Scattering Model with Crossing Symmetry and Minimal Inelastic Effects*

FRANKLIN F. K. CHEUNG AND JOHN S. TOLL
Institute for Theoretical Physics, State University of New York, Stony Brook, New York
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The iterative method of solving the dispersion relations and unitarity relations for the two-body scattering amplitude of a neutral scalar field is investigated. A consistent and nontrivial approximating scheme is obtained by incorporating enough inelastic effects that crossing symmetry is maintained at every stage of calculations. If one has initially a twice-subtracted dispersion relation, it can be shown that because of crossing symmetry, the subtraction term linear in s, t, and u cannot be present, so that it is sufficient to start the iteration process by just one subtraction. The unitarity relations are simplified by taking the scattering particle to have zero mass. Questions concerning infrared divergencies and convergency properties in iterating the single and double density functions are fully discussed. At low energies, the amplitude is given by the first few iterations, which will be carried out explicitly to fourth order. Higher orders of iteration can be carried out in a straightforward but increasingly tedious manner. We study the behavior of the scattering amplitude in the high-energy region, where s, t, and u are all large in magnitude, while the scattering angles are held at fixed values. The leading terms of the scattering amplitude and its spectral functions are obtained for each order of iteration, and the resulting leading series are subsequently summed into closed forms for weak coupling. The ratio of the elastic cross section to the total cross section is also obtained in the same region. The result indicates that the amount of inelastic effects being included is considerable.

I. INTRODUCTION

A SOLVABLE scattering model satisfying the basic principles of the local quantum field theory such as Lorentz invariance, crossing, and unitarity relations is still nonexistent. The intricate complexity of the unitarity relations make it necessary to approximate them in some way. The simplest possibility, of course, is to replace the full unitarity by the elastic unitarity. Unfortunately, if elastic unitarity is assumed to hold for all energies in one channel, one will arrive at the trivial result that the S matrix is an identity[1]. As a consequence, the complication of inelastic processes is an inevitable feature of any relativistic field theory; and for a nontrivial result, some inelastic effects must be included in all three channels. Hence it is a question of whether one can construct a model which incorporates the minimal amount of inelastic effects as required by analyticity and crossing relations, and whether such a model can be solved.

We shall study here the iterative method of obtaining approximate solutions for dispersion relations and the unitarity relations for the two-body scattering amplitude satisfying the analyticity and crossing requirements of the Mandelstam representation.[2] To avoid nonessential complications from spin and isotopic spins, we shall consider a neutral scalar particle with pairing symmetry for which the physics is the same in all three channels. We shall incorporate some but not all inelastic processes, so that crossing symmetry is always maintained at all stages of our calculations. This is the minimum amount of inelastic contributions to the unitarity integrals for nontrivial results as demanded by crossing symmetry alone. In this manner we get a consistent approximation scheme by replacing all the single density functions by the elastic single density functions and the double density functions by the symmetrized elastic double density functions.

For computational purposes, we shall also construct a new kind of single dispersion relation that is explicitly symmetric in all three channels and which involves only elastic absorptive parts. We shall then set out to iterate this approximate dynamical system. Although two subtractions may be needed,[3] we shall show that because of crossing symmetry, the linearly divergent subtraction terms can always be reduced to a constant. Thus it is sufficient to start our iteration with a once-subtracted dispersion relation. The first-order iteration amplitude is the subtraction constant itself, which also plays the role of the coupling constant. The calculation is further simplified by taking the mass of the particle to be zero,[4] which enables one to eliminate certain complicating kinematic factors in the unitarity integrals. This simplification comes about only at the expense of introducing logarithmic infrared divergencies in some of the integrals. However, such infrared divergencies can be naturally absorbed into the subtraction terms in such a way that we get an nth-order amplitude which is regular in the finite s-t planes but which has logarithmic singularities at thresholds and at infinity. The dual characteristics of these essential singularities at the origin and at infinity will be made clear. Further

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[1] S. Aks, J. Math. Phys. **6**, 516 (1965); F. K. Cheung and J. S. Toll, Phys. Rev. **160**, 1072 (1967); F. K. Cheung, *ibid.* **166**, 1828 (1968). We will consider the neutral scalar field with pairing symmetry, so that there are no pole terms in the corresponding Mandelstam representation in Eq. (1).

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958). For simplicity, our discussion refers explicitly to the Mandelstam representation. However, in the approximations being used in the following, the Mandelstam representation actually follows from the axioms of the local field theory. See Cheung and Toll (Ref. 1).

[3] A. Martin and Y. S. Jin, Phys. Rev. **135**, B1375 (1964).

[4] With a fair amount of algebraic complications everything done in this paper can be carried out for the finite mass case.

complications in iterating the double density functions are fully discussed. At low energies, the solution is described by the first few orders of iterations, which will be carried out explicitly to fourth order. Higher orders of iteration can be carried out, but they are not expected to bring new features into the model. Therefore, we shall go on to study the behavior of the scattering amplitude when s, t, and u are all large in magnitude while the scattering angles are held at fixed values. In this region, the leading terms of the scattering amplitude and its spectral density functions are obtained in each order of iteration. This is made possible because in this limit $\ln s \simeq \ln t \simeq \ln u$. The series so obtained will be convergent when the coupling is weak and can be summed to close analytic forms. The ratio of the elastic cross section to the total cross section will also be obtained in the same region, and the result indicates that the amount of inelastic processes we have incorporated is considerable. In the zero-mass limit, the initial subtraction term is not necessarily a real number. The cases of $\bar{A}'=i$ and more generally of $\bar{A}'=e^{i\phi}$ will be considered. However, the solutions so generated are all alike and have similar properties. A brief summary and a few remarks then follow in the conclusion.

II. APPROXIMATE DYNAMICAL SYSTEM

Let us consider the scattering amplitude for two neutral scalar particles of mass m, satisfying the analyticity requirement of a once-subtracted Mandelstam representation[2]

$$A(s,t,u)=\lambda+\frac{s-\alpha}{\pi}\int_{4m^2}^{\infty}\frac{\sigma_s(s')ds'}{(s'-s)(s'-\alpha)}+\frac{t-\alpha}{\pi}\int_{4m^2}^{\infty}\frac{\sigma_t(t')dt'}{(t'-t)(t'-\alpha)}+\frac{u-\alpha}{\pi}\int_{4m^2}^{\infty}\frac{\sigma_u(u')du'}{(u'-u)(u'-\alpha)}+\frac{(s-\alpha)(t-\alpha)}{\pi^2}$$
$$\times\int^{\infty}\int_{4m^2}\frac{A_{st}(s',t')ds'dt'}{(s'-s)(s'-\alpha)(t'-t)(t'-\alpha)}+\frac{(t-\alpha)(u-\alpha)}{\pi^2}\int^{\infty}\int_{4m^2}dt'du'\frac{A_{tu}(t',u')}{(t'-t)(t'-\alpha)(u'-u)(u'-\alpha)}$$
$$+\frac{(u-\alpha)(s-\alpha)}{\pi^2}\int^{\infty}\int_{4m^2}du'ds'\frac{A_{us}(u',s')}{(u'-u)(u'-\alpha)(s'-s)(s'-\alpha)},\qquad(1)$$

where, as in all the following dispersion integrals, each factor $s'-s$, $t'-t$, and $u'-u$ in the denominators is taken to have a small negative imaginary part. λ is the subtraction constant $\lambda=A(s_0,t_0,u_0)$, with $s_0=t_0=u_0=\alpha=\frac{4}{3}m^2$. Crossing symmetry now requires that

$$A(s,t,u)=A(t,s,u)=A(u,t,s),\qquad(2)$$

which implies the following crossing relations for the single density function:

$$\sigma_s(x)=\sigma_t(x)=\sigma_u(x)=\sigma(x).\qquad(3)$$

If we define

$$\rho(\xi,\eta)\equiv A_{st}(\xi,\eta),\qquad(4)$$

then Eq. (2) also requires that[2]

$$\rho(\xi,\,4m^2-\xi-\eta)=A_{su}(\xi,\eta),\qquad(5)$$

$$\rho(4m^2-\xi-\eta,\,\eta)=A_{ut}(\xi,\eta),\qquad(6)$$

and furthermore $\rho(\xi,\eta)$ is symmetric with respect to its two arguments,

$$\rho(\xi,\eta)=\rho(\eta,\xi).\qquad(7)$$

In terms of σ and ρ, Eq. (1) may be made manifestly crossing-symmetric by rewriting it as

$$A(s,t,u)=\lambda+\frac{1}{\pi}\int_{4m^2}^{\infty}d\xi\,\sigma(\xi)\left[\frac{s-\alpha}{(\xi-s)(\xi-\alpha)}+\text{c.p.}\right]$$
$$+\frac{1}{\pi^2}\int^{\infty}\int_{4m^2}d\xi\,d\eta\,\rho(\xi,\eta)$$
$$\times\left[\frac{(s-\alpha)(t-\alpha)}{(\xi-\alpha)(\xi-s)(\eta-t)(\eta-\alpha)}+\text{c.p.}\right],\qquad(8)$$

where c.p. means similar terms obtained by cyclic permutation of s, t, and u. We see that the scattering amplitude is entirely determined by a subtraction constant λ, a single density function $\sigma(\xi)$, and a symmetric density function $\rho(\xi,\eta)=\rho(\eta,\xi)$.

If we let $\nu(s,t)$ be the s-channel absorptive part, then

$$\nu(s,t)=\sigma(s)+\frac{1}{\pi}\int_{4m^2}^{\infty}d\eta\,\rho(s,\eta)$$
$$\times\left[\frac{t-\alpha}{(\eta-t)(\eta-\alpha)}+\frac{u-\alpha}{(\eta-u)(\eta-\alpha)}\right],\qquad(9)$$

and by crossing symmetry the t- and u-channel absorptive parts are given by $\nu(t,s)$ and $\nu(4m^2-t-s,\,t)$, respectively.

