Covariant M Functions for Higher Spin*

M. D. SCADRON† AND H. F. JONES
Physics Department, Imperial College, London SW7, England
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We give a systematic treatment of high-spin M functions in the Dirac-Rarita-Schwinger formalism. The main difficulty in writing such M functions is that apparently independent covariants are in fact related. We derive these relations (equivalence theorems) and show how they may be used to obtain a kinematic-singularity-free expansion of the M function. Many examples are given and the general pattern is discussed. We also analyze the restrictions due to discrete symmetries and show how to choose invariant amplitudes whose discontinuities are given by unitarity. The connection with the helicity formalism is stressed throughout.

1. INTRODUCTION

In recent years high-spin processes have been increasingly dealt with in terms of helicity amplitudes[1] rather than the invariant amplitudes of the M function.[2] The main reasons for this have been the generality of the helicity approach, the simplicity of the partial-wave expansion, and the very direct connection with cross sections.

These advantages are, however, gained at the cost of rather complicated analytic[3] and crossing[4] properties, together with constraints which must be satisfied at the boundary of the physical region[5] and at thresholds and pseudothresholds.[6] When concentrating on one particular aspect of the problem, one can sometimes circumvent these difficulties, as in the use of reduced t-channel helicity amplitudes[7] for superconvergence[8] or in the use of s-channel helicity amplitudes for Reggeization.[9] Nevertheless, it is clear that in general it would be desirable to work in a formalism in which the difficulties do not arise.

The M-function approach in principle provides such a framework. That is to say, a judiciously chosen set of invariant amplitudes will satisfy simple crossing and analyticity properties and will be free of threshold constraints; the difficulty of writing a partial-wave expansion has to a large extent been overcome.[10,11]

It is the purpose of this paper to provide a systematic treatment of high-spin M functions in the Rarita-Schwinger[12] (RS) formalism, thus complementing a previous analysis[10] of on-shell propagators and vertex functions. We have chosen to work with the RS formalism rather than with $SL(2,C)$[13] because of the extremely simple way in which parity invariance can be incorporated. Again, however, this simplicity is not without its cost; the redundancy of components in the formalism, implicitly removed by subsidiary conditions, leads to complications which are the main stumbling block in writing M functions for higher spin. Namely, apparently independent covariants are, in fact, dependent when taken between RS wave functions, giving rise to "equivalence theorems" (Sec. 5). This difficulty is closely associated with the problem of choosing invariant amplitudes without kinematic singularities (KSF) and certainly constitutes a drawback to the M-function approach. We would, however, like to make a sharp distinction between these difficulties and the corresponding ones in the helicity formalism: Our difficulties are those of a choice of amplitudes, which must be solved *prior* to any dynamical calculation; on the other hand, the forward scattering conditions on helicity amplitudes,[5] for example, have to be imposed *afterwards* on quantities which are the result of such a calculation.

Section 2 is concerned with setting up the notation that we use throughout the paper, and the general kinematics of 4-particle scattering.

In Sec. 3 we discuss the conditions on the M function imposed by the discrete symmetries P, C, and T and, where applicable, the statistics of identical particles. It is shown how PT invariance provides a criterion for choosing invariant amplitudes which are real analytic.

† National Science Foundation Postdoctoral Fellow. Present address: Physics Department, Northwestern University, Evanston, Ill.

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In the next two sections we turn to the problem of writing the M function explicitly as a sum of invariant amplitudes multiplied by kinematic covariants constructed from the vectors and γ matrices of the problem.

Section 4 is concerned with the two simplest classes of M functions: those for the reactions $0+0\to 0+s_B'$, $\frac{1}{2}+0\to s_F'+0$ (class I) and those for the reactions $0+1\to 0+s_B'$, $\frac{1}{2}+0\to\frac{1}{2}+s_B'$ (class II), where s_B' (s_F') can be any integer (half-integer). M functions of the first kind can be written immediately; for M functions of the second kind one needs the "abnormal reductions"—decompositions of products of abnormal covariants into sums of normal covariants. These are derived from general identities given in Appendix A.

Section 5 is concerned with higher-spin processes, where even after performing the abnormal reductions one is left with a surfeit of covariants, which must therefore be related, as mentioned above, by equivalence theorems. We divide the higher-spin M functions into two classes: those for which the number of equivalence theorems is small compared with the number of covariants (class III) and those for which these two numbers are of the same order of magnitude (class IV). For M functions of the third kind, which turn out to be those for BF reactions with two vector indices, normal BB reactions with four vector indices, and the FF reaction $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$, we show how the general equivalence theorems (derived in Appendix B) provide us with a means of reducing the covariants to an independent set without introducing kinematic singularities; for the process $1+1\to 1+1$ this turns out to be quite involved. In principle, one could deal with M functions of the fourth kind in a similar manner, but in practice the algebra would become unmanageable; for example, for the process $\frac{3}{2}+0\to\frac{3}{2}+1$ one can write 38 possible covariants, of which 14 have to be removed by equivalence theorems.

Finally, in Sec. 6 we compare our procedure with other possible ways of proving invariant amplitudes free of kinematic singularities.

2. HELICITY AMPLITUDES AND M FUNCTIONS —NOTATION AND KINEMATICS

We begin by discussing the generalized πN kinematic notation which we will use throughout the paper.

