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Relativistic Thermodynamics of Moving Systems

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Thermodynamics is extended to systems moving with relativistic velocities. It is shown that one is led naturally, although not necessarily, to the thermodynamics of Ott, if one maintains the first and second law in their original form. The classical theory of Planck *et al.* can also be obtained in the case of a homogeneous fluid; the difference with Ott's theory is that the fluid alone is regarded as the thermodynamic system, rather than the fluid together with the box in which it is enclosed. Subsequently, a third form of relativistic thermodynamics is obtained by replacing the first law with a covariant equation expressing conservation of both energy and momentum. This leads to a formulation in which not only S but also T and $đQ$ are scalars. The discussion of heat transfer between systems with different velocities is thereby simplified. It is shown that such processes are irreversible even for equal temperatures, unless the velocities are equal too.

1. INTRODUCTION

The problem is to extend the laws of thermodynamics to a system moving with relativistic velocity $\vec{u}$. If $\vec{u}$ is taken to be a constant, this is simply the problem of transforming the thermodynamic properties of the system to a different frame of reference. However, *one wants to treat* $\vec{u}$ *as an additional thermodynamic variable* subject to adiabatic variations. That amounts to extending the usual space of thermodynamic variables by adding three dimensions, corresponding to the three components of $\vec{u}$. Thermodynamics of moving systems is therefore more than simply an exercise in Lorentz transformations.

The first law of thermodynamics is affected, because the work $đA$ now consists of the usual term representing the work done by expanding, $đA_V$, plus an additional term $đA_u$, representing the work involved in varying $\vec{u}$, that is, in accelerating or decelerating the whole system.

The second law is affected because Kelvin's principle, "No cyclic engine can convert heat into work," receives an extended meaning: "not even when the engine employs acceleration and deceleration of the system (to relativistic velocities)."

2. THE NONRELATIVISTIC CASE

The above remarks also apply to the nonrelativistic case, but that case is trivial. One has $đA_u = -M\vec{u}\cdot d\vec{u}$, where M is the mass of the system and is constant.[1] For the "internal energy" U one simply takes

$$U = U^0 + M\vec{u}^{\,2},$$

where U^0 is the internal energy of the system at rest. The first law then states

$$\begin{aligned} đQ &= dU + đA, \\ &= dU + đA_u + đA_V, \\ &= dU^0 + đA_V. \end{aligned} \tag{1}$$

Thus the two additional terms cancel and everything reduces to the usual thermodynamics of systems at rest.

Incidentally, we note for future use that, for a homogeneous system, (1) may be written in the form

$$\text{đ}Q = dU - \vec{u}\cdot d\vec{G} + PdV, \tag{2}$$

where V is the volume, P is the pressure, and $\vec{G} = M\vec{u}$ is the total momentum of the moving system.

The relativistic case is less trivial for the following reasons.

(*i*) U cannot be decomposed in a kinetic energy term and a term U^0 depending only on the internal state.

(*ii*) The rest mass M is not constant, since any heat transfer $\text{đ}Q$ represents energy and therefore mass. Similarly the work $\text{đ}A_V$ done on expanding decreases the rest mass.

(*iii*) The transfer of heat and work between moving systems implies the transfer of mass and hence of momentum.

3. EXTENSION OF THE THERMODYNAMIC SPACE

For simplicity we take as our system a gas enclosed in a box. When at rest, it has two thermodynamic variables, the volume V^0 and the pressure P. Moreover, of the three components of $\vec{u}$ we only consider the x-component, to be denoted by u. The generalization to more variables is obvious.

The thermodynamic space now has three dimensions, corresponding to the variables V^0, P, and u. In the plane $u = 0$, the thermodynamic quantities U^0, T^0, and S^0 are given as functions of V^0 and P by the usual thermodynamics. Moreover, two differential forms $\text{đ}A^0$ and $\text{đ}Q^0$ in the variables V^0 and P are given, obeying

$$\text{đ}Q^0 = dU^0 + \text{đ}A^0,$$

$$T^0 dS^0 = \text{đ}Q^0. \tag{3}$$

Our task is to extend the definition of these quantities to the domain[2] $-1 < u < 1$, in such a way that the first and second law remain valid.