We shall represent the unitarity condition in the s channel (similarly for the t and u channels) by the following[2]:

$$\nu(s,t)=\tfrac{1}{8}\theta(s-4m^2)\left(\frac{s-4m^2}{s}\right)^{1/2}\times\int dz_1\int dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}A^*(s;z_1)A(s;z_2)+\nu^{\mathrm{in}(s)}(s,t)$$
$$\equiv\nu^{\mathrm{el}(s)}(s,t)+\nu^{\mathrm{in}(s)}(s,t) \tag{10}$$

and

$$\rho(s,t)=\tfrac{1}{2}\theta(s-4m^2)\left(\frac{s-4m^2}{s}\right)^{1/2}\times\int dz_1\int dz_2\frac{\theta(K_s)}{\sqrt{K_s}}2\nu^*(t_1,s)\nu(t_2,s)+\rho^{\mathrm{in}(s)}(s,t)$$
$$\equiv\rho^{\mathrm{el}(s)}(s,t)+\rho^{\mathrm{in}(s)}(s,t)\,, \tag{11}$$

where $K_s=z_s^2+z_1^2+z_2^2-1-2z_sz_1z_2$, with $z_s=1+2t/(s-4m^2)$ being the s-channel elastic unitarity kernal; $\nu^{\mathrm{in}(s)}(s,t)$ and $\rho^{\mathrm{in}(s)}(s,t)$ are the s-channel inelastic contribution to $\nu(s,t)$ and $\rho(s,t)$, respectively. On the other hand, $\rho(s,t)$ is also given by

$$\rho(s,t)=\tfrac{1}{2}\theta(t-4m^2)\left(\frac{t-4m^2}{t}\right)^{1/2}\times\int dz_1\int dz_2\frac{\theta(K_t)}{\sqrt{K_t}}2\nu^*(s_1;t)\nu(s_2;t)+\rho^{\mathrm{in}(t)}(s,t)$$
$$\equiv\rho^{\mathrm{el}(t)}(s,t)+\rho^{\mathrm{in}(t)}(s,t)\,, \tag{12}$$

where K_t is K_s with z_s replaced by $z_s=1+2s/(t-4m^2)$.

Now $A(s,t,u)$ is taken to be analytic in the joint domain of s-t planes, with cuts only on the real axis. The double discontinuities across the s and t cuts should not depend on the order of crossing these cuts; hence

$$\rho(s,t)=\rho^{\mathrm{el}(s)}(s,t)+\rho^{\mathrm{in}(s)}(s,t)=\rho^{\mathrm{el}(t)}(s,t)+\rho^{\mathrm{in}(t)}(s,t)=\rho(s,t)\,. \tag{13}$$

For a finite-mass particle, setting either $\rho^{\mathrm{in}(s)}(s,t)$ or $\rho^{\mathrm{in}(t)}(s,t)$ equal to zero will lead to the trivial solution that the amplitude itself is zero.[1] For a zero-mass particle, this is not necessarily the case. However, in any event it is evident from Eqs. (11) and (12) that neither $\rho^{\mathrm{el}(s)}(s,t)$ nor $\rho^{\mathrm{el}(t)}(s,t)$ is crossing-symmetric in s and t, which we insist on for all approximations used in the following. Thus to satisfy crossing, the minimum amount of inelastic contributions to $\rho^{\mathrm{in}(s)}(s,t)$ is $\rho^{\mathrm{el}(t)}(s,t)$ and vice versa. In other words, there must contain in $\rho^{\mathrm{in}(s)}(s,t)$ a part which is elastic in t. This is most easily seen if one writes all the unitarity diagrams contributing to $\rho^{\mathrm{el}(t)}(s,t)$. From the lack of a three-neutral-scalar-particle vertex,[1] these diagrams are always inelastic in s. This part of inelastic effects is included to satisfy crossing while the rest of $\rho^{\mathrm{in}(s)}$ is neglected, and similarly for $\rho^{\mathrm{in}(t)}$. We define

$$\bar\rho(s,t)\equiv\rho^{\mathrm{el}(s)}(s,t)+\rho^{\mathrm{el}(t)}(s,t)\,, \tag{14}$$

where an overbar means a symmetrized elastic quantity. Therefore an approximation scheme consistent with crossing is to replace all the double density functions in Eq. (8) by the symmetrized elastic double density functions and the single density functions by the elastic single density functions,

$$A(s,t,u)\simeq\bar A(s,t,u)=\lambda+\frac{1}{\pi}\int d\xi\,\bar\sigma(\xi)\left[\frac{s-\alpha}{(\xi-s)(\xi-\alpha)}+\text{c.p.}\right]+\frac{1}{\pi^2}\int\int d\xi\,d\eta\,\bar\rho(\xi,\eta)\times\left[\frac{(s-\alpha)(t-\alpha)}{(\xi-s)(\xi-\alpha)(\eta-\alpha)(\eta-t)}+\text{c.p.}\right], \tag{15}$$

where $\bar\sigma$ is now determined by the approximate s-channel absorptive part

$$\bar\nu(s,t)=\bar\sigma(s)+\frac{1}{\pi}\int d\eta\,\bar\rho(s,\eta)\times\left[\frac{t-\alpha}{(\eta-t)(\eta-\alpha)}+\frac{u-\alpha}{(\eta-u)(\eta-\alpha)}\right] \tag{16}$$

and from now on, the A's and ν's in Eqs. (10), (11), and (12) must be replaced by the $\bar A$'s and $\bar\nu$'s, respectively. For computational purposes, let us also define the s-channel elastic absorptive parts by

$$\hat\nu(s,t)=\bar\sigma(s)+\frac{t-\alpha}{\pi}\int d\eta\frac{\rho^{\mathrm{el}(t)}(\eta,s)}{(\eta-t)(\eta-\alpha)}+\frac{u-\alpha}{\pi}\int\frac{\rho^{\mathrm{el}(u)}(s,\eta)du}{(\eta-u)(\eta-\alpha)}\,. \tag{17}$$

Equation (15) can then be rewritten in the following symmetric form:

$$\bar A(s,t,u)=\lambda+\frac{s-\alpha}{\pi}\int\frac{\hat\nu(\xi,t)d\xi}{(\xi-s)(\xi-\alpha)}+\frac{t-\alpha}{\pi}\int\frac{\hat\nu(\xi,s)d\xi}{(\xi-t)(\xi-\alpha)}+\frac{u-\alpha}{\pi}\int\frac{\hat\nu(\xi,t)d\xi}{(\xi-u)(\xi-\alpha)}\,. \tag{18}$$

We should note the difference between this crossing-

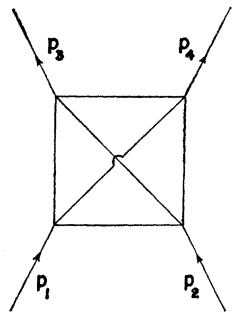

FIG. 1. The lowest-order dispersion which is simultaneously inelastic in all three channels.

symmetric approximate single dispersion relation and the usual fixed-energy or fixed-momentum-transfer dispersion relation. The convenience in using $\hat{\rho}$ instead of $\bar{\nu}$ comes from the fact that $\hat{\rho}$ is entirely given by the elastic unitarity integral in the s channel.

Equations (17) and (18) together with Eqs. (10) and (11) now form an approximate dynamical system for the scattering amplitude of two neutral scalar particles which we are considering.

We note that the inelastic effects being neglected correspond to those processes that are simultaneously inelastic in all three channels. In terms of dispersion diagrams, the lowest-order contribution to such processes comes from the one shown in Fig. 1. On the other hand, the s-channel inelastic effects that we have included are those that are elastic in either t or u channels.

Presumably in a complete theory one has to add these two kinds of inelastic amplitudes coherently. Nevertheless, it is easy to see that the inelastic effects included, as well as those neglected, always give a positive contribution to the s-channel absorptive part in the forward direction. Let us consider[4]

$$\mathrm{Im}\langle i|T|i\rangle=\tfrac{1}{2}\sum_n \frac{1}{(2\pi)^{3n-4}}\times\int\cdots\int dq_i\cdots dq_n\prod_{i=1}^{n}\delta(q_i^2+\mu_i^2)\times\theta(q_i)\delta(Q-\textstyle\sum_i q_i)|\langle n|T|i\rangle|^2, \quad (19)$$

where the summation over n corresponds to summing over all possible intermediate states; $n=2$ is the elastic scattering contribution. Our approximation includes partial contributions from each possible inelastic intermediate states. Since all factors in the integrand are positive, it is certainly true that the inelastic processes included give a positive contribution to $\hat{\rho}(s,0)$ and likewise for the neglected inelastic effects. If we denote the inelastic cross section of the included and neglected parts by $\sigma_{\mathrm{in}}^{(1)}$ and $\sigma_{\mathrm{in}}^{(2)}$, respectively, then $\sigma_{\mathrm{in}}^{(1)}$, $\sigma_{\mathrm{in}}^{(t)}\geq 0$. Let us also denote the elastic scattering cross section given in this model by $\bar{\sigma}_{\mathrm{el}}$ and define

$$\bar{\sigma}_{\mathrm{tot}}=\bar{\sigma}_{\mathrm{el}}+\sigma_{\mathrm{in}}^{(1)}. \quad (20)$$

Then by the optical theorem[5] we have

$$\frac{\bar{\sigma}_{\mathrm{el}}}{\bar{\sigma}_{\mathrm{tot}}}=\frac{1}{8}\left(\frac{s-4m^2}{s}\right)^{1/2}\int(\bar{A})^2d\Omega\Big/\bar{\nu}(s,0). \quad (21)$$

This ratio will serve as an indication of the amount of inelastic effects that have been incorporated.

III. ITERATIVE SOLUTION

We shall study in the following the iterative method of solving the approximate dynamical system given in Sec. II. To do this, let us expand the amplitude and its spectral functions into powers of λ.

$$\bar{A}(s,t,u)=\sum_{n=1}^{\infty}\lambda^n\bar{A}^n(s,t,u), \quad (22a)$$

$$\bar{\nu}(s,t,u)=\sum_{n=2}^{\infty}\lambda^n\bar{\nu}^n(s,t,u), \quad (22b)$$

$$\bar{\rho}(s,t,u)=\sum_{n=4}^{\infty}\lambda^n\bar{\rho}^n(s,t,u). \quad (22c)$$

Here λ plays the same role as the coupling constant in a usual perturbative series; whether the series in Eq. (22) will converge or not depends on the magnitude of λ. It is a tacit premise of the iterative method that λ is not too large; more precise conditions on λ will be given as we go along.