The s channel is taken to be the process $s_1+s_2\to s_1'+s_2'$. Momenta and masses are defined in Fig. 1, where by writing p_μ, $q'_{\nu'}$, etc., we mean that the particles with these momenta are described by RS wave functions[10] $\psi_{\mu_1\cdots\mu_{J1}}{}^{(\lambda_1)}(p)$, $\bar\psi_{\nu_1'\cdots\nu_{J2}'}{}^{(\lambda_2')}(q')$, etc. The most convenient set of momenta ("natural" momenta) for

FIG. 1. The s channel.

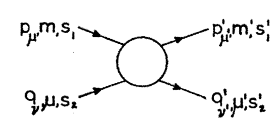
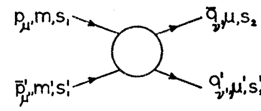


FIG. 2. The t channel.



this channel is

$$K\equiv p+q=p'+q',$$
$$\Lambda\equiv\tfrac{1}{2}(p-q),\qquad(1)$$
$$\Lambda'\equiv\tfrac{1}{2}(p'-q'),$$

with $s=K^2$.

The t channel (cf. Fig. 2) is taken to be the process $\bar s_1'+s_1\to s_2'+\bar s_2$, obtained from the s channel by the crossing $p'\to\bar p'=-p'$, $\lambda_1'\to\lambda_1'$; $q\to\bar q=-q$, $\lambda_2\to\lambda_2$. The natural momenta in this channel are

$$\Delta\equiv p'-p=q-q',$$
$$P\equiv\tfrac{1}{2}(p+p'),\qquad(2)$$
$$Q\equiv\tfrac{1}{2}(q+q'),$$

with $t=\Delta^2$.

Finally, the u channel (cf. Fig. 3) is taken to be the process $s_1+\bar s_2'\to s_1'+\bar s_2$, obtained from the s channel by the crossing $q\to\bar q=-q$, $\lambda_2\to\lambda_2$; $q'\to\bar q'=-q'$, $\lambda_2'\to\lambda_2'$.

The natural momenta are now

$$\bar K\equiv p-q'=p'-q,$$
$$\bar\Lambda\equiv\tfrac{1}{2}(p+q'),\qquad(3)$$
$$\bar\Lambda'\equiv\tfrac{1}{2}(p'+q),$$

with $u=\bar K^2$ and $s+t+u=m^2+m'^2+\mu^2+\mu'^2$.

We shall always label the particles so that the t channel is a boson channel (i.e., the initial and final states have baryon number zero). For BF reactions the fermions will always be associated with the momenta p, p', in analogy with πN scattering. To obtain simple crossing properties for the case of elastic scattering (cf. Sec. 3), it is customary to write the general M function in terms of the t-channel natural momenta P, Q, and Δ. We note that

$$K=P+Q,\quad \bar K=P-Q,\qquad(4)$$

while the various scalar products are

$$P\cdot Q\equiv\nu=\tfrac{1}{4}(s-u),$$
$$P\cdot\Delta=\tfrac{1}{2}(m'^2-m^2),\qquad(5)$$
$$Q\cdot\Delta=-\tfrac{1}{2}(\mu'^2-\mu^2),$$

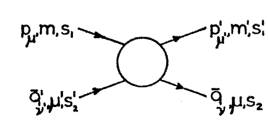
and

$$P^2=\tfrac{1}{2}(m'^2+m^2)-\tfrac{1}{4}t,\quad Q^2=\tfrac{1}{2}(\mu'^2+\mu^2)-\tfrac{1}{4}t,\qquad(6)$$

so that

$$P^2+Q^2=\tfrac{1}{2}(s+u).\qquad(7)$$

FIG. 3. The u channel.



Finally, we define the covariant normal vector N_μ as[14]

$$N_\mu = \epsilon_\mu(PQ\Delta)\,, \tag{8}$$

which can also be written

$$N_\mu = -\epsilon_\mu(p'Qp) = -\epsilon_\mu(\Lambda' K \Lambda)\,. \tag{9}$$

Then $-N^2$ is the determinant of Kibble,[15] whose vanishing determines the boundary of the physical region. It is related to c.m. scattering angles by

$$-N^2/s = p_s^2 p_s'^2 \sin^2\theta_s\,, \tag{10}$$

and similarly for the t and u channels.

The T matrix for particles with spin has rather complicated transformation properties under the action of the Lorentz group, the helicity components being "shuffled" by the so-called Wigner rotations.[16] By the well-known device of introducing wave functions (matrix elements of boosts) which carry the helicity labels, one can define an amplitude (M function) which transforms simply, in our case like a tensor-spinor, under Lorentz transformations.