More precisely, by "extending the definition" we mean defining three functions $U(V^0, P, u)$, $S(V^0, P, u)$, and $T(V^0, P, u)$, and two differential forms $\text{đ}Q$ and $\text{đ}A$ in the same three variables V^0, P, and u, such that

$$\text{đ}Q = dU + \text{đ}A, \tag{4}$$

$$TdS = \text{đ}Q. \tag{5}$$

For $u = 0$, they should reduce to U^0, S^0, T^0, and $\text{đ}Q^0$, $\text{đ}A^0$. Moreover, in irreversible processes, S should always increase.

4. EXTENSION OF THE FUNCTIONS U AND S

Let the system be at rest in a state (V^0, P). Its rest mass equals U^0 (when the arbitrary constant in U^0 is chosen appropriately). In principle, the system can be accelerated to any velocity at the expense of mechanical work, without affecting the internal state of the system.[3] The amount of work required is

$$-\text{đ}A_u = d[U^0/(1-u^2)^{1/2}] = U^0 d\gamma,$$

where $\gamma = (1-u^2)^{-1/2}$. If we now *choose* as a natural extension[4] of the definition of internal energy

$$U(V^0, P, u) = \gamma U^0(V^0, P), \tag{6}$$

we have for pure accelerations

$$dU + \text{đ}A = 0.$$

It follows that the first law remains valid if one postulates that $\text{đ}Q = 0$ for pure accelerations. That is, the (as yet unknown) differential form $\text{đ}Q$ in three variables contains no term with du, in agreement with the physical interpretation of $\text{đ}Q$ as heat supplied to the system.

In order to maintain the second law in its customary form (5) one must have

$$\partial S(V^0, P, u)/\partial u = 0, \text{ or } S(V^0, P, u) = S^0(V^0, P).$$

Thus the entropy is independent of u, that is, the entropy transforms as a scalar.[5] In the three-dimensional thermodynamical space, the surfaces S = constant are cylinders parallel to the u axis. (Fig. 1).

5. EXTENSION OF T AND $\text{đ}Q$

So far we are still free to extend the definition of temperature by choosing for T any positive function of V^0, P, and u, or alternatively of V^0, T^0, and u, provided that it reduces to T^0 for $u = 0$. The second law (5) implies that one must then put

$$\text{đ}Q = \text{đ}Q^0\,\frac{T(V^0, T^0, u)}{T^0}.$$

However, if $\text{đ}Q$ is to be an extensive quantity, the ratio T/T^0 cannot depend on V^0, so that

$$T = T^0 g(T^0, u), \quad \text{đ}Q = \text{đ}Q^0 g(T^0, u).$$

Here $g(T^0, u)$ is an arbitrary positive function, subject only to the condition that $g(T^0, 0) = 1$. The

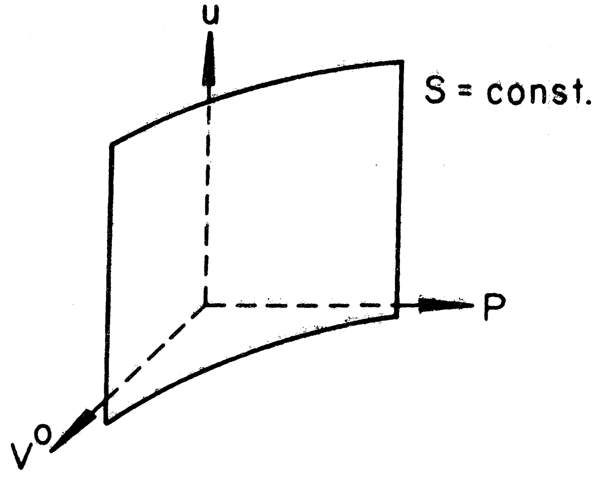

FIG. 1. The extended thermodynamic space.

second law can tell no more about g.

For further information one has to return to the first law, which one would like to maintain in the form (4). On substituting the results obtained so far, this becomes

$$g\,\text{đ}Q^0 = \gamma dU^0 + \text{đ}A_V, \tag{7}$$

where $\text{đ}A_V = \text{đ}A - \text{đ}A_u$ is the work done when the system, moving with the velocity u, expands. Since such work is typically measured as the increase of mechanical energy of a second system, connected with the first one and moving with the same velocity u, one has

$$\text{đ}A_V = \gamma\,\text{đ}A_V{}^0. \tag{8}$$

Hence, if one chooses g to be equal to γ, Eq. (7) coincides with (3), so that indeed (4) is satisfied. Thus, for this g, and only for this g, the first law remains valid in the form (4).