From here on, we will also take $m=0$.[4] This simplifies the kinematics of Eqs. (10) and (11) at the price of introducing infrared divergencies in some of the integrals later. Whatever ambiguities occur, it will be understood that m is small and is allowed to go to zero after all mathematical calculations are performed. The results of Sec. IV concerning large-energy behavior of the scattering amplitude are the same whether $m=0$ or is of some finite value. The infrared divergencies from the zero-mass limit can be conveniently absorbed into the subtraction term. For this purpose, let us rewrite Eq. (18) as

$$\sum_{n=1}\bar{A}^n\lambda^n-\sum_{n=1}\bar{A}^n(\alpha,\alpha)\lambda^n=\frac{1}{\pi}\int_0^{\infty}\sum_{n=2}\hat{\rho}^n\lambda^n\left(\frac{1}{s'-s}-\frac{1}{s'-\alpha}\right)ds'+\text{c.p.}, \quad (23)$$

from which we obtain the dispersion relation for the nth-order scattering amplitude

$$\bar{A}^n(s,t,u)=\bar{A}^n(\alpha,\alpha)+\frac{1}{\pi}\int_0^{\infty}\hat{\rho}^n\left(\frac{1}{s'-s}-\frac{1}{s'-\alpha}\right)ds'+\text{c.p.}, \quad (24)$$

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[5] G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964), p. 18. Our normalization of the unitarity condition in Eqs. (10) and (11) corresponds to $8\pi^2A(s,t)=\langle f|T|i\rangle$.

where, unlike in Eq. (18), $A^n(\alpha,\alpha)$ is no longer set equal to zero for $n\geq 2$. Its value will depend on the nature of the threshold singularity and will be determined by the requirement that the nth-order iterative amplitude is regular in the finite s-t planes.

When the scattering particles are of finite mass, $A(\frac{4}{3}m^2,\frac{4}{3}m^2,\frac{4}{3}m^2)$ is real, and the iteration is usually initiated by letting $\bar{A}^1(s,t)=\bar{A}^1(\alpha,\alpha)=1$. In the zero-mass limit, this leads to a particular solution. More general solutions will be considered in a later section by taking $\bar{A}^1$ to be imaginary or complex.

With $\bar{A}^1=1$, $\hat{\rho}^2(s,t)$ is given by the elastic unitarity integral

$$\hat{\rho}^2(s,t)=\tfrac{1}{8}\theta(s)\int_{-1}^{1}dz_1\int_{-1}^{1}dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}\times\bar{A}^1(s,z_1)\bar{A}^1(s,z_2)^*=\tfrac{1}{4}\pi\theta(s). \quad (25)$$

By crossing symmetry, the corresponding quantities in the t and u channels must be $\frac{1}{4}\pi\theta(t)$ and $\frac{1}{4}\pi\theta(u)$, respectively. Equation (24) now gives

$$\bar{A}^2(s,t,u)=\bar{A}^2(\alpha,\alpha)+\frac{1}{4}\int_0^\infty\left(\frac{1}{s'-s}-\frac{1}{s'-\alpha}\right)ds'+\text{c.p.}$$
$$=\bar{A}^2(\alpha,\alpha)+\tfrac{1}{4}\ln\left(\frac{-\alpha}{-s}\right)+\text{c.p.},$$

which we rewrite as

$$\bar{A}^2(s,t,u)+\tfrac{1}{4}[\ln(-s)+\text{c.p.}]=\bar{A}^2(\alpha,\alpha)+\tfrac{3}{4}\ln(-\alpha). \quad (26)$$

In the limit $m\to 0$, α is zero. The appearance of $\ln(-\alpha)$ shows the logarithmic infrared divergency characteristic of a theory for a zero-mass particle. If $\bar{A}^2(\alpha,\alpha)$ is set equal to zero, as in Eq. (18), $\bar{A}^2(s,t,u)$ is logarithmically divergent everywhere in the finite s-t planes except at the subtraction point. However, we recognize that physically the divergencies come from the fact that the threshold at the origin is an essential singularity when the scattering particles have zero mass. For this reason, the subtraction should be normalized in such a way that we get finite nth-order amplitude in the finite s-t planes but which is logarithmically divergent at the threshold. This is done by noticing that since the right-hand side of Eq. (26) does not depend on s or t, it must be equal to a constant, and we may consistently take the constant to be zero, so that

$$\bar{A}^2(s,t)=-\tfrac{1}{4}[\ln(-s)+\ln(-t)+\ln(-u)], \quad (27)$$

where that branch of $\ln(-s)$ is taken such that for fixed t, $\bar{A}^2$ has a cut along the real axis $s>0$ with discontinuity $\hat{\rho}^2=\frac{1}{4}\pi\theta(s)$ and that

$$\ln[-(s+i\epsilon)]=\ln s-i\pi \quad \text{for} \quad s>0,\ \text{real}. \quad (28)$$

Infrared divergencies in higher-order iterations will be treated in the same way. Thus suppose that the dispersion integrals in Eq. (24) have been evaluated; the nth-order amplitude will then take the form

$$\bar{A}^n(s,t,u)-f^n(s,t,u)=\bar{A}^n(\alpha,\alpha)-f^n(\ln(-\alpha))=k^n, \quad (29)$$

where k^n is some constant independent of s and t, and may be taken to be zero. In this manner[6] we have removed the logarithmic divergencies from the finite s-t planes at the expense of a threshold singularity.

In the third-order iteration,

$$\hat{\rho}^3(s,t)=\tfrac{1}{8}\theta(s)\int_{-1}^{1}dz_1\int_{-1}^{1}dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}\times[\bar{A}^2(s,z_1)+\bar{A}^{2*}(s,z_2)]. \quad (30)$$

From Eq. (27), evaluation of the integrals gives

$$\hat{\rho}^3(s,t)=-\tfrac{3}{8}\pi\ln s+\tfrac{1}{4}\pi, \quad (31)$$

and from the dispersion integrals Eq. (24), we have

$$\bar{A}^3(s,t)=\bar{A}^3(\alpha,\alpha)+\tfrac{3}{4}\ln(-\alpha)-\tfrac{9}{16}\ln^2(-\alpha)+\tfrac{3}{16}[\ln^2(-s)+\ln^2(-t)+\ln^2(-u)]-\tfrac{1}{4}[\ln(-s)+\ln(-u)+\ln(-t)], \quad (32)$$

which is normalized to

$$\bar{A}^3(s,t)=\tfrac{3}{16}[\ln^2(-s)+\ln^2(-t)+\ln^2(-u)]-\tfrac{1}{4}[\ln(-s)+\ln(-t)+\ln(-u)]. \quad (33)$$

We note that, up to the third order, the absorptive parts depend only on one variable and the scattering amplitude has no cross terms in s, t, or u. Consequently, there are no double density functions. In these orders, a single bar or caret over an absorptive part makes no difference. Before we go into the complication of the double density functions we wish to clear up one point about the asymptotic behavior of the scattering amplitude in iterating the single density functions. From Jin and Martin,[3] it is more appropriate to start in Eq. (18) a twice-subtracted dispersion relation for the amplitude. Now if we initially start with the subtraction term

$$\bar{A}^1=\lambda_1+s\lambda_2,$$

the elastic unitarity integral will give

$$\bar{\rho}^2=\tfrac{1}{8}\theta(s)\int\int_{-1}^{1}dz_1dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}\bar{A}^1\bar{A}^{1*}\simeq s^2, \quad (34)$$

so that the dispersion relation for $\bar{A}^2(s,t,u)$ will require three subtractions. It is easy to see that because of the nonlinear nature of the unitarity integrals, more and more subtractions may be needed as we go to higher-order iterations. However, here crossing symmetry

[6] We note that now λ is no longer given by $\bar{A}(\alpha,\alpha)$. From the fact that $f^n(\ln(-\alpha))=0$ for $\ln(-\alpha)=0$, we have $\lambda=\bar{A}(-1,-1,-1)$, which means that λ can be obtained from the amplitude itself only by extrapolating it off the mass shell.

comes into play, and to satisfy it $\bar{A}^1$ must take the form

$$\bar{A}^1=\lambda_1+(s+t+u)\lambda_2. \quad (35)$$

Since $s+t+u=0$, therefore $\bar{A}^1=\lambda_1$, i.e., the linearly divergent term cannot be present because of crossing symmetry. Even for the finite-mass case, $s+t+u=4m^2$, so that the linearly divergent input can always be included in the constant term λ_1, provided that there is complete symmetry in s, t, and u. Furthermore, we have seen that at least up to the third order the subtraction constant λ can generate through iterations an amplitude and its absorptive parts which behave only like a power of logarithms in s, t, and u, which is consistent with the initial assumption that only one subtraction is needed. If a superfluous subtraction is made on the single density function integrals,

$$\bar{A}(s,t,u)=\bar{A}(\alpha,\alpha)+\frac{(s-\alpha)^2}{\pi}\int_0^\infty\frac{\bar{\sigma}(s')ds'}{(s'-s)(s'-\alpha)^2}+\text{c.p.}$$
$$+\text{contributions from the double density functions}, \quad (36)$$

then by crossing symmetry the contribution of second subtraction is

$$\int_0^\infty\bar{\sigma}(\xi)d\xi\left(\frac{s-\alpha}{(\xi'-\alpha)^2}+\frac{t-\alpha}{(\xi'-\alpha)^2}+\frac{u-\alpha}{(\xi'-\alpha)^2}\right)$$
$$=\frac{s+t+u-3\alpha}{\pi}\int_0^\infty\bar{\sigma}(\xi)\frac{d\xi}{(\xi-\alpha)^2}\equiv 0. \quad (37)$$

Hence the extra subtraction gives no contributions, even for a finite-mass particle. We see that in any case a linearly divergent subtraction term cannot be present due to crossing symmetry and there is no need to assume initially a twice-subtracted dispersion relation for $A(s,t,u)$. However, we shall see in the following that quite different situations may come up in iterating the double density functions.