Thus in the s channel, omitting momentum labels, we define M from the helicity amplitude by[17]

$$\langle s_1's_2'; \lambda_1'\lambda_2' | T | s_1 s_2; \lambda_1\lambda_2 \rangle \equiv \bar{\psi}_{\mu'}^{(\lambda_1')} \bar{\psi}_{\nu'}^{(\lambda_2')} \mathfrak{M}_{\mu'\nu';\mu\nu} \psi_\mu^{(\lambda_1)} \psi_\nu^{(\lambda_2)}\,. \tag{11}$$

One can go further and introduce *scalar* amplitudes by writing the M function as a linear combination of tensor spinors (kinematic covariants) constructed explicitly from momenta and γ matrices[18]:

$$\mathfrak{M}_{\mu'\nu';\mu\nu}(P,Q) = \sum_\kappa A_\kappa(s,t) \mathcal{K}_{\mu'\nu';\mu\nu}^{\kappa}(P,Q)\,. \tag{12}$$

It is important (cf. Sec. 6) that the kinematic covariants should be chosen in such a way that the decomposition (12) does not introduce kinematic singularities into the invariant amplitudes A_κ. There will be as many *independent* kinematic covariants as there are helicity amplitudes, viz., $N = \prod(2s_i+1)$. However, in high-spin processes one can in the first instance write down more than this number of kinematic covariants. Care must then be exercised to ensure that the reduction to an independent set does not introduce unwanted kinematical singularities.

In Eq. (12) we have expanded the M function in terms of t-channel kinematic covariants[19] depending on the t-channel natural momenta: This is the most convenient form for imposing discrete symmetries or calculating helicity amplitudes in that channel. Since the M function is channel-independent, we can also expand it in terms of s-channel covariants:

$$\mathfrak{M}(\Lambda',\Lambda) = \sum A^{(s)}(s,t) \mathcal{K}(\Lambda',\Lambda)\,, \tag{13}$$

and similarly for the u channel.

For FF scattering we have to define a convention for the order of the spinors. We define t-channel covariants by $\bar{u}(q')\mathcal{O}u(q)\bar{u}(p')\mathcal{O}u(p)$, s-channel covariants by $\bar{u}(p')\mathcal{O}C\bar{u}^T(q')u^T(q)C^{-1}\mathcal{O}u(p)$, and u-channel covariants by $\bar{u}(p')\mathcal{O}u(q)\bar{u}(q')\mathcal{O}u(p)$.

3. SYMMETRY TRANSFORMATIONS ON M FUNCTIONS

Before discussing the actual forms that kinematic covariants must have in various high-spin reactions, we establish restrictions on them due to invariance under discrete transformations.

We shall write the M function in a symbolic way as $\mathfrak{M}_{\beta\alpha}$, where α and β stand for the set of initial and final vector labels. For fermion reactions it is imagined as being sandwiched between one BF or two FF pairs of Dirac spinors.

We remind the reader that high-spin wave functions may be compounded from elementary spin-1 and spin-$\frac{1}{2}$ wave functions according to[10]

$$\begin{aligned} \epsilon_{\alpha_1\cdots\alpha_J}^{(\lambda)}(p) &= \sum_{\lambda_1\cdots\lambda_J} \langle \lambda_1\cdots\lambda_J | J\lambda \rangle \times \epsilon_{\alpha_1}^{(\lambda_1)}(p)\cdots\epsilon_{\alpha_J}^{(\lambda_J)}(p)\,, \\ u_{\alpha_1\cdots\alpha_J}^{(\lambda)}(p) &= \sum_{\lambda_1\cdots\lambda_J\lambda_{\frac{1}{2}}} \langle \lambda_1\cdots\lambda_J\lambda_{\frac{1}{2}} | J+\tfrac{1}{2}, \lambda \rangle \times \epsilon_{\alpha_1}^{(\lambda_1)}(p)\cdots\epsilon_{\alpha_J}^{(\lambda_J)}(p) u^{(\lambda_{\frac{1}{2}})}(p)\,, \end{aligned} \tag{14}$$

where $J = [s]$, the integral part of s.

We shall follow the phase conventions of Jacob and Wick,[1] in which the antiparticle helicity wave functions are given by

$$\begin{aligned} \epsilon_\alpha^{*(\lambda)}(p) &= -\xi_\lambda \epsilon_\alpha^{(-\lambda)}(p)\,, \\ v^{(\lambda)}(p) &= C\bar{u}^{T(\lambda)}(p) = i\gamma_5\xi_\lambda u^{(-\lambda)}(p)\,, \end{aligned} \tag{15}$$

where $\xi_\lambda = (-1)^{s-\lambda}$ and C is the Dirac matrix[20] such that $C^{-1}\gamma_\mu C = -\gamma_\mu^T$, with $\gamma_5^2 = -1$.

A. Parity

Under the action of the parity operator, helicity states[20] transform as $|\mathbf{p}\lambda\rangle \to \eta_P \xi_\lambda |-\mathbf{p}, -\lambda\rangle$, where η_P is the intrinsic parity. Hence the statement of parity conservation for helicity amplitudes is

$$\begin{aligned} \langle \mathbf{p}_1'\lambda_1'\mathbf{p}_2'\lambda_2' | T | \mathbf{p}_1\lambda_1\mathbf{p}_2\lambda_2 \rangle &= n(-1)^{\lambda'-\lambda} \langle -\mathbf{p}_1', -\lambda_1', -\mathbf{p}_2', -\lambda_2' | \\ &\quad \times T | -\mathbf{p}_1, -\lambda_1, -\mathbf{p}_2, -\lambda_2 \rangle\,, \end{aligned} \tag{16}$$

where $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_1' - \lambda_2'$, and n is the "over-all

[14] The notation here is $\epsilon_\mu(ABC) \equiv \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma$. Our ϵ symbol is defined so that $\epsilon_{0123} = 1$.