6. THE THERMODYNAMICS OF OTT

We have thus rederived the transformation formulas of Ott[6]

$$T = \gamma T^0, \quad \text{đ}Q^0 = \gamma\,\text{đ}Q^0. \tag{9}$$

The first law (4) can be written more explicitly as

$$\text{đ}Q = dU - U^0 d\gamma + \gamma P dV^0. \tag{10}$$

If one desires, one may express U^0, V^0, and P in terms of the quantities referring to the laboratory frame, $U = \gamma U^0$ and $V = V^0/\gamma$. The pressure P is not changed by a Lorentz transformation.[7]

It is possible to write (10) in a form similar to (2). The momentum G of the system is $\gamma u U^0$ and one has

$$u\,dG = \gamma u^2 dU^0 + U^0 d\gamma = \gamma u^2 dU^0 + u(dG)^*,$$

where $(dG)^*$ denotes the variation of G while the rest mass U^0 is kept constant. Thus (10) may be written

$$\text{đ}Q = dU - u(dG)^* + \text{đ}A_V.$$

The fact that one has to write $(dG)^*$ instead of dG is the essential remark of Ott. It exhibits clearly that any change in momentum due to the change in rest mass should not be included in the acceleration work $\text{đ}A_u$. Such changes of rest mass occur when heat or work is exchanged with another system.

7. THE THERMODYNAMICS OF PLANCK

The classical theory of Planck *et al.*[8] amounts to taking $g = 1/\gamma = (1-u^2)^{1/2}$, so that

$$T = T^0 (1-u^2)^{1/2}, \tag{11}$$

$$\text{đ}Q = \text{đ}Q^0 (1-u^2)^{1/2} = (1-u^2)^{1/2} (dU^0 + P dV^0). \tag{12}$$

On the other hand, they chose to define the internal energy for $u \neq 0$ by

$$\tilde{U}(V^0, P, u) = \gamma (U^0 + u^2 P V^0), \tag{13}$$

instead of Ott's definition (6). Similarly the momentum was defined by

$$\tilde{G}(V^0, P, u) = \gamma u (U^0 + P V^0). \tag{14}$$

With these definitions, Eq. (12) may be written

$$\text{đ}Q = d\tilde{U} - u\,d\tilde{G} + P dV. \tag{15}$$

The question is whether this is a reasonable extension of the first law to moving systems.

The essential difference with Ott is that *one now regards as the thermodynamic system the gas in the box alone*, and not the gas and box together. In fact, U and G are the energy and momentum of the gas molecules. They do not form a four-vector[9] because the gas alone is not a closed, self-contained system, but is subject to forces exerted by the walls. The walls of the box contain stresses necessary to counterbalance the gas pressure. In the rest frame these stresses do not carry energy or momentum (provided that the walls are rigid), but viewed from a moving frame they do. The reason is that the time components of the four-dimensional stress tensor represent energy and momentum density[10]; these components are zero in the rest frame, but become nonzero after a Lorentz transformation. This gives rise to an energy and momentum of the walls. Hence one must make a distinction between the $\tilde{U}$ and $\tilde{G}$ of the gas, and the four-vector U and G of the total system.

It follows that the energy and momentum due to the wall stresses are

$$U - \tilde{U} = -\gamma u^2 P V^0, \quad G - \tilde{G} = -\gamma u P V^0$$

Variation of V^0 with constant u and P entails an increase of the wall energy,

$$d(U - \tilde{U}) = -\gamma u^2 P dV^0 = (\gamma^{-1} - \gamma) P dV^0.$$

This is the difference between the term $PdV = \gamma^{-1} P dV^0$ in (15), and the term $\gamma P dV^0$ in (10). The term PdV represents the work done by the gas on the wall plus surroundings, whereas PdV^0 is the work done by the gas plus wall on surroundings.[11]

Nevertheless, (15) is not a satisfactory formulation of the first law. One objection is that it only applies to homogeneous systems (or to systems at least having the same pressure throughout their volume), inasmuch as the definitions (13) and (14) are confined to that case. The other objection is that, for other processes than a variation of V^0, the separate terms in (15) have no clear-cut physical meaning. For example, a pure acceleration gives $\text{đ}Q = 0$ only by a clever cancellation of the variations of all three terms. The term $-u\,d\tilde{G}$ is *not* just the acceleration energy $\text{đ}A_u$, since $d\tilde{G}$ involves a term with dU^0 and is therefore nonzero when heat or work is exchanged at constant u. The term PdV is *not* just the work done on expanding, since dV involves a term with du and is therefore nonzero when the system is accelerated with constant V^0.