Coming back to our iteration procedure, we remarked that starting from the fourth order, the s-channel absorptive part is no longer just a function of s because there are now contributions from the double density functions. From Eq. (16), the single density function $\bar{\sigma}(s)$ is that part of $\bar{\nu}(s,t)$ which depends on s alone. Since $\bar{\nu}(s,t)$ as well as $\bar{\rho}(s,t)$ can be generated by the unitarity relations, $\bar{\sigma}(s)$ can be obtained once these quantities are known. However, it will be more convenient if we can calculate the nth-order single density function $\bar{\sigma}^n$ from the unitarity relation directly. This is easily done using the fact that $\bar{\sigma}^n$ is that part of $\bar{\nu}^n(s,t)$ which is only a function of s; it is given by

$$\bar{\sigma}^n(s)=\tfrac{1}{8}\theta(s)\int_{-1}^1 dz_1\int_{-1}^1 dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}$$
$$\times[\bar{A}^{(n-1)}(s,z_1)\bar{A}^{1*}(s,z_2)+(\bar{A}^{(n-2)}\bar{A}^{2*}-\bar{A}'^{(n-2)}\bar{A}'^{2*})$$
$$+\cdots+\bar{A}^1\bar{A}^{(n-1)*}], \quad (38)$$

where a prime means only those terms that have z_i, $i=1$, 2, or equivalently either t_i or u_i dependence are included. This definition for $\bar{\sigma}^n(s)$ actually follows uniquely from Eqs. (16), (10), and (11) because if another function $g^n(s)$ is added to $\bar{\sigma}^n(s)$, then consistency with these equations requires that

$$\sum_{n=1}^\infty\lambda^n g^n(s)=0. \quad (39)$$

Because this is true for all values of λ, we must have $g^n(s)=0$ for each n.

To show that the integrals on the right-hand side of Eq. (38) give a function of s alone, we note that within each superscript parentheses in the integrand, the quantity depends only on either z_1 or z_2 but not both. If we make use of the expansion[7]

$$\frac{\theta(-K_s)}{\sqrt{K_s}}=\tfrac{1}{2}\pi\sum_{l=0}(2l+1)P_l(z_s)P_l(z_1)P_l(z_2), \quad (40)$$

then one of the integrals can be easily carried out and because of the orthogonality of the Legendre polynomials, only the $l=0$ term is left, so that the z dependence drops out of the remaining integrals. It follows that $\bar{\sigma}^n(s)$ depends only on s. This also means that $\bar{\sigma}^n(s)$ has no t or u cut and is trivially symmetric in t and u. We also note that in our approximation the inelastic processes being incorporated do not contribute to $\bar{\sigma}^n(s)$ since these contributions have explicitly a t or u cut.

Let us now apply Eq. (38) to the fourth-order iteration

$$\bar{\sigma}^{(4)}(s)=\tfrac{1}{8}\theta(s)\int_{-1}^1 dz_1$$
$$\times\int_{-1}^1 dz_2[\bar{A}^3\bar{A}^{1*}+(\bar{A}^2\bar{A}^{2*}-\bar{A}'^2\bar{A}'^2)+\bar{A}^1\bar{A}^{3*}]$$
$$=\tfrac{1}{64}\pi[23\ln^2 s-52\ln s+40-7\pi^2], \quad (41)$$

which gives after symmetrization the following contribution to $\bar{A}^{(4)}$:

$$\bar{A}^{(4)}(\bar{\sigma}^{(4)};s,t)=\tfrac{1}{64}\{-(23/3)\ln^3(-s)+26\ln^2(-s)$$
$$-[40-(44/3)\pi^2]\ln(-s)\}+\text{c.p.} \quad (42)$$

There are now also contributions coming from the double density functions which greatly complicate the calculation. From Eq. (11), $\rho^{\text{el}(s);(4)}(s,t)$ is given by

$$\rho^{\text{el}(s);(4)}(s,t)=\tfrac{1}{2}\theta(s)\int\int dz_1dz_2\frac{\theta(K_s)}{\sqrt{K_s}}[2\bar{\nu}^2(t_1,s)\bar{\nu}^{2*}(t_2,s)]$$
$$=\tfrac{1}{16}\pi^2\int_1^z dz_1\int_1^{z_2^-}dz_2\frac{1}{\sqrt{K_s}}, \quad (43)$$

[7] R. Omnes, Nuovo Cimento **25**, 806 (1962).

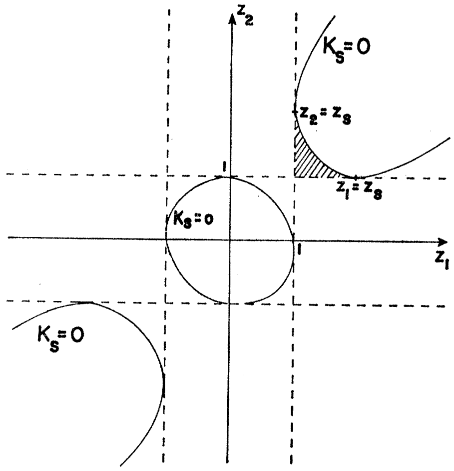

FIG. 2. The curves shown correspond to $K_s(z_s,z_1,z_2)=0$ on the z_1-z_2 plane. The shaded area is the region where the s-channel elastic unitarity integrals for the double density function $\rho^{\mathrm{el}(s)}(s,t)$ should be evaluated.

where the integration region is restricted by $\theta(K_s)$ and is given by the shaded region of Fig. 2, and where

$$z_2^- = zz_1-(z^2-1)^{1/2}(z_1^2-1)^{1/2}. \tag{44}$$

The integrals in Eq. (43) can be carried out explicitly to give

$$\rho^{\mathrm{el}(s);(4)}(s,t)=\tfrac{1}{16}\pi^2[\ln(z+1)-\ln 2], \tag{45}$$

or, since $z=1+2t/s$,

$$\rho^{\mathrm{el}(s);(4)}(s,t)=\tfrac{1}{16}\pi^2[\ln(s+t)-\ln s]. \tag{46}$$

After symmetrization, this gives the following contribution to $\bar{A}^{(4)}$:

$$A^{(4)}(\bar{\rho}^{(4)};s,t)=\bar{A}^4(\alpha,\alpha)+\int_0^\infty \frac{\bar{\rho}(\xi,\alpha)d\xi}{(\xi-s)(\xi-\alpha)}(s-\alpha)$$
$$+\text{c.p.}+\tfrac{1}{16}(s-\alpha)(t-\alpha)$$
$$\times\int\int\frac{2\ln(s'+t')-\ln s'-\ln t'}{(s'-s)(s'-\alpha)(t'-t)(t'-\alpha)}ds'dt'+\text{c.p.} \tag{47}$$

The integrals involving $\ln s'+\ln t'$ are easy to perform:

$$\tfrac{1}{16}(s-\alpha)(t-\alpha)\int_0^\infty\int_0^\infty\frac{(\ln s'+\ln t')ds'dt'}{(s'-s)(s'-\alpha)(t'-t)(t'-\alpha)}$$
$$=\tfrac{1}{32}\ln^2(-\alpha)[2\ln(-\alpha)-\ln(-t)-\ln(-s)]$$
$$+\tfrac{1}{32}\ln(-\alpha)[2\ln^2(-\alpha)-\ln^2(-s)-\ln^2(-t)]$$
$$+\tfrac{1}{32}\ln^2(-s)\ln(-t)+\tfrac{1}{32}\ln^2(-t)\ln(-s). \tag{48}$$

As remarked after Eq. (26), the first two terms may be absorbed into the subtraction terms of $\bar{\rho}(s,\alpha)$ and $\bar{\rho}(\alpha,t)$. The integration with $\ln(s'+t')$ is more tedious to carry out. After leaving out infrared divergencies it gives

$$\tfrac{1}{8}(s-\alpha)(t-\alpha)\int^\infty\int_0\frac{\ln(s'+t')ds'dt'}{(s'-s)(s'-\alpha)(t'-t)(t'-\alpha)}=\frac{1}{32}\Bigg[\frac{1}{3}\pi^2\ln(t+s)-\tfrac{1}{3}\ln^3(-t-s)-\frac{t}{2t+s}\ln^2(-t)+2\ln(-t)\ln\frac{2t+s}{t+s}$$
$$+\sum_{n=1}^\infty\frac{1}{n^2}\left(\frac{t}{t+s}\right)^n\Bigg]-\tfrac{1}{16}\ln(-t)\left[-\tfrac{1}{2}\ln^2(-s-t)+\ln(s+t)\ln\frac{s+t}{s}-\ln(s+t)+\sum_{n=1}^\infty\frac{1}{n^2}\left(\frac{s}{s+t}\right)^n\right]$$
$$+\frac{1}{16}\sum_{n=1}^\infty\frac{1}{n^2}\left(\frac{t}{s+t}\right)^n\left[-\ln\left(\frac{-s}{t}\right)+\sum_{m=1}^{n-1}\frac{1}{m}\left(\frac{s+t}{t}\right)^m\right]+\{s\leftrightarrow t\}. \tag{49}$$

Collecting all the results, we obtain the fourth-order amplitude as

$$A^4(s,t)=\{\tfrac{1}{64}\{-(23/3)\ln^3(-s)+26\ln^2(-s)-[40-(44/3)\pi^2]\ln(-s)\}+\text{c.p.}\}$$
$$+\tfrac{1}{32}\{\ln^2(-s)\ln(-t)+\ln^2(-t)\ln(-s)+\text{c.p.}\}+\frac{1}{16}\Bigg\{\Bigg[\frac{1}{3}\pi^2\ln(t+s)-\tfrac{1}{3}\ln^2(-t-s)-\frac{t}{2t+s}\ln^2(-t)$$
$$+2\ln(-t)\ln\frac{2t+s}{t+s}+\sum_{n=1}^\infty\frac{1}{n^2}\left(-\frac{t}{t+s}\right)^n\Bigg]+\left[\frac{1}{2}\ln^2(-s-t)+\ln(s+t)\ln\frac{s+t}{s}+\ln(s+t)\right][\ln(-t)+\ln(-s)]$$
$$-\sum_{n=1}\frac{1}{n^2}\left[\ln(-t)\left(\frac{s}{s+t}\right)^n+\left(\frac{t}{s+t}\right)^n\ln(-s)\right]+\sum_{n=1}\frac{1}{n^2}\left(\frac{t}{t+s}\right)^n\left[-\ln\left(\frac{-s}{t}\right)+\sum_{m=1}^{n-1}\frac{1}{m}\left(\frac{s+t}{t}\right)^m\right]$$
$$+\sum_{n=1}\frac{1}{n^2}\left(\frac{s}{s+t}\right)^n\left[-\ln\left(\frac{-t}{s}\right)+\sum_{m=1}^{n-1}\frac{1}{m}\left(\frac{s+t}{s}\right)^m\right]+\text{c.p.}\Bigg\}. \tag{50}$$