[15] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

[16] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[17] Here the covariant label μ, for example, is symbolic for the set of labels $\mu_1\cdots\mu_J$ for a particle of spin J or $J+\frac{1}{2}$.

[18] Because of the subsidiary condition $p\cdot\psi(p) = 0$, the explicit dependence of the $\mathcal{K}$'s on Δ can be removed except for abnormal BB reactions.

[19] Therefore we strictly should have written $A_\kappa(s,t)$ as $A_\kappa^{(t)}(s,t)$.

[20] In what follows we shall take $\mathbf{p}$ to lie in the 1-3 plane, in which case C can be taken as $\gamma_5\sigma_2$.

normality," the product of the individual normalities $\eta_P(-1)^J$.

By relating the parity-reversed wave functions to the original wave functions,

$$\epsilon_\alpha^{(-\lambda)}(-\mathbf{p}) = -\xi_\lambda g_{(\alpha)}\epsilon_\alpha^{(\lambda)}(\mathbf{p}),$$
$$u^{(-\lambda)}(-\mathbf{p}) = \xi_\lambda \gamma_0 u^{(\lambda)}(\mathbf{p}), \tag{17}$$

where $g_{(\alpha)}=(1,-1,-1,-1)$, we can rewrite (16) as a parity statement on the M function:

$$\mathfrak{M}_{\beta\alpha}(f,i) = n g_{(\beta)} g_{(\alpha)} \gamma_0 \mathfrak{M}_{\beta\alpha}(Pf,Pi)\gamma_0, \tag{18}$$

which just means that if the reaction is normal ($n=1$), $\mathfrak{M}$ should be expanded in a set of proper tensors $\mathcal{K}^+$, whereas if it is abnormal ($n=-1$), $\mathfrak{M}$ should be expanded in a set of pseudotensors $\mathcal{K}^-$, containing one over-all γ_5 or one over-all Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$. The number of independent kinematic covariants $\mathcal{K}^\pm$ is then $N^\pm=\frac{1}{2}N$ for BF or FF reactions and $N^\pm=\frac{1}{2}(N\pm1)$ for BB reactions, agreeing with the number of independent helicity amplitudes under parity conservation.

B. Time Reversal

Under time reversal, a one-particle helicity state transforms as $|\mathbf{p},\lambda\rangle \to \eta_T(-1)^{2s}|-\mathbf{p},\lambda\rangle$. Invariance of the S matrix under time reveral, $S_{fi}=\eta_T S_{Ti,Tf}$, then reads for helicity amplitudes

$$\langle \mathbf{p}_1'\lambda_1'\mathbf{p}_2'\lambda_2'|T|\mathbf{p}_1\lambda_1\mathbf{p}_2\lambda_2\rangle = \eta_T\langle -\mathbf{p}_1,\lambda_1,-\mathbf{p}_2,\lambda_2|T|-\mathbf{p}_1',\lambda_1',-\mathbf{p}_2',\lambda_2'\rangle, \tag{19}$$

where η_T is the product of the four time-reversal phases.

Using the relations

$$\epsilon_\alpha^{*(\lambda)}(-\mathbf{p}) = g_{(\alpha)}\epsilon_\alpha^{(\lambda)}(\mathbf{p}),$$
$$\bar{u}^{(\lambda)T}(-\mathbf{p}) = Tu^{(\lambda)}(\mathbf{p}), \tag{20}$$

where T is the Dirac matrix such that $T\gamma_\mu T^{-1}=g_{(\mu)}\gamma_\mu^T$ ($T=i\gamma_0\gamma_5 C^{-1}$), we write (19) as a time-reversal condition on the M function:

$$\mathfrak{M}_{\beta\alpha}(f,i) = \eta_T g_{(\beta)} g_{(\alpha)} T^{-1}\mathfrak{M}_{\alpha\beta}^T(Ti,Tf)T. \tag{21}$$

This only restricts the number of covariants for elastic processes (cf. Sec. 3 E); otherwise it relates the amplitudes of a reaction to the inverse reaction.

For future reference we note the condition for invariance under the combined transformation PT:

$$\mathfrak{M}_{\beta\alpha}(f,i) = n\eta_T B^{-1}\mathfrak{M}_{\alpha\beta}^T(i,f)B, \tag{22}$$

where B is the Dirac matrix such that $B\gamma_\mu B^{-1}=\gamma_\mu^T$ ($B=i\gamma_5 C^{-1}$).

C. Charge Conjugation

Using the relation

$$\bar{v}^{T(\lambda)}(p) = C^{-1}u^{(\lambda)}(p), \tag{23}$$

we convert the statement of charge-conjugation invariance for the S matrix, $S_{f,i}=\eta_C S_{Cf,Ci}$, into the corresponding relation for the M function:

$$\mathfrak{M}_{\beta\alpha}(f,i) = \eta_C(-C)\mathfrak{M}_{\beta\alpha}^T(Cf,Ci)C^{-1}. \tag{24}$$

As in the case of time reversal, this is in general a relation between the M functions of different processes. However, it will give restrictions on the number of invariants when applied in the t channel of an elastic process, where both initial and final states are essentially self-conjugate (cf. Sec. 3 E).