Because of these objections to Planck's formulation, we now return to the choice (6) for the internal energy.

8. HOMOTACHIC AND HETEROTACHIC PROCESSES

So far only two kinds of processes have been considered: (*i*) exchange of heat and work between systems *having the same velocity* (which are the processes of ordinary thermodynamics and will be called "homotachic"); and (*ii*) processes in which u varies adiabatically with constant V^0, and P ("pure accelerations"). They sufficed to define the thermodynamic quantities in the whole thermodynamic (V^0, P, u) space, because any point in that space can be reached from an arbitrary starting point by a chain of such processes. Yet a number of "natural" but not logically compelling choices had to be made.

(*i*) We chose to extend the definition of U by writing (6).

(*ii*) We chose to maintain the first law in the form (4).

(*iii*) We chose to maintain the second law in the form (5).

The task remains to show that these choices are compatible with Kelvin's principle for other processes too. Such other processes are those which involve a direct heat transfer between systems *moving with different velocities* ("heterotachic processes"). In this case, however, the momentum transfer inherent in exchange of heat and work does affect the velocities of the systems. Hence at least part of this momentum transfer should be counted as work $\text{đ}A_u$. This complication has been analyzed by Ott, but we shall now propose a different formulation of relativistic thermodynamics, in which this momentum transfer need not be taken into account explicitly.

9. A THIRD FORMULATION OF RELATIVISTIC THERMODYNAMICS

Let $u_\mu = (\gamma\,; \gamma\vec{u})$ be the four-velocity of the system, and $U_\mu = u_\mu U^0$ be its energy-momentum four-vector. Let K_μ be the Minkowski force acting on the system; if there are no other interactions, K_μ gives rise to a pure acceleration described by

$$dU_\mu/d\tau = U^0\,du_\mu/d\tau = K_\mu, \tag{16}$$

where τ is the proper time of the system. Note that[12] $u_\mu u_\mu = 1$, and therefore

$$u_\mu K_\mu = 0. \tag{17}$$

Let $\text{đ}A_{V\mu}$ represent the mechanical energy and momentum transferred to another system, for instance, by expanding. We now write as the relativistic extension of the first law

$$\begin{aligned}\text{đ}Q_\mu &= dU_\mu + \text{đ}A_\mu\\ &= dU_\mu - K_\mu d\tau + \text{đ}A_{V\mu}\,.\end{aligned} \tag{18}$$

The four-vector $\text{đ}Q_\mu$ defined by this equation will be called the *"thermal energy-momentum transfer."* On the other hand, the *"heat supply"* will be defined as the component of this four-vector along the four-velocity,

$$u_\mu\,\text{đ}Q_\mu = \text{đ}Q^0. \tag{19}$$

As it is a scalar, it is identical to the heat supply measured in the rest frame. The second law (5) remains valid when T is defined as equal to the temperature T^0 in the rest frame.

Thus we have obtained a different formulation of relativistic thermodynamics, in which both $\text{đ}Q$ and T are scalars.[13] However, the first law is now replaced with the covariant equation (18). This equation does not give the heat supply $\text{đ}Q^0$ itself, but instead gives the four-vector $\text{đ}Q_\mu$, from which $\text{đ}Q^0$ can be found according to (19). Alternatively one may write directly

$$\text{đ}Q^0 = u_\mu dU_\mu + u_\mu \text{đ}A_{V\mu}\,. \tag{20}$$

10. VERIFICATION FOR HOMOTACHIC PROCESSES

When heat is transferred between two systems having the same velocity, then in their common rest frame the thermal energy transfer is $\text{đ}Q^0$, and the thermal momentum transfer is zero. As $\text{đ}Q_\mu$ is a four-vector, one must therefore have in an arbitrary frame $\text{đ}Q_\mu = u_\mu \text{đ}Q^0$. Similarly, transfer of mechanical work $\text{đ}A_V{}^0$ in the rest frame does not entail momentum transfer, because, when the gas expands, the force on the piston is always balanced by an equal and opposite force, if the expansion is to be reversible. Hence one also has in an arbitrary frame $\text{đ}A_{V\mu} = u_\mu \text{đ}A_V{}^0$.