We note that the fourth-order double density function $\bar{\rho}^{(4)}$ behaves like a logarithm of z. This is a coincidence. More generally, there are also terms behaving like a power of z, because suppose $\bar{A}^n(s,t)$ with $n\leq N$ have only terms behaving like a power of logarithm in s, t, and u, then

$$\rho^{\mathrm{el}(s);(N+2)}=\int_1^z dz_1\int_1^{z_2^-} dz_2\frac{1}{\sqrt{K_s}}(\bar{\nu}^N\bar{\nu}^{2*}+\cdots+\bar{\nu}^2\nu^{N*}) \underset{z\to\infty}{\sim} z\ln(z^2-1)+\cdots. \quad (51)$$

Our above comment is already made evident in the fifth-order iteration,

$$\begin{aligned}\rho^{\mathrm{el}(s);(5)}(s,t) &=\tfrac{1}{8}\pi^2\int_1^z dz_1\int_1^{z_2^-} dz_2\frac{1}{\sqrt{K_s}}(2\ln t_1-2)\\ &=-\tfrac{1}{16}\pi^2(3\ln\tfrac{1}{2}s-2)\ln(t/s+1)\\ &\quad-\tfrac{3}{16}\pi^2[(z-1)\ln^2(z-1)-3(z-1)\ln(z-1)\\ &\quad+2(z-1)\ln(z+1)+(z+1)\ln(z+1)\\ &\quad+2(z-1)+2\ln 2]-\tfrac{3}{8}\pi^2\int_1^z\frac{\ln(z_1-1)}{z_1+1}dz_1. \quad (52)\end{aligned}$$

This would mean that in writing a dispersion relation for $\bar{\rho}^{(5)}$, two subtractions are needed. It is not difficult to see that as we go to higher and higher order iterations with the double density functions, more and more subtractions may be needed. This is to be compared with iterations with the single density functions, where we found that the number of subtractions required in each iteration is unchanged. This difference comes from the fact that the elastic unitarity integral in the s channel, say, is expressed as a function of z for a given s. The domain of integration for the double density functions is linearly dependent on z, making the double density function $\rho^{\mathrm{el}(s),(n)}(s,t)$ dependent in general on a power of z, while the z dependence drops out[8] when the nth-order single density functions is obtained from the unitarity integral. This means, for instance, that in integrating $\rho^{\mathrm{el}(s);(n)}(s,t)$ over t, enough subtractions must be made to insure convergency. We should note, however, that these subtraction terms are not a power of s, t, or u but a power of z_s, z_t, and z_u. If the infinity in the s-t planes is reached along lines of fixed z, these subtraction terms will remain finite; indeed, when they are weighted by λ^n their contributions should become vanishingly small as n gets large insofar as λ is assumed to be small. Thus instead of carrying out further algebraic details of iterations, which can be done in an increasingly tedious manner, we shall study in Sec. IV the high-energy behavior of the scattering amplitude at fixed angles, namely, in the region where s, t, and u are all large in magnitude while the c.m. scattering angles in all three channels are held at fixed values. We shall obtain the leading terms for the scattering amplitude and its spectral functions in each order of iteration and subsequently sum the resulting series.

[8] See the discussion after Eq. (39).

IV. HIGH-ENERGY BEHAVIOR

To study the high-energy behavior of the scattering amplitude at fixed angles, let us first note that in the region where s, t, and u are large in magnitude, $\ln(-t)$ and $\ln(-u)$ are of the same order of magnitude as $\ln(-s)$, since for $s>0$ real,

$$\begin{aligned}\ln(-t)&=\ln s+\ln[\tfrac{1}{2}(1-z)]\simeq\ln s,\\ \ln(-u)&=\ln s+\ln[\tfrac{1}{2}(1+z)]\simeq\ln s,\\ \ln(-s)&+\ln s-i\pi\simeq\ln s.\end{aligned} \quad (53)$$

We see that in the region of large energies, the leading term of the amplitude can always be reduced to a power of $\ln s$. To make crossing symmetry explicit, we may take as the leading term $\ln^m(-s)+\ln^m(-t)+\ln^m(-u)$, which also has the correct analyticity properties. Now we already have the complete expressions of the amplitude for the first few orders of iteration;

$$\begin{aligned}\bar{A}^1&=1,\\ \bar{A}^2&=-\tfrac{1}{4}[\ln(-s)+\ln(-t)+\ln(-u)],\\ \bar{A}^3&=\tfrac{3}{16}[\ln^2(-s)+\ln^2(-t)+\ln^2(-u)]+O(\ln s).\end{aligned}$$

Furthermore, from the fact that the leading contribution of the integral

$$\int_0^\infty \ln^m s'\left(\frac{1}{s'-s}-\frac{1}{s'-\alpha}\right)ds'$$

is of order $\ln^{m+1}(-s)$, we see that the leading term of $\bar{A}^n$ may be taken to be

$$\bar{A}^n(s,t)=C_n[\ln^{n-1}(-s)+\ln^{n-1}(-t)+\ln^{n-1}(-u)]+O(\ln^{n-2}s), \quad (54)$$

where C_n is some real constant to be determined by the unitarity condition. The leading contribution to the double density functions must be obtained separately from the unitarity integrals because in Eq. (54) we have already reduced the crossed terms by use of Eq. (53), so that although Eq. (54) gives the correct high-energy behavior for the amplitude and its absorptive parts, it has in fact no double density functions, the leading contributions of which will be obtained later in this section. From Eq. (54), the leading term for the nth-order s-channel absorptive part is

$$\bar{\nu}^n(s,t)=-C_n\pi(n-1)\ln^{n-2}s+O(\ln^{n-3}s). \quad (55)$$

On the other hand, $\bar{\nu}^n(s,t)$ is also given by the unitarity integral

$$\bar{\nu}^n(s,t)=\tfrac{1}{8}\theta(s)\int_{-1}^{1}dz_1\int_{-1}^{1}dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}\times\Big[\sum_{j=1}^{n-1}\bar{A}^{n-j}(s,z_1)\bar{A}^{j*}(s,z_2)+\sum_{j=2}^{n-2}\bar{A}'^{n-j}(s,z_1)A'^{j*}(s,z_2)\Big]. \quad (56)$$

Now Eq. (54) gives

$$\bar{A}^{n-j}(s,z_1)\bar{A}^{j*}(s,z_2)=9C_{n-j}C_j\ln^{n-2}s+O(\ln^{n-3}s) \quad (57)$$

and

$$\bar{A}'^{n-j}(s,z_1)\bar{A}'^{j*}(s,z_2)=4C_{n-j}C_j\ln^{n-2}s+O(\ln^{n-3}s), \quad (58)$$

so that in the limit of large s,

$$\bar{\nu}^n(s,t)=\left(\frac{9\pi}{4}\sum_{j=1}^{n-1}C_jC_{n-j}+\pi\sum_{j=2}^{n-2}C_jC_{n-j}\right)\ln^{n-2}s+O(\ln^{n-3}s). \quad (59)$$

Consistency between Eqs. (56) and (59) requires that

$$-(n-1)C_n=-\frac{13}{4}\sum_{j=1}^{n-1}C_jC_{n-j}-\tfrac{2}{3}C_{n-1} \quad \text{for} \quad n\geq 3 \quad (60\text{a})$$

and

$$-(n-1)C_n=-\frac{13}{4}\sum_{j=1}^{n-1}C_jC_{n-j}-\tfrac{1}{3}C_{n-1} \quad \text{for} \quad n=2. \quad (60\text{b})$$

Together with the initial condition that $C_1=\frac{1}{3}$, these recurrence relations give C_n for all higher n. The reason for $n=2$ to be different from other cases comes from the fact that in the above equations the summations are carried out from $j=1$ up to and including $j=n-1$, except for $n=2$, in which case the terms $j=1$ and $j=n-1$ are just one and the same term. Our purpose now is to be able to sum

$$\bar{A}(s,t)\simeq\lambda\sum_{n=1}^{\infty}C_n\{[\lambda\ln(-s)]^n+\text{c.p.}\}. \quad (61)$$

When the coupling is weak, or more precisely, for $|\lambda\ln s|<1$, the series obviously converges. To sum it into closed analytic form, let us consider the function

$$E(x)=\sum_{n=1}^{\infty}C_nx^{n-1} \quad (62)$$

within the unit circle $|x|<1$, and with the boundary condition

$$E(0)=\tfrac{1}{3}. \quad (63)$$

From Eq. (62), it follows that

$$\frac{dE(x)}{dx}=\sum_{n=2}^{\infty}C_n(n-1)x^{n-2} \quad (64)$$

and

$$E(x)^2=\sum_{n=2}^{\infty}x^{n-2}\sum_{j=1}^{n-1}C_{n-j}C_j. \quad (65)$$

On the other hand, the recurrence relations imply that

$$\sum_{n=2}(n-1)C_nx^n=-\frac{13}{4}\sum_{n=2}x^n\sum_{j=1}^{n-1}C_{n-j}C_j+\tfrac{2}{3}\sum_{n=2}C_{n-1}x^n-\tfrac{1}{3}C_1x^2. \quad (66)$$

We see that $E(x)$ satisfies the following differential equation:

$$\frac{dE(x)}{dx}=-\frac{13}{4}E^2(x)+\tfrac{2}{3}E(x)-\tfrac{1}{9}, \quad (67)$$

which has as its solution

$$E(x)=\frac{2}{13}\left(\frac{2}{3}+\tan\frac{x_0-x}{2}\right), \quad (68)$$

where x_0 is the constant of integration and is determined by the boundary condition Eq. (63),

$$x_0=2\tan^{-1}(\tfrac{3}{2}), \quad (69)$$

so that

$$E(x)=\frac{2}{13}\left(\frac{2}{3}+\frac{\frac{3}{2}-\tan\frac{1}{2}x}{1+\frac{3}{2}\tan\frac{1}{2}x}\right). \quad (70)$$