Combining the results of C and PT, we get the CPT relation

$$\mathfrak{M}_{\beta\alpha}(f,i) = n\eta_C\eta_T(-i\gamma_5)\mathfrak{M}_{\alpha\beta}(Ci,Cf)(i\gamma_5). \tag{25}$$

On the other hand, crossing of all four particles gives

$$\mathfrak{M}_{\beta\alpha}(f,i) = i^F\mathfrak{M}_{\alpha\beta}(Ci,Cf;\{-p\}), \tag{26}$$

where F is the number of fermions and the notation $\{-p\}$ means that all 4-momenta are reversed. Since this reversal can be achieved for the kinematic covariants by

$$\mathcal{K}_{\beta\alpha}(P,Q) = (-)^J i\gamma_5\mathcal{K}_{\beta\alpha}(-P,-Q)i\gamma_5, \tag{27}$$

where J is the total number of indices, the scalar invariants remaining unaltered, we can write (26) as

$$\mathfrak{M}_{\beta\alpha}(f,i) = i^F(-)^J(i\gamma_5)\mathfrak{M}_{\alpha\beta}(Ci,Cf)(i\gamma_5), \tag{28}$$

which by comparison with (25) gives a constraint on the over-all C, P, T phases:

$$\eta_P\eta_C\eta_T = 1. \tag{29}$$

D. Discontinuity Condition

The unitarity condition for the T matrix, defined in terms of the S matrix by

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f-p_i)T_{fi}, \tag{30}$$

reads

$$-\tfrac{1}{2}i(T_{fi}-T_{if}^*) = \tfrac{1}{2}\sum_n T_{nf}^* T_{ni}(2\pi)^4\delta^4(p_f-p_n). \tag{31}$$

We would like to show that the kinematic covariants can be chosen in such a way that the corresponding invariant amplitudes are real analytic and that the left-hand side of Eq. (31) corresponds directly to discontinuities of invariant amplitudes.

In terms of M functions, Eq. (31) reads

$$-\tfrac{1}{2}i[\mathfrak{M}_{\beta\alpha}(f,i) - \bar{\mathfrak{M}}_{\alpha\beta}(i,f)] = \tfrac{1}{2}\sum_n \bar{\mathfrak{M}}_\beta(n,f)\mathcal{P}_n\mathfrak{M}_\alpha(n,i)(2\pi)^4\delta^4(p_f-p_n), \tag{32}$$

where $\mathcal{P}_n$ is the product of the spin projection operators for the intermediate state and $\bar{\mathfrak{M}}\equiv\gamma_0\mathfrak{M}^\dagger\gamma_0$.

The second term on the left-hand side can be related back to $\mathfrak{M}(f,i)$ by use of the PT relation (22). Taking into account Eq. (29), we can write the left-hand side of Eq. (32) as

$$-\tfrac{1}{2}i[\mathfrak{M}_{\beta\alpha}(f,i)-\eta_C(-)^J B^{-1}\bar{\mathfrak{M}}_{\alpha\beta}{}^T(f,i)B].$$

Then, if we always choose our kinematic covariants to satisfy

$$\bar{\mathcal{K}}_{\beta\alpha}(P,Q)=\eta_C(-)^J B^{-1}\mathcal{K}_{\beta\alpha}{}^T(P,Q)B\,, \tag{33}$$

the unitarity condition can be cast in the desired form

$$\sum_\kappa (\mathrm{Im}A_\kappa)\mathcal{K}_{\beta\alpha}{}^\kappa(f,i) =\tfrac{1}{2}\sum_n \bar{\mathfrak{M}}_\beta(n,f)\mathcal{P}_n\mathfrak{M}_\alpha(n,i)(2\pi)^4\delta^4(p_f-p_n)\,. \tag{34}$$

The discontinuity condition, Eq. (33), determines the number of factors of i accompanying each kinematic covariant. If we choose to write the M function in terms of the set of γ matrices $(1,\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5,[\gamma_\mu,\gamma_\nu])$, then for processes such as elastic scattering where $\eta_C=(-)^J$ the M function will contain no explicit factors of i. There will be one over-all factor of i for processes such as $\pi N\to A_1N$, where $\eta_C=-(-)^J$. For other processes, such as $\pi N\to K\Sigma$, η_C is not observable. In such cases we can *choose* η_C to be $(-)^J$; the phases of effective Hamiltonian (vertex functions) will correspondingly be affected so that Born terms will be real.

As a byproduct of the discontinuity condition, we can rederive the Hermitian analyticity theorem of Olive.[21] Writing in an obvious notation

$$T_{fi}{}^{(\lambda',\lambda)}(s,t)=\sum A_\kappa(s,t)\mathcal{K}_\kappa{}^{(\lambda',\lambda)}(P,Q)\,, \tag{35}$$

we have shown that

$$\begin{aligned}T_{if}{}^{(\lambda',\lambda)*}(s,t)&=\sum A_\kappa{}^*(s,t)\mathcal{K}_\kappa{}^{(\lambda',\lambda)}(P,Q)\\&=\sum A_\kappa(s^*,t)\mathcal{K}_\kappa{}^{(\lambda',\lambda)}(P,Q)\,,\end{aligned}$$

in the s channel, or

$$T_{if}{}^*(s,t)=T_{fi}(s^*,t)\,, \tag{36}$$

so that again the left-hand side of Eq. (31) corresponds to a discontinuity when the A_κ are real analytic.

