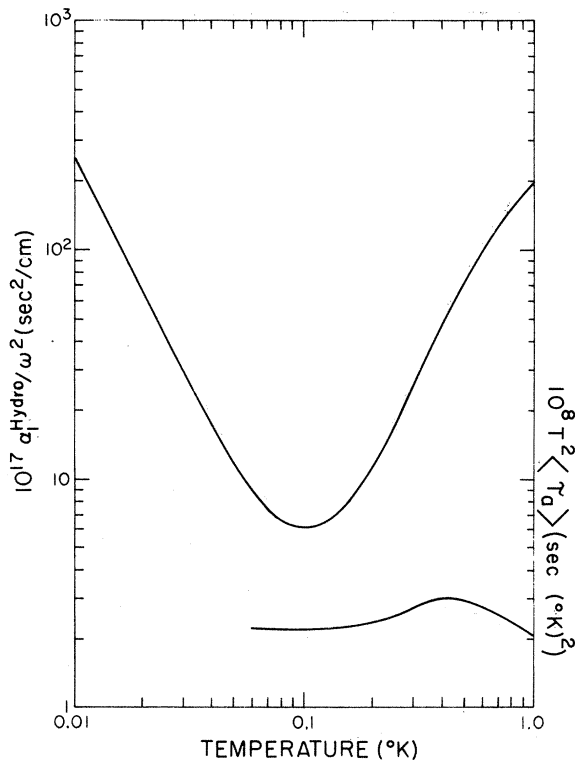


FIG. 5. Calculated attenuation of first sound in a 5% solution of He³ in He⁴ in the hydrodynamic limit (upper curve) and average value of phonon absorption lifetime (lower curve).

ACKNOWLEDGMENT

We wish to thank Dr. Bernard Abraham and Dr. Yakov Eckstein for providing us with detailed information about their sound attenuation experiments.

*Research supported in part by National Science Foundation Grants NSF GP-4937 and NSF GP-7765.

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