As a result, the leading series for the scattering amplitude can be summed to the following close form, which we write as

$$\bar{A}(s,t)\simeq\frac{2\lambda}{13}\left[2+\frac{\frac{3}{2}-\tan(\frac{1}{2}\lambda\ln(-s))}{1+\frac{3}{2}\tan(\frac{1}{2}\lambda\ln(-s))}+\text{c.p.}\right]. \quad (71)$$

Similarly, in the region $\ln s\gg 1$ but $|\lambda\ln s|<1$ the leading series for the s-channel absorptive part is

$$\bar{\nu}(s,t)=-\sum_{n=2}^{\infty}\lambda^n\pi(n-1)[C_n\ln^{n-2}s+O(\ln^{n-3}s)]. \quad (72)$$

From Eq. (64) we see at once that

$$\bar{\nu}(s,t)\simeq-\pi\lambda^2\frac{dE(x)}{dx}\bigg|_{x=\lambda\ln s}=\frac{2\pi\lambda^2}{13}\left[1+\left(\frac{\frac{3}{2}-\tan(\frac{1}{2}\lambda\ln s)}{1+\frac{3}{2}\tan(\frac{1}{2}\lambda\ln s)}\right)^2\right]. \quad (73)$$

This is to be compared with that obtained directly from Eq. (71),

$$\bar{\nu}(s,t)=\mathrm{Im}\left[\frac{2\lambda}{13}\tan(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s)\right]$$

$$\simeq\frac{2\pi\lambda^2}{13}\frac{1+\tan^2(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s)}{1+\tfrac{1}{4}\pi\lambda^2\tan^2(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s)}. \quad (74)$$

Neglecting the term $\frac{1}{4}\pi\lambda^2\tan^2(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)$ in the denominator, it becomes just Eq. (73), which shows that in the high-energy limit, taking the sum and the imaginary parts are interchangeable.

Having obtained an asymptotic form for the scattering amplitude and its absorptive parts, we can go back to Eq. (11) to get the nth-order double density function

$$\rho^{\mathrm{el}(s);(n)}(s,t)=\int dz_1\int dz_2\frac{\theta(K_s)}{\sqrt{K_s}}[\bar{\nu}^{n-2}\bar{\nu}^{2*}+\bar{\nu}^{n-3}\bar{\nu}^{3*}+\cdots+\bar{\nu}^{2}\bar{\nu}^{n-2*}]. \quad (75)$$

The leading contribution is obtained by substituting the leading term of $\bar{\nu}^j$ as obtained from Eq. (73) into Eq. (75) and subsequently reducing them to powers of $\ln s$ with Eq. (53);

$$\rho^{\mathrm{el}(s);(n)}(s,t)=\pi^2(\ln^{n-4}s)\sum_{j=2}^{n-2}(j-1)(n-j-1)C_jC_{n-j}\times\int dz_1\int dz_2\frac{\theta(K_s)}{\sqrt{K_s}}[1+O(\ln^{-1}s)]. \quad (76)$$

Again for weak coupling, $|\lambda\ln s|<1$, the leading series for $\rho^{\mathrm{el}(s)}$ is then

$$\rho^{\mathrm{el}(s)}(s,t)=\sum_{n=4}\lambda^n\rho^{\mathrm{el}(s);(n)}(s,t)=\pi^2\lambda^4\ln\left(\frac{t+s}{s}\right)\sum_{n=4}^{\infty}(\lambda\ln s)^{n-4}\sum_{j=2}^{n-2}C_jC_{n-j}, \quad (77)$$

which can likewise be summed to a closed form. In fact, the series is nothing more than $[dE(x)/dx]^2|_{x=\lambda\ln s}$, and hence

$$\rho^{\mathrm{el}(s)}(s,t)=\left(\frac{2\pi}{13}\lambda^2\right)^2\ln\frac{t+s}{s}\left[1+\left(\frac{\frac{3}{2}-\tan(\frac{1}{2}\lambda\ln s)}{1+\frac{3}{2}\tan(\frac{1}{2}\lambda\ln s)}\right)^2\right]^2. \quad (78)$$

Comparing this with Eq. (73), we see that the leading series of $\bar{\nu}(s,t)$ and $\rho^{\mathrm{el}(s)}(s,t)$ satisfy the following simple relation:

$$\rho^{\mathrm{el}(s)}(s,t)\simeq[\bar{\nu}(s,t)]^2\ln[(s+t)/s]. \quad (79)$$

Finally, from crossing symmetry we have in the large-energy region,

$$\bar{\rho}(s,t)\simeq[\bar{\nu}(s,t)]^2\ln\frac{s+t}{s}+[\bar{\nu}(t,s)]^2\ln\frac{s+t}{t}. \quad (80)$$

Another important check in the model is the ratio of the elastic cross section to the total cross section, both approximated as given by Eq. (21). For weak coupling, the high-energy form of $\bar{\sigma}_{\mathrm{el}}$ can be obtained by integration of Eq. (71):

$$|\bar{A}(s,t)|^2\simeq(\pi\lambda^2/13)^2(1+\tan^2(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s))^2+(4\lambda^2/13^2)\{2+3\tan(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s)+\lambda(1+\tan(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s))\ln[\tfrac{1}{4}(1-z^2)]\}^2 \quad (81)$$

and

$$\bar{\sigma}_{\mathrm{el}}(s)=\int|A|^2d\Omega\simeq4\pi(4\lambda^2/13^2)[2+3\tan(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s)-2\lambda(1+\tan(\tfrac{1}{2}x_0-\tfrac{1}{2}\lambda\ln s))]^2. \quad (82)$$

Hence the ratio in Eq. (21) becomes

$$\frac{\bar{\sigma}_{\mathrm{el}}}{\bar{\sigma}_{\mathrm{tot}}}\simeq\frac{1}{13}\frac{[2+3\tan(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)-2\lambda(1+\tan(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s))]^2}{1+\tan^2(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)}, \quad (83)$$

where $x_0\simeq2.08$, as given by Eq. (69). For $\lambda\ln s\simeq1$, the above ratio is approximately equal to 11/13; for $\tan(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)=0$, it is 4/13; for $\tan(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)=0$, it is 4/13; for $\tan(\frac{1}{2}x_0-\frac{1}{2}\lambda\ln s)=\infty$, it tends to 9/13. Equation (83) shows that in the high-energy limit, the inelastic effects which we have incorporated are considerable.

The above process of generating leading series for the scattering amplitude in the region of large s, t, and u can be carried out further. The point here is that because of Eq. (53), we can expand $\bar{A}^n$ as simple powers of logarithm of s, t, and u,

$$\bar{A}^n=\sum_{j=1}^{n}C_{nj}[\ln^{n-j}(-s)+\ln^{n-j}(-t)+\ln^{n-j}(-u)], \quad (84)$$

where the coefficients C_{nj} will be uniquely determined by requiring that the absorptive parts obtained from this equation must be the same as the ones given by the unitarity condition. Let us illustrate this by considering the next leading series for $\bar{A}(s,t)$,

$$\bar{A}^n(s,t)=C_n[\ln^{n-1}(-s)+\mathrm{c.p.}]+D_n[\ln^{n-2}(-s)+\mathrm{c.p.}]+O(\ln^{n-3}s). \quad (85)$$

Then

$$\bar{\nu}^n(s,t)=-\pi(n-1)C_n\ln^{n-2}s-\pi(n-2)D_n\ln^{n-3}s+O(\ln^{n-4}s), \quad (86)$$

while the unitarity condition gives

$$\bar{\nu}^n(s,t)=\left[\frac{9\pi}{4}\sum_{j=1}^{n-1}C_{n-j}C_j+\pi\sum_{n=2}^{n-2}C_{n-j}C_j\right]\ln^{n-2}s$$
$$+[\tfrac{3}{2}\pi\sum_{j=1}^{n-1}(C_{n-j}D_j-(n-2)C_{n-j}C_j)$$
$$+\pi\sum_{j=2}^{n-2}(C_{n-j}D_j-(n-2)C_{n-j}C_j)]\ln^{n-3}s$$
$$+O(\ln^{n-4}s). \quad (87)$$

Consistency between these two expressions gives, besides Eq. (60), the following recurrence relations for D_n:

$$-(n-2)D_n=\tfrac{5}{2}\sum_{j=1}^{n-1}D_jC_{n-j}+20/26(n-1)(n-2)C_n$$
$$-\tfrac{1}{3}D_{n-1}-(46/39)(n-2)C_{n-1},\quad n\geq 3 \quad (88)$$

with $D_1=D_2=0$. If we define

$$G(x)=\sum_{n=2}^{\infty}D_nx^{n-2}, \quad (89)$$

then $G(x)$ satisfies the following differential equation:

$$\frac{dG(x)}{dx}=[\tfrac{1}{3}-\tfrac{5}{2}E(x)]G(x)+\frac{5}{2}\frac{dE^2}{dx}+\frac{2}{3}\frac{dE(x)}{dx}. \quad (90)$$

Making use of the boundary condition $G(O)=0$ and the integrating factor

$$W(x)=\{\sec[\tfrac{1}{2}(x_0-x)]\}^{-10/13}e^{-x/13}, \quad (91)$$

we get for $G(x)$,

$$G(x)=\frac{5}{2}\frac{dE(x)}{dx}+\tfrac{3}{2}E(x)+\frac{1}{W(x)}\frac{325}{26}\int_0^x E^3(x')W(x')dx'$$
$$-\frac{35}{4W(x)}\int_0^x E(x')^2W(x')dx'+\frac{43}{36W(x)}$$
$$\times\int_0^x E(x')W(x')dx'+\frac{1}{8W(x)}\left(\frac{4}{13}\right)^{13/5}. \quad (92)$$

If we wish, the integrals appearing on the right-hand side can be easily performed numerically. However, the point to be made here is that the sum of the second leading series,

$$\bar{A}_{II}(s,t)=\sum_{n=2}^{\infty}D_n\lambda^n[\ln^{n-2}(-s)+\text{c.p.}]$$
$$=\lambda^2[G(\lambda\ln(-s))+\text{c.p.}], \quad (93)$$

is seen to be suppressed by a power of λ as compared with the sum of the first leading series obtained previously,

$$\bar{A}_I(s,t)=\lambda[E(\lambda\ln(-s))+\text{c.p.}].$$

This is a general feature of the iteration solution, namely, that for large s, t, and u, the summation of successive series formed from the elements appearing in Eq. (84) is consecutively suppressed by a factor of λ. Now although the series in Eq. (62) converges only for $|\lambda\ln s|<1$, the analytic form in Eq. (71) is regular to at least $|\lambda\ln s|<5$. From our discussion and Eq. (92), one can expect that for small λ, $\bar{A}_I(s,t)$ will still be dominant even beyond the radius of convergence of the original series.