E. Elastic Scattering

As remarked above, for a general process only P reduces the number of covariants, C and T relating one process to another. For elastic scattering, however, C and T do lead to a reduction in the number of covariants, as is well known.

For elastic scattering, with $n=\eta_C=\eta_T=1$, PT invariance, Eq. (22), gives

$$\mathcal{K}_{\beta\alpha}(P,Q)=B^{-1}\mathcal{K}_{\alpha\beta}{}^T(P,Q)B\,. \tag{37}$$

Combined with the restriction due to parity that $\mathcal{K}=\mathcal{K}^+$, this reduces the number of independent kinematic covariants to

$$N_{\mathrm{el}}{}^+=\tfrac{1}{2}[\tfrac{1}{2}(2s'+1)^2(2s+1)^2+(2s'+1)(2s+1)]$$

for BF or FF elastic scattering and to

$$N_{\mathrm{el}}{}^+=\tfrac{1}{2}[\tfrac{1}{2}(2s'+1)^2(2s+1)^2+(2s'+1)(2s+1)+\tfrac{1}{2}]$$

for BB elastic scattering $(s+s'\to s+s')$.[22]

The above restriction can also be obtained from charge-conjugation invariance in the t channel. This gives

$$\mathcal{K}_{\beta\alpha}(P,Q)=C\mathcal{K}_{\alpha\beta}{}^T(-P,-Q)C^{-1}\,, \tag{38}$$

which because of the CPT condition (27) is equivalent to (37).

F. Identical Particles and Crossing

When all four particles are identical (in the s channel), the number of covariants is further reduced by the constraint

$$\mathcal{K}_{\mu'\nu',\mu\nu}(P,Q)=\mathcal{K}'_{\nu'\mu',\nu\mu}(Q,P)\,, \tag{39}$$

where in $\mathcal{K}'$ the spinor sets $\gamma^{(1)}$ and $\gamma^{(2)}$ are interchanged.

If only two of the particles are identical, crossing relates the invariant amplitudes of one channel to those of another. If we choose our covariants in the proper way, each invariant amplitude will satisfy crossing symmetry. We distinguish three cases.

(a) If the t-channel initial state consists of two identical bosons (momenta $\bar{q}$ and q'), crossing symmetry $(s\leftrightarrow u)$ states that if we choose our covariants to satisfy

$$\mathcal{K}_{\nu'\nu}{}^\kappa(P,Q)=\xi_\kappa{}^Q\mathcal{K}_{\nu\nu'}{}^\kappa(P,-Q)\,, \tag{40}$$

where $\xi_\kappa{}^Q=\pm1$, then

$$A_\kappa{}^{(t)}(\nu,t)=\xi_\kappa{}^Q A_\kappa{}^{(t)}(-\nu,t)\,. \tag{41}$$

(b) If instead the s-channel initial state consists of identical particles, crossing symmetry $(t\leftrightarrow u)$ states that if we choose the s-channel covariants to satisfy

$$\mathcal{K}_{\mu\nu}{}^\kappa(\Lambda',\Lambda)=\pm\xi_\kappa{}^\Lambda\mathcal{K}_{\nu\mu}{}^\kappa(\Lambda',-\Lambda) \tag{42}$$

for identical bosons or fermions, respectively, then

$$A_\kappa{}^{(s)}(\bar{\nu},s)=\xi_\kappa{}^\Lambda A_\kappa{}^{(s)}(-\bar{\nu},s)\,, \tag{43}$$

where $\bar{\nu}=\Lambda\cdot\Lambda'=\frac{1}{4}(t-u)$.

(c) Finally, if in the s channel one initial (p) and one final (p') particle are identical and, furthermore, if the other two particles are self-conjugate bosons,[23] then charge conjugation and crossing imply that

$$A_\kappa(\nu,t)=\eta_C\xi_\kappa{}^P A_\kappa(-\nu,t) \tag{44}$$

if

$$\mathcal{K}_{\mu'\mu}(P,Q)=\xi_\kappa{}^P C\mathcal{K}_{\mu\mu'}{}^T(-P,Q)C^{-1}\,. \tag{45}$$

Such constraints as Eqs. (40), (42), and (45) indicate

[21] D. I. Olive, Nuovo Cimento **26**, 73 (1962).

[22] P. L. Csonka, M. J. Moravcsik, and M. D. Scadron, Ann. Phys. (N. Y.) **41**, 1 (1967).

[23] For example, $\pi^0p\to\omega^0p$. Crossing symmetry for πN scattering can be obtained from either (a) or (c).

TABLE I. BB reactions of the first kind.

Reaction	$\mathcal{K}^+$	$\mathcal{K}^-$
$0+0\to 0+1$	P, Q	N
$0+0\to 0+2$	PP, PQ, QQ	NP, NQ
$0+0\to 0+3$	PPP, PPQ, PQQ, QQQ	NPP, NPQ, NQQ

that symmetrized forms of covariants such as

$$P_\mu Q_\nu+Q_\mu P_\nu\equiv\{P,Q\}_{\mu\nu},$$
$$P_\mu Q_\nu-Q_\mu P_\nu\equiv[P,Q]_{\mu\nu},\ [\gamma_{\nu'},\gamma_\nu],\ [\gamma_\nu,Q\ ],\ \text{etc.},$$

should be used as far as possible.