Thus, if all exchange of heat and work takes place homotachically, the general equation (18) reduces to

$$u_\mu \text{đ}Q^0 = dU_\mu - K_\mu d\tau + u_\mu \text{đ}A_V{}^0. \tag{21}$$

This equation can be decomposed into two equations by separating the components parallel and orthogonal to the four-vector u_μ. With the aid of the identity

$$dU_\mu = d(u_\mu U^0) = U^0 du_\mu + u_\mu dU^0$$

one finds for the component parallel to u_μ

$$\text{đ}Q^0 = dU^0 + dA_V{}^0\,,$$

which is simply the first law for the system at rest. The orthogonal component yields

$$0 = U^0 du - K_\mu d\tau\,,$$

which is simply the equation of motion (16) for the system as a whole.

This shows that, for homotachic processes and pure accelerations, (18) reproduces the familiar results. The entropy is, according to Sec. 9, given by $dS = \text{đ}Q^0/T^0$, which is of course also the familiar entropy.

11. APPLICATION TO HETEROTACHIC PROCESSES

Suppose two systems, a and b, interact with each other, but with nothing else. Then $U_\mu{}^a + U_\mu{}^b$ is constant and $\text{đ}A_{V\mu}{}^a + \text{đ}A_{V\mu}{}^b = 0$. Hence

$$\text{đ}Q_\mu{}^a + \text{đ}Q_\mu{}^b = 0. \tag{22}$$

However, unless the systems have the same velocity,

$$\text{đ}Q^a + \text{đ}Q^b \neq 0.$$

When thermal energy and momentum are transferred, the heat lost by one system is not necessarily equal in amount to the heat gained by the other system.[14] The reason is that the heat contents of the transmitting agency (e.g. electromagnetic waves) is not the same for all observers.[15] For they use different frames when decomposing the thermal energy-momentum four-vector in energy and momentum.

More precisely, let b emit radiation, which in its rest frame has energy q, and is isotropic and therefore carries no momentum. In this frame $\text{đ}Q_\mu{}^b = (-q;\ 0,0,0)$. In an arbitrary laboratory frame

$$\text{đ}Q_\mu{}^b = -u_\mu{}^b q.$$

If this radiation is absorbed by a, one has according to (22) and (19)

$$\text{đ}Q_\mu{}^a = u_\mu{}^b q, \quad dQ^a = u_\mu{}^a u_\mu{}^b q.$$

Note that $u_\mu{}^a u_\mu{}^b = (1-u^2)^{-1/2} = \gamma$, where u is the relative velocity of a and b. It follows that

$$\text{đ}Q^a + \text{đ}Q^b = (\gamma - 1)q > 0.$$

The increase in total entropy is

$$dS = \frac{\text{đ}Q^a}{T_a{}^0} + \frac{\text{đ}Q^b}{T_b{}^0} = \left(\frac{\gamma}{T_a{}^0} - \frac{1}{T_b{}^0}\right) q.$$

This is negative for $T_a{}^0 > \gamma T_b{}^0$, but that is not a paradox. It is not possible without the aid of Maxwell's demon to let radiation go from b to a, without radiation from a to b. For this reason Ott was not justified in using the above example to prove that his transformation formula (9) for the temperature is the only one consistent with Kelvin's principle for heterotachic processes.[16]

12. THERMAL CONTACT BETWEEN MOVING SYSTEMS

It is possible to further specify the example, such that the exchange of thermal energy and momentum can be calculated explicitly. Let two black bodies a and b be separated by a thin metallic sheet (Fig. 2). With respect to the laboratory frame, a and b have velocities u^a and u^b parallel to the sheet, and the relative velocity of b with respect to a is $u = (u^b - u^a)/(1 - u^a u^b)$. Body a fills the space between it and the sheet with blackbody radiation; in the rest frame of a the energy density ρ_a is a universal function of the rest temperature $T_a{}^0$. Similarly the space between b and the sheet is filled with blackbody radiation moving with velocity u^b. So far there is no interaction between a and b.

Let a small hole of area ΔA be opened in the sheet during Δt. The transmitted radiation will be proportional to $\Delta A \Delta t$; since this is a scalar it does not matter with respect to which frame one measures ΔA and Δt. Also the velocity of the sheet is irrelevant.

The radiation leaking from a to b can be computed in the rest frame of a to be $\frac{1}{4}\rho_a \Delta A \Delta t$. It carries no momentum component parallel to the sheet, as is clear from symmetry.[17] Thus one has in the rest frame of a

$$\text{đ}A_{V\mu}{}^{a0} = 0, \quad \text{đ}U_\mu{}^{a0} = (-\tfrac{1}{4}\rho_a \Delta A \Delta t;\ 0,0,0).$$

For brevity we absorb the factor $\frac{1}{4}\Delta A \Delta t$ in ρ_a, and obtain

$$\text{đ}Q_\mu{}^{a0} = dU_\mu{}^{a0} = -(\rho_a;\ 0,0,0).$$

Transformation to the laboratory frame yields for this process

$$\text{đ}Q_\mu(a \to b) = -u_\mu{}^a \rho_a.$$

In addition, radiation leaks from b to a, yielding a second contribution to the thermal energy-momentum transfer

$$\text{đ}Q_\mu(b \to a) = -u_\mu{}^b \rho_b.$$

The total thermal energy-momentum transfer is therefore

$$\text{đ}Q_\mu{}^a = -\text{đ}Q_\mu{}^b = -u_\mu{}^a \rho_a + u_\mu{}^b \rho_b.$$

The *heat supply* to each system is, according to (19),

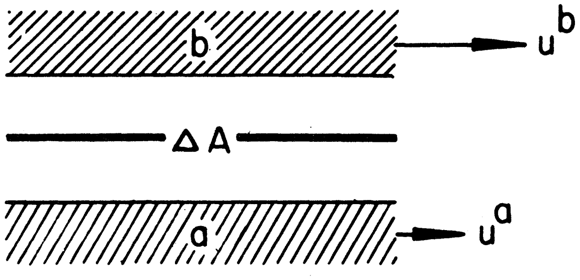

FIG. 2. Direct energy-momentum transfer between moving systems.

$$\text{đ}Q^a = -\rho_a + \gamma\rho_b\,, \quad \text{đ}Q^b = -\rho_b + \gamma\rho_a\,.$$

Again one finds that their sum is not zero:

$$\text{đ}Q^a + \text{đ}Q^b = (\gamma - 1)(\rho_a + \rho_b) > 0.$$

13. THE INCREASE OF ENTROPY

The variation of the total entropy is

$$dS = \frac{\text{đ}Q^a}{T_a{}^0} + \frac{\text{đ}Q^b}{T_b{}^0} = \left(\frac{\gamma}{T_b{}^0} - \frac{1}{T_a{}^0}\right)\rho_a + \left(\frac{\gamma}{T_a{}^0} - \frac{1}{T_b{}^0}\right)\rho_b$$

$$= \left(\frac{1}{T_a{}^0} - \frac{1}{T_b{}^0}\right)(\rho_b - \rho_a) + (\gamma - 1)\left(\frac{\rho_a}{T_b{}^0} + \frac{\rho_b}{T_a{}^0}\right).$$

As ρ increases with T^0, the first term is positive unless $T_a{}^0 = T_b{}^0$. The second term is positive unless $\gamma = 1$, that is, unless the velocities are equal. Thus we find that *the entropy increases, unless both bodies have the same temperature and velocity*. Exchange of radiation between bodies with the same temperatures (in their respective rest frames), but different velocities, is an irreversible process. In other words, there is friction, due to the momentum transfer inherent in the transfer of radiation, as also mentioned by Ott.

Unless the two bodies are completely isolated from each other, they will exchange radiation and thereby approach equal temperature and velocity. This equilibrium state can be found from the condition of maximum entropy:

$$dS = 0 \quad \text{when } dU_\mu = 0. \tag{23}$$

Indeed, this variational principle leads to

$$dS = u_\mu{}^a dU_\mu{}^a / T_a{}^0 + u_\mu{}^b dU_\mu{}^b / T_b{}^0\,,$$

$$dU_\mu = dU_\mu{}^a + dU_\mu{}^b\,.$$

In order that the first line vanish for all variations for which the second line vanishes, one must have

$$u_\mu{}^a / T_a{}^0 = u_\mu{}^b / T_b{}^0\,.$$

From this one finds easily the equilibrium condition

$$T_a{}^0 = T_b{}^0\,, \quad u_\mu{}^a = u_\mu{}^b\,.$$

Note that the supplementary condition in (23) states that S has to be maximized for fixed total energy *and total momentum*. If one keeps the momentum of each system separately fixed, one finds only one equilibrium condition

$$T_a{}^0[1 - (u^a)^2]^{\frac{1}{2}} = T_b{}^0[1 - (u^b)^2]^{\frac{1}{2}} \tag{24}$$

This result has been used[18] as an argument in favor of the classical transformation formula (11). However, keeping the separate momenta fixed not only ignores the fact that momentum is transferred, it is not even a relativistically invariant condition. Nor is the resulting equilibrium condition (24).

14. ADDITIONAL REMARKS

A slightly modified way of expressing the same thermodynamic formulas is the following. For each system define a four-vector $\beta_\mu = u_\mu / T^0$. Having written the first law in the covariant form (18), one may then write the second law in the invariant form

$$dS = \beta_\mu \text{đ}Q_\mu\,. \tag{25}$$

Thus the concept of heat supply and its transformation properties is completely eliminated, although the physical results are the same.

Although this paper is concerned with macroscopic thermodynamics, a final remark on the connection with statistical mechanics should be made. For an ideal gas in equilibrium the number of particles per unit volume in a momentum range $d^3\vec{p}^{\,0}$ is given in the rest frame by the Boltzmann distribution

$$f^0(p^0) = \text{const. } e^{-\epsilon^0/T^0}\,, \tag{26}$$

where $\epsilon^0 = [m^2 + (\vec{p}^{\,0})^2]^{\frac{1}{2}}$ is the energy in the rest frame. If the gas moves with a velocity $\vec{u}$ with respect to the laboratory frame, the distribution observed in the laboratory is obtained by expressing (26) in the transformed variables,[19]

$$f(\vec{p}) = f^0(\vec{p}^{\,0}) = \text{const. } \exp\left[-\frac{\epsilon - \vec{u}\cdot\vec{p}}{T^0(1 - u^2)^{1/2}}\right]\,, \tag{27}$$

$$= \text{const. } \exp[-u_\mu p_\mu / T^0] \tag{28}$$

$$= \text{const. } \exp[-\beta_\mu p_\mu]\,, \tag{29}$$

where p_μ is the energy-momentum four-vector of a particle. It is easily seen that a similar remark applies to Bose and Fermi statistics.

Equation (27) has been used[20] as an argument in favor of the classical transformation formula (11). It should be emphasized, however, that absolute temperature is a thermodynamic concept, defined by the second law. Only when thermodynamics does not uniquely determine the relativistic extension of temperature is one free to make that choice that is most convenient in statistical mechanics.

We have shown that the choice made by Planck *et al.* is limited to homogeneous systems and leads to an awkward formulation of the first law. Ott's choice is satisfactory for homotachic processes and pure accelerations, but its application to heterotachic processes is complicated by the necessity of taking the guiding forces into account. The simplest choice is to define the temperature as a scalar, $T = T^0$, provided that one replaces the first law by the covariant Eq. (18) and distinguishes between thermal energy-momentum transfer and heat. This choice is also clearly favored by the statistical formula (28). An equivalent, but slightly more streamlined formulism consists in eliminating the concept of heat altogether, using the second law in the form (25), and using (29) for the statistical distribution.

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[1] Processes in which the number of particles varies will not be considered.

[2] The velocity of light is taken equal to unity.

[3] The acceleration process should of course be quasi-stationary, i.e., so slow that at all times the system is in equilibrium. That implies that also the stresses in the walls have time to adjust themselves. It is therefore justified to treat the box as rigid, even while u varies.

[4] An alternative choice is discussed in Sec. 7.

[5] The often repeated argument that S must be independent of u because it is the logarithm of a number of states, appears to dissolve itself when one tries to formulate it in precise terms.

[6] H. Ott, Z. Physik 175, 70 (1963).

[7] This is not a thermodynamical statement, but follows from the transformation properties of force and area.

[8] M. Planck, Sitzber. Kl. Preuss. Akad. Wiss., p. 542 (1907) or Ann. Physik 26, 1 (1908); A. Einstein, Jahrb. Radioaktivität und Elektronik 4, 411 (1907); M. von Laue, Die Relativitätstheorie (Friedrich Vieweg und Sohn, Braunschweig, Germany, 1921), 4th ed.

[9] This fact is well known and has recently been emphasized in this connection by A. Staruszkiewicz, Acta Phys. Polon. 29, 249 (1966) and F. Rohrlich, Nuovo Cimento 45B, 76 (1966).

[10] See, e.g., C. Møller, The Theory of Relativity (Clarendon Press, Oxford, 1952), p. 181.

[11] This distinction explains the difference between our formula (8) [see also: H. Arzeliès, Nuovo Cimento 35, 792 (1965); S. Putterman, unpublished] and the formula $\text{đ}A_V = \gamma^{-1}\text{đ}A_V^{0}$ derived by T. W. B. Kibble, Nuovo Cimento 41B, 72 (1966).

[12] The scalar product $u_\mu u_\mu$ is defined as $u_0^2 - u_1^2 - u_2^2 - u_3^2$.

[13] That means $g = 1$ in the notation of Sec. 5. Of course one is still free to *define* $T = T^0$, $\text{đ}Q = \text{đ}Q^0$, at the expense of modifying the first law with a factor γ. This possibility was mentioned by Ott, and advocated by P. T. Landsberg, Nature 212, 571 (1966) 214, 903 (1967).

[14] This is the reason why we refer to $\text{đ}Q$ as "heat supply" rather than "heat transfer".

[15] This was already noticed by F. Hasenöhrl, Ann. Physik. 15, 344 (1904).

[16] Ref. 15, Sec. 4.

[17] Momentum perpendicular to the sheet, however, is transferred, but this can be cancelled by two guiding forces, which prevent the two bodies from moving away from each other. One may verify that this momentum transfer does not appear in the results. For convenience we leave it out altogether.

[18] E. A. Guggenheim, Elements and Formulae of Special Relativity (Pergamon, Ltd., Oxford, 1967).

[19] P. A. M. Dirac, Proc. Roy. Soc. (London) 104A, 581 (1924).

[20] J. H. Eberly and A. Kujawski, Phys. Rev. 155, 10 (1967).

Low Temperature Measurements of the He^3 and He^4 Melting Curves*

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The melting curve of He^3 has been measured to 12 m°K in order to see whether the exchange interaction has caused the solid entropy to fall below its completely disordered value of $R\ln 2$. For $12\ \mathrm{m°K} \leqslant T \leqslant 600\ \mathrm{m°K}$, the melting curve data are described by $P = 29.107 + 0.255\,37(t)^2 - 0.057\,00(t)^3 + 0.016\,25(t)^4$ atm, where $100t = T - 299.028$ and T is the absolute temperature in m°K. The data points fit this equation with an rms deviation of 0.06 atm. Using this information and previous results for the liquid and solid molar volumes and the liquid entropy, the Clausius-Clapeyron relation shows that $S_{\mathrm{solid}} = R\ln 2 \pm 16\%$ for $12\ \mathrm{m°K} \leqslant T \leqslant 320\ \mathrm{m°K}$. This places an upper limit of 3 m°K on the magnitude of the exchange interaction in solid He^3 for molar volumes along the melting curve. The He^4 melting curve was measured between 12 m°K and 300 m°K and found to have a slope of 0 ± 0.007 atm/°K.

INTRODUCTION

The melting curve of any material provides information which relates thermodynamic properties of the liquid and solid phases along the curve. This is a consequence of the Clausius-Clapeyron equation

$$(dP/dT)_{\mathrm{melt}} = \Delta S/\Delta V, \qquad (1)$$

where P is absolute pressure, T is absolute temperature, *melt* indicates that both phases are in equilibrium, ΔS is the difference in entropy between the two phases, and ΔV is the difference in