This suggests that, in the asymptotic region, we may take Eq. (71) as the first-order approximation to $\bar{A}(s,t)$, starting from which we can again iterate the dispersion relation with the unitarity relation. Thus to second-order iteration, we can approximate $\bar{\nu}$ by $\frac{1}{4}\pi\lambda+\frac{3}{16}\pi\lambda^2\ln s$ for $s<e^{1/(10\lambda)}$, say, and

$$(2\pi\lambda^2/13)\{1+\tan^2[\tfrac{1}{2}(x_0-\lambda\ln s)]\}$$

for $s>e^{1/(10\lambda)}$, then for $s\gg e^{1/\lambda}$ we obtain from the dispersion relation,

$$\bar{A}_I^{(2)}(s,t)\simeq(2\lambda/13)\{\tfrac{2}{3}+\tan\tfrac{1}{2}[x_0-\lambda\ln(-s)]\}+\text{c.p.}$$
$$+\left(\frac{3}{e^{1/\lambda}}-\frac{1}{s}-\frac{1}{t}-\frac{1}{u}\right)(\pi\lambda e^{1/(10\lambda)})$$
$$\times[(2/13)x_0\lambda(x_0+\lambda-\tfrac{1}{10})-(43/160-3\lambda/16)], \quad (94)$$

which is normalized in such a way that it coincides with $\bar{A}_I(s,t)$ for $s=t=-\frac{1}{2}u=\frac{1}{2}e^{1/\lambda}$.

As mentioned before, another general feature of the iterative solution is that in the zero-mass limit it has essential singularities of logarithmic nature at both the origin and infinity. The dual character of these two singular points is made transparent if we change variables to

$$x=1/s,\quad y=1/t\quad\text{and}\quad z=1/u,\quad \alpha=1/\beta,$$

so that points at infinity are mapped to the origin and vice versa. Since $\bar{A}(x,y,z)$ diverges logarithmically at infinities, it satisfies a similar once-subtracted dispersion relation as that for $\bar{A}(s,t,u)$. In fact, from Eq. (18) we have

$$\bar{A}(x,y,z)=\gamma+\frac{1}{\pi}\int_0^\infty dw\,\bar{\nu}(w,1/y)\left(\frac{1}{x-w}-\frac{1}{\beta-w}\right)$$
$$+\frac{1}{\pi}\int_0^\infty dw\,\bar{\nu}(1/x,w)\left(\frac{1}{y-w}-\frac{1}{\beta-w}\right)$$
$$+\frac{1}{\pi}\int_0^\infty dw\,\bar{\nu}(w,1/y)\left(\frac{1}{z-w}-\frac{1}{\beta-w}\right), \quad (95)$$

where we let γ to be in general different from λ.

The elastic unitarity for fixed s is also that for fixed x because it is assumed that once symmetrized, it is valid for all s and $\theta(s)=\theta(x)$, so that

$$\bar{\rho}\begin{pmatrix}1 & 1\\ -, & -\\ x & y\end{pmatrix}=\tfrac{1}{8}\theta(x)\int_{-1}^{1}dz_1\int_{-1}^{1}dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}\bar{A}\begin{pmatrix}1\\ -,z_1\\ x\end{pmatrix}\times A^*\begin{pmatrix}1\\ -,z_2\\ x\end{pmatrix}. \quad (96)$$

Expanding $\bar{A}(x,y)$ and $\bar{\rho}(x,y)$ in powers of γ,

$$\bar{A}(x,y)=\sum_{n=1}\gamma^n\bar{A}^n(x,y),\qquad \bar{\rho}(x,y)=\sum_{n=2}\gamma^n\bar{\rho}^n(x,y), \quad (97)$$

then the nth-order amplitude satisfies the following dispersion relation:

$$\bar{A}^n(x,y)=\bar{A}^n(\beta,\beta)+\frac{1}{\pi}\int_0^\infty dw\,\bar{\rho}^n(x,w)\times\left(\frac{1}{x-w}-\frac{1}{\beta-w}\right)+\text{c.p.} \quad (98)$$

Iteration of this dispersion relation with the unitarity integral Eq. (96) again give rise to infrared divergencies. Using our previous normalization procedure, the origin is again found to be an essential singularity of logarithmic nature, and the first few orders of interactions give

$$\begin{aligned}&\bar{A}^1=1,\\ &\bar{A}^2=\tfrac{1}{4}[\ln(-x)+\text{c.p.}],\\ &\bar{A}^3=\tfrac{3}{16}[\ln^2(-x)+\text{c.p.}]+\tfrac{1}{4}[\ln(-x)+\text{c.p.}],\ \text{etc.},\end{aligned} \quad (99)$$

where that branch of $\ln(-x)$ must be taken such that $\ln[-(x+i\epsilon)]=\ln x+i\pi$ for x real>0. Now for $s=t=1$, $x=y=1$. At this point, we should require $\bar{A}(s,t)=\bar{A}(x,y)$. Then it can be shown immediately that $\gamma=\lambda$ and the solution in Eq. (99) can be obtained directly from that for $\bar{A}(s,t,u)$ if we simply replace s and t by $1/x$ and $1/y$, respectively. To conclude, we see that for s and t not far from unity or $|\lambda \ln s|<\delta$ for some small δ, the scattering amplitude is well approximated by the first few iteration terms given by Eq. (99); while in the region of very large energies or very small energies,

$$|\ln s|,\quad |\ln t|,\quad |\ln u|\gg 1\quad \text{but}\quad |\lambda \ln s|<5,$$

the amplitude is represented by Eq. (71). In the immediate neighborhood of the origin and infinity, the scattering amplitude is singular, and naturally the iterative method is no longer suitable to use for finding a solution.

V. OTHER ITERATIVE SOLUTIONS

We remarked in Sec. III that when the scattering particles have zero mass, it is not necessary to start the iteration with $\bar{A}^1=1$. In the following, we discuss briefly other solutions that can be generated by assuming different values for $\bar{A}^1$. Let us first consider $\bar{A}^1=i$, and denote the amplitude so generated by $\bar{B}(s,t)$; then $\bar{B}^1=i$. Using the same method and subtraction normalization as in Sec. III, it is easy to show that

$$\bar{B}^2(s,t)=-\tfrac{1}{4}[\ln(-s)+\text{c.p.}] \quad (100)$$

and

$$\bar{B}^3(s,t)=-\tfrac{1}{8}\pi[\ln(-s)+\text{c.p.}]. \quad (101)$$

Note that $\bar{B}^2=\bar{A}^2$, while for $\bar{B}^3$, the terms $\ln^2(-s)+\text{c.p.}$ present in $\bar{A}^3$ are lacking. This comes from the fact that in the elastic unitarity relation for the nth-order single density function

$$\bar{\sigma}^n(s)=\tfrac{1}{8}\theta(s)\iint dz_1dz_2\frac{\theta(-K_s)}{\sqrt{(-K_s)}}[\bar{B}^{n-1}(s,t_1)\bar{B}^{1*}(s,t_2)+(\bar{B}^{n-2}\bar{B}^{2*}-\bar{B}'^{n-2}\bar{B}'^{2*})+\cdots+\bar{B}^1\bar{B}^{n-1*}]. \quad (102)$$

The contribution from $\bar{B}^{n-1}\bar{B}^{1*}+\bar{B}^1\bar{B}^{n-1*}$ is in general one power of $\ln s$ lower than if $\bar{B}^1$ were 1, except for the case $n=2$.

In the fourth-order iteration, we have

$$\bar{\sigma}^{(4)}(s)=5\pi\frac{\ln^2 s+\pi^2}{64}-\tfrac{1}{16}\pi\ln s \quad (103)$$

and

$$\rho^{\text{el}(s);(4)}(s,t)=\tfrac{1}{2}\theta(s)\iint dz_1dz_2\frac{\theta(K_s)}{\sqrt{K_s}}(2\bar{\rho}^2\bar{\rho}^{2*})=\tfrac{1}{16}\pi^2\{\ln[(s+t)/s]\}, \quad (104)$$

so that

$$\bar{B}^{(4)}(\bar{\rho}^{(4)},s,t,u)=\bar{A}^{(4)}(\bar{\rho}^{(4)};s,t), \quad (105)$$

while

$$\bar{B}^{(4)}(\bar{\sigma}^{(4)};s,t)=\tfrac{1}{64}[-(5/3)\ln^3(-s)+2\ln^2(-s)-5\pi^2\ln(-s)]+\text{c.p.} \quad (106)$$

In a region where all s, t, and u, or their reciprocals, are very large in magnitude, the leading term for $\bar{B}^n(s,t)$ may be taken to be

$$\bar{B}^n(s,t)=\mathfrak{C}_n[\ln^{n-1}(-s)+\text{c.p.}]+O(\ln^{n-2}s), \quad (107)$$

where $\mathfrak{C}_n$ is some real number, except for $\mathfrak{C}_1$, which is $\frac{1}{3}i$, and

$$\bar{\rho}^n(s,t)=-\mathfrak{C}_n\pi(n-1)\ln^{n-2}s+O(\ln^{n-3}s), \quad (108)$$

while from the unitarity condition

$$\bar{\rho}^n(s,t)=\frac{13\pi}{4}\sum_{j=2}^{n-2}\mathfrak{C}_{n-j}\mathfrak{C}_j\ln^{n-2}s+O(\ln^{n-3}s), \quad (109)$$

from which we get the recurrence relation for $\mathfrak{C}_n$,

$$-\mathfrak{C}_n(n-1)=\frac{13}{4}\sum_{j=2}^{n-2}\mathfrak{C}_j\mathfrak{C}_{n-j},\quad n>3$$

with

$$\mathcal{C}_2=-\tfrac{1}{4},\quad \mathcal{C}_3=0, \tag{110}$$

and the scattering amplitude is approximated by

$$\bar{B}(s,t)\simeq\lambda i+\lambda\sum_{n=2}^{\infty}\mathcal{C}_n\{[\lambda\ln(-s)]^{n-1}+\text{c.p.}\}. \tag{111}$$

To sum this series for weak coupling, $|\lambda\ln s|<1$, let us consider

$$\mathcal{E}(x)=\sum_{n=2}^{\infty}\mathcal{C}_n x^{n-1}, \tag{112}$$

with the boundary condition

$$\mathcal{E}(0)=0. \tag{113}$$

The recurrence relation for $\mathcal{C}_n$ implies that $\mathcal{E}(x)$ satisfies the following differential equation:

$$-d\mathcal{E}(x)/dx+\mathcal{C}_2+\mathcal{C}_3x=(13/4)\mathcal{E}(x)^2 \tag{114}$$

or

$$\mathcal{E}(x)=\frac{-1}{\sqrt{13}}\tan\frac{(\sqrt{13})x}{4}. \tag{115}$$

Hence from Eq. (113) we have

$$\bar{B}(s,t)\simeq\lambda i-\frac{\lambda}{\sqrt{13}}\left(\tan\frac{(\sqrt{13})\lambda\ln(-s)}{4}+\text{c.p.}\right) \tag{116}$$

and

$$\bar{\nu}(s,t)=\lambda-\sum_{n=2}^{\infty}\pi\lambda^n(n-1)\mathcal{C}_n\ln^{n-2}s$$

$$=\lambda+\frac{\pi\lambda^2}{\sqrt{13}}\left(1+\tan^2\frac{(\sqrt{13})\lambda\ln s}{4}\right). \tag{117}$$

We note that for small λ, the imaginary part of $\bar{B}(s,t)$ comes mainly from the constant term λi. The ratio of elastic cross section to total cross section is now given by

$$\frac{\bar{\sigma}_{\text{el}}}{\bar{\sigma}_{\text{tot}}}\simeq\tfrac{1}{2}\pi\lambda\left(1+\frac{9}{13}\tan^2\frac{(\sqrt{13})\lambda\ln s}{4}\right). \tag{118}$$

For $\lambda\ln s\sim1$, the ratio is approximately $2\pi\lambda$, which is a rather small number for small λ. For instance, if $\lambda\sim0.02$, the ratio is roughly 0.13, so that the inelastic contribution to $\bar{\sigma}_{\text{tot}}$ is about seven times that of the elastic contribution.

More general solutions can be obtained by taking $\bar{A}^1$ to be complex. Denoting by $\bar{H}(s,t)$ the scattering amplitude so generated, we let $\bar{H}^{(1)}=e^{i\phi}$ and

$$\bar{H}(s,t)=\sum_{n=1}^{\infty}\lambda^n\bar{H}^n(s,t). \tag{119}$$

Following the same method that we have been using, it is easy to obtain

$$\bar{H}^{(2)}(s,t)=-\tfrac{1}{4}[\ln(-s)+\text{c.p.}],$$
$$\bar{H}^{(3)}(s,t)=\tfrac{3}{16}\cos\phi[\ln^2(-s)+\text{c.p.}]-\tfrac{1}{4}(\cos\phi+\tfrac{1}{2}\pi\sin\phi)[\ln(-s)+\text{c.p.}],$$
$$\bar{H}^{(4)}=-\tfrac{3}{4}(\cos^2\phi+5/18)[\ln^3(-s)+\text{c.p.}]+[2\cos^2\phi+\pi\sin\phi\cos\phi+1][\ln^2(-s)+\text{c.p.}]+[\tfrac{1}{2}\pi\sin^2\phi-\cos^2\phi-\tfrac{3}{4}\pi^2\cos^2\phi-\tfrac{1}{8}\pi^2]\times[\ln(-s)+\text{c.p.}]+\bar{A}^{(4)}(\bar{\rho}^{(4)};s,t),\ \text{etc.} \tag{120}$$

The behavior of $\bar{H}(s,t)$ at fixed scattering angles but in the region of very large (or very small) energies can be studied in the same manner as we did for $\bar{A}(s,t)$ and $\bar{B}(s,t)$. Thus taking

$$\bar{H}^n=\mathcal{C}_n(\phi,\lambda)[\ln^{n-1}(-s)+\text{c.p.}]+O(\ln^{n-2}s), \tag{121}$$

$\mathcal{C}_n(\phi,\lambda)$ is then to be determined by the unitarity condition and satisfies the following recurrence relation:

$$-(n-1)\mathcal{C}_n=\frac{13}{4}\sum_{j=2}^{n-2}\mathcal{C}_j\mathcal{C}_{n-j}+\tfrac{3}{2}\cos\phi\,\mathcal{C}_{n-1},\quad\text{for } n\geq3. \tag{122}$$

Together with

$$\mathcal{C}_1=\tfrac{1}{3}e^{i\phi},$$
$$\mathcal{C}_2=-\tfrac{1}{4},$$

Eq. (126) gives $\mathcal{C}_n$ for all n. Defining

$$T(x)=\sum_{n=2}^{\infty}\mathcal{C}_nx^{n-1},\quad T(0)=0 \tag{123}$$

we have

$$\bar{H}(s,t)=e^{i\phi}+[\lambda T(\lambda\ln(-s))+\text{c.p.}] \tag{124}$$

and $T(x)$ satisfies the differential equation

$$-\frac{dT(x)}{dx}+2\mathcal{C}_3x+\mathcal{C}_2=\frac{13}{4}T^2(x)+\tfrac{3}{2}\cos\phi\,[T(x)-\mathcal{C}_2x]$$

or

$$-4dT/dx=13T^2+6\cos\phi\,T+1, \tag{125}$$

the solution of which is

$$T(x)=a\tan\left(\frac{13a}{4}(x_0-x)\right)-\frac{3}{13}\cos\phi, \tag{126}$$

where

$$a=\left(\frac{1}{13}-\frac{9}{13^2}\cos^2\phi\right)^{1/2},$$

$$x_0=\frac{4}{13a}\tan^{-1}\left(\frac{3}{13a}\cos\phi\right).$$

As a result we obtain

$$\bar{H}(s,t)=\lambda e^{i\phi}-\frac{9\lambda}{13}\cos\phi+\lambda a\left[\tan\left(\frac{13a}{4}[x_0-\lambda\ln(-s)]\right)+\text{c.p.}\right] \quad (127)$$

and

$$\bar{\nu}(s,t)=\lambda\sin\phi+\lambda\sum_{n=2}^{\infty}\lambda^{n-1}\mathcal{C}_n(n-1)(-\pi)\ln^{n-2}s$$
$$=\lambda\sin\phi+\frac{13\pi a^2\lambda^2}{4}\left[1+\tan^2\left(\frac{13a}{4}(x_0-x)\right)\right]. \quad (128)$$

It is easy to check that for $\phi=0$ and $\frac{1}{2}\pi$, $\bar{H}(s,t)$ reduces to $\bar{A}(s,t)$ and $\bar{B}(s,t)$, respectively.

The ratio of the elastic cross section to the total cross section now takes the form

$$\frac{\bar{\sigma}_{\text{el}}}{\bar{\sigma}_{\text{tot}}}=\left\{\lambda\sin\phi+\lambda\left[\frac{4}{13}\cos\phi+3\lambda a\tan\left(\frac{13a}{4}(x_0-\lambda\ln s)\right)\right]^2\right\}\Big/\left\{\sin\phi+\frac{13\pi a^2\lambda}{4}\left[1+\tan^2\left(\frac{13a}{4}(x_0-x)\right)\right]\right\}, \quad (129)$$

from which it is again easy to check that the inelastic $\bar{\sigma}_{\text{tot}}$ is always comparable to the elastic contribution for all ϕ.

VI. CONCLUSIONS AND REMARKS

We have studied the iterative method of obtaining a solution for the two-particle scattering amplitude of a neutral scalar field based on the dispersion relation and unitarity integrals. A minimal but considerable amount of inelastic processes has been included in such a way that crossing symmetry is maintained at all stages of our calculations. It has been shown that because of crossing symmetry, the iterative procedure can be initiated with one subtraction constant even if one assumed at the beginning that the scattering amplitude satisfied a twice-subtracted dispersion relation. To simplify the kinematic factors in the unitarity integrals, the actual calculation was done in the limit that the scattering particles have zero mass.[9] In this limit, the solution has essential singularities of logarithmic nature both at infinity and at the origin. For s and t not far away from unity (assuming that s and t are normalized to some convenient unit, like the mass of the pion), the solution is described by the first few orders of iteration, which was carried out explicitly up to the fourth order. In the very high, or very low, energy region where s, t, and u, or their reciprocals, are all large in magnitude while the c.m. scattering angles are held at fixed values, the leading terms of the scattering amplitude and its spectral functions have been obtained, and the resulting leading series were subsequently summed to close analytic forms in the weak-coupling limit. It was also shown that the sum of secondary series is suppressed by at least a factor of λ, which was assumed small. The ratio of the elastic cross section to the total cross section has also been obtained, and the result indicates that the inelastic effects which were incorporated into the scheme are considerable. In the zero-mass limit, $\bar{A}^1$ is not necessarily a real number; consequently we have also considered in some detail the cases with $\bar{A}^1=i$ or more generally $e^{i\phi}$. The solutions so generated have the same general features as that for $\bar{A}^1=1$. In particular, the nth-order amplitude behaves like a power of logarithm in s, t, and u; the solutions have essential singularities at the origin and at infinity, and the inelastic contribution to $\bar{\sigma}_{\text{tot}}$ is comparable to the elastic contribution for all cases.

If pole terms appear in the model,[10] questions concerning subtractions and convergency properties will be of a rather different nature. The residue of the pole terms now plays the role of the coupling constant and the inputs to the iteration are now inversely proportional to s, t, or u. The iteration procedure discussed here will be applicable, although the solution so obtained will have rather different properties than the ones obtained in this paper.

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[9] The iteration solution for scattering particles with finite mass will not have an essential singularity at the threshold; otherwise it has the same main feature as that for the zero mass. In particular, the behavior of the scattering amplitude in the large-energy region will be the same for both cases.

[10] The iterative method for pion-nucleon scattering with pole terms was first discussed by Mandelstam (Ref. 2).