Comments and Addenda

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Note on Nonlinear Realizations of Chiral Symmetry

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The general solution is found for Weinberg's equation for the Lagrangian of chiral-symmetry breaking. The result is an elementary function, and in the particular case of a tensor of rank 2 the effect of the nonlinearity is to multiply the ordinary Lagrangian by a simple factor.

In a recent paper, Weinberg[1] has investigated the nonlinear approach to chiral invariance, and has derived a differential equation [Ref. 1, Eq. (6.3)] for a chiral-symmetry-breaking term in the Lagrangian which transforms as a traceless symmetric tensor of rank N. He then solves this equation in a power series in the pion field. This note demonstrates that Weinberg's differential equation has a simple closed-form solution for all N.

Following the notation of Ref. 1, let $\boldsymbol{\pi}$ represent the pion field, $\mathfrak{L}_N(\boldsymbol{\pi}^2)$ be the symmetry-breaking term in the Lagrangian, and λ be a constant. Then Weinberg's differential equation is

$$(1+\lambda^2\boldsymbol{\pi}^2)^2\boldsymbol{\pi}^2\mathfrak{L}_N''(\boldsymbol{\pi}^2)+\tfrac{1}{2}(1+\lambda^2\boldsymbol{\pi}^2)(3+\lambda^2\boldsymbol{\pi}^2)\mathfrak{L}_N'(\boldsymbol{\pi}^2)+N(N+2)\lambda^2\mathfrak{L}_N(\boldsymbol{\pi}^2)=0, \quad (1)$$

where the primes denote derivatives with respect to $\boldsymbol{\pi}^2$. Define a variable z by the relation

$$z=\lambda^2\boldsymbol{\pi}^2/(1+\lambda^2\boldsymbol{\pi}^2). \quad (2)$$

After algebraic simplification there results

$$z(1-z)d^2\mathfrak{L}_N/dz^2+(\tfrac{3}{2}-3z)d\mathfrak{L}_N/dz+N(N+2)\mathfrak{L}_N=0. \quad (3)$$

Equation (3) is in the standard form of the hypergeometric differential equation. One of its solutions behaves like a constant at $z=0$, the other like $z^{-1/2}$. Choosing the first solution yields

$$\mathfrak{L}_N(z)=c_2F_1(-N,\, N+2,\, \tfrac{3}{2},\, z). \quad (4)$$

In terms of the variable z, this solution is a polynomial of degree N, which may be identified as a Chebyshev polynomial of the second kind. If a new variable x be defined as

$$z=\sin^2 x, \quad (5)$$

the solution takes the closed form

$$\mathfrak{L}_N(x)=c[\sin(2N+2)x]/(N+1)\sin 2x. \quad (6)$$

Returning to the original variable $\boldsymbol{\pi}$, the Lagrangian takes the form

$$\mathfrak{L}_N(\boldsymbol{\pi}^2)=\frac{c(1+\lambda^2\boldsymbol{\pi}^2)\sin[2(N+1)\tan^{-1}\lambda\boldsymbol{\pi}]}{2(N+1)\lambda\boldsymbol{\pi}}. \quad (7)$$

The coefficient c is determined by requiring that the coefficient of $\boldsymbol{\pi}^2$ be $-\tfrac{1}{2}m_\pi^2$, whence

$$c=3m_\pi^2/4N(N+2)\lambda^2. \quad (8)$$

The Lagrangian Eq. (7) equals c at $\boldsymbol{\pi}=0$, equals $(-1)^N c$ at $\boldsymbol{\pi}=\infty$, and has N zeros. The special case $N=2$ leads to a particularly simple form. Dropping the constant term and combining with the pion kinematic term yields

$$L_2(\boldsymbol{\pi}^2)=-\tfrac{1}{2}[\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}+m_\pi^2\boldsymbol{\pi}^2]/(1+\lambda^2\boldsymbol{\pi}^2)^2. \quad (9)$$

Thus, for $N=2$ the effect of the nonlinearity is entirely contained in a simple factor multiplying the Lagrangian in the absence of nonlinearity. In contrast, the case $N=1$ modifies the kinematic and mass terms differently. As indicated in Ref. 1, only experiment can decide which particular choice of N, if any, is correct. However, the solution (7) permits calculation of all scattering lengths, and indicates the closed form for the Lagrangian if some definite value of N be established.

[1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).