4. M FUNCTIONS OF THE FIRST AND SECOND KINDS—ABNORMAL REDUCTIONS

We build up the kinematic covariants first for "simple" and then for "compound" reactions. We shall return to the problem of proving that the resulting invariant amplitudes are free of KSF in Sec. 6.

M functions of the first kind are those for the basic reactions (in spin space) $0+0\to 0+s_B'$ and $\frac{1}{2}+0\to s_F'+0$, where s_B' (s_F') can take any integer (half-integer) value.

For the basic boson-boson (BB) reactions one simply writes all possible covariants (in terms, say, of the momenta P, Q); the number of kinematic covariants will always be the correct number $N^\pm$. In Table I we give a few examples.

Metric tensors $g_{\nu_1'\nu_2'}$ do not appear in these covariants because they vanish due to the tracelessness of the spin s_B' wave function. Similarly, because of the subsidiary condition, the momentum $\Delta_{\nu'}$ is equivalent to $2Q_{\nu'}$.

For the basic boson-fermion (BF) reactions we give the first few sets of normal covariants in Table II.

Here the notation $(P,Q)(1,\mathbf{Q})$ means the set of covariants $P_{\mu'}$, $Q_{\mu'}$, $P_{\mu'}\mathbf{Q}$, $Q_{\mu'}\mathbf{Q}$, and so on (with $\mathbf{Q}\equiv\gamma_\mu Q^\mu$). We have not written the abnormal covariants $\mathcal{K}^-$ explicitly since these are simply $\gamma_5\mathcal{K}^+$.

In place of 1 or $\mathbf{Q}$ we could have used $\gamma_5\mathbf{N}$; however, this would lead to kinematic singularities, being related to $(1,\mathbf{Q})$ by the identity (cf. Appendix A)

$$\gamma_5\mathbf{N}=2(P^2-m_-^2)\mathbf{Q}-2m_+\nu+m_-Q\cdot\Delta\,, \tag{46}$$

where $m_+\equiv\frac{1}{2}(m'+m)$ and $m_-\equiv\frac{1}{2}(m'-m)$. We shall call identities such as Eq. (46) "abnormal reductions."

M functions of the second kind are those for the reactions $0+1\to 0+s_B'$ and $\frac{1}{2}+0\to\frac{1}{2}+s_B'$, which are products in spin space of the above basic reactions.

TABLE II. BF reactions of the first kind.

Reaction	$\mathcal{K}^+$
$\frac{1}{2}+0\to\frac{1}{2}+0$	1, $\mathbf{Q}$
$\frac{1}{2}+0\to\frac{3}{2}+0$	$(P,Q)(1,\mathbf{Q})$
$\frac{1}{2}+0\to\frac{5}{2}+0$	$(PP,PQ,QQ)(1,\mathbf{Q})$

TABLE III. BB reactions of the second kind.

Reaction	$\mathcal{K}^+$	$\mathcal{K}^-$
$0+1\to 0+1$	$(P,Q)_{\nu'}(P,Q)_\nu$	$(\epsilon(PQ),\epsilon(P\Delta),\epsilon(Q\Delta))_{\nu'\nu}$
	$g_{\nu'\nu}$	$\{N_{\nu'},P_\nu\}$
$0+1\to 0+2$	$(PP,PQ,QQ)_{\nu_1'\nu_2'}(P,Q)_\nu$	$(\epsilon(PQ),\epsilon(P\Delta),\epsilon(Q\Delta))_{\nu_1'\nu}(P,Q)_{\nu_2'}$
	$g_{\nu_1'\nu}(P,Q)_{\nu_2'}$	$g_{\nu_1'\nu}N_{\nu_2'}$

One might have hoped to obtain the kinematic covariants for these reactions as a direct product of the basic covariants according to the factorization rule[22]

$$\mathcal{K}_c^\pm=\mathcal{K}_a^+\otimes\mathcal{K}_b^\pm\oplus\mathcal{K}_a^-\otimes\mathcal{K}_b^\mp\,. \tag{47}$$

However, although one obtains the correct number of covariants in this way, the corresponding invariant amplitudes would have kinematic singularities. This can be seen from the fact that, using these covariants, the s-channel helicity amplitudes, as calculated from Eq. (11), do not automatically have the right θ_s dependence $(\cos\frac{1}{2}\theta_s)^{|\lambda'+\lambda|}(\sin\frac{1}{2}\theta_s)^{|\lambda'-\lambda|}$. Thus, for example, the factorized normal covariants for $0+1\to 0+1$ are PP, PQ, QP, QQ, and NN. Then one or more of the corresponding invariant amplitudes must have a kinematic singularity, since otherwise a simple calculation gives $\langle\lambda'=1|T|\lambda=1\rangle\propto\sin^2\theta_s$, which is incorrect. However, by replacing the factorized covariant $N_{\nu'}N_\nu$ by $g_{\nu'\nu}$ and assuming no kinematic singularities of the new set of invariant amplitudes, we obtain the correct angular dependence $\langle\lambda'=1|T|\lambda=1\rangle\propto\cos^2(\frac{1}{2}\theta_s)$.

We therefore draw the general conclusion that at least one new nonfactorizable covariant must be included in the set of covariants for high-spin (compound) reactions if one is to obtain KSF invariant amplitudes. We relate factorized covariants to the new covariants by means of abnormal reductions such as Eq. (46) or

$$\gamma_5 N_\nu=(\tfrac{1}{4}t-m_-^2)[\gamma_\nu,\mathbf{Q}]+(m_+Q\cdot\Delta-2m_-\nu)\gamma_\nu+2(m_+Q_\nu+m_-P_\nu)\mathbf{Q}-(2\nu Q_\nu+Q\cdot\Delta P_\nu)\,, \tag{48}$$

$$N_{\nu'}N_\nu=N^2g_{\nu'\nu}+(tQ^2-(Q\cdot\Delta)^2)P_{\nu'}P_\nu+(P\cdot\Delta Q\cdot\Delta-t\nu)\times\{P,Q\}_{\nu'\nu}-2(\nu Q\cdot\Delta-Q^2P\cdot\Delta)[P,Q]_{\nu'\nu}+(4(\nu^2-P^2Q^2)+tP^2-(P\cdot\Delta)^2)Q_{\nu'}Q_\nu\,, \tag{49}$$

etc., obtained by saturating the identities of Appendix A with various momenta.

We give the first few KSF covariants of the second kind for BB reactions in Table III.

TABLE IV. BF reactions of the second kind.

Reaction	$\mathcal{K}^+$
$\frac{1}{2}+0\to\frac{1}{2}+1$	$(P,Q)(1,\mathbf{Q})$
	$\gamma,[\gamma,\mathbf{Q}]$
$\frac{1}{2}+0\to\frac{1}{2}+2$	$(PP,PQ,QQ)(1,\mathbf{Q})$
	$(\gamma,[\gamma,\mathbf{Q}])(P,Q)$

Here, for the abnormal reactions, we used the abnormal reductions

$$\begin{aligned}
[N,P]_{\nu'\nu} &= P\cdot\Delta\epsilon_{\nu'\nu}(PQ)+P^2\epsilon_{\nu'\nu}(Q\Delta)+\nu\epsilon_{\nu'\nu}(\Delta P)\,,\\
[N,Q]_{\nu'\nu} &= Q\cdot\Delta\epsilon_{\nu'\nu}(PQ)+\nu\epsilon_{\nu'\nu}(Q\Delta)\\
&\quad +Q^2\epsilon_{\nu'\nu}(\Delta P)\,, \qquad (50)\\
-2\{N,Q\}_{\nu'\nu} &= t\epsilon_{\nu'\nu}(PQ)+P\cdot\Delta\epsilon_{\nu'\nu}(Q\Delta)+Q\cdot\Delta\epsilon_{\nu'\nu}(\Delta P)\,,
\end{aligned}$$

which follow from Eq. (A2).[24]

The first few KSF covariants of the second kind for BF reactions are given in Table IV. Again $\mathcal{K}^-$ is simply $\gamma_5\mathcal{K}^+$.

Finally, we may include in class II the normal reaction $0+1\rightarrow 1+1$. Only abnormal reductions are required to reduce all possible covariants to the KSF set, which is $(P,Q)_{\mu'}(P,Q)_{\nu'}(P,Q)_\nu$, $g_{\mu'\nu'}(P,Q)_\nu$, $g_{\nu'\nu}(P,Q)_{\mu'}$, and $g_{\mu'\nu}(P,Q)_{\nu'}$.

5. *M* FUNCTIONS OF THE THIRD AND FOURTH KINDS—EQUIVALENCE THEOREMS

So far, in obtaining KSF covariants of the first and second kind, we have followed the prescription of Hearn,[25] viz., simply to remove the "obvious" covariants such as $\gamma_5 N_\mu$, $N_\mu N_\nu$, etc., by abnormal reductions. However, a problem arises when, after all abnormal reductions have been made, one is left with more covariants than there are independent helicity amplitudes. Such is the case for the covariants of the third kind, occurring in the reactions

BF: $\quad \frac{3}{2}+0\rightarrow\frac{3}{2}+0,\quad \frac{1}{2}+0\rightarrow\frac{3}{2}+1,\quad \frac{1}{2}+1\rightarrow\frac{1}{2}+1,$

FF: $\quad \frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2},$

BB^+: $\quad 0+2\rightarrow 0+2,\quad 0+1\rightarrow 1+2,\quad 1+1\rightarrow 1+1.$

For example, for $N^*\pi$ scattering there are eight "natural" covariants $(PP,\{P,Q\},QQ,g)\ (1,\boldsymbol{Q})$, of which only six can be independent, while in NN scattering there are again eight natural covariants (the combined sets of GNO,[26] GGMW,[27] and ALV[28]), of which only five are independent.

Thus seemingly independent covariants must in fact be related when taken between Dirac-Rarita-Schwinger (DRS) wave functions. The relations between covariants for the examples above have already been given in the literature.

For $N^*\pi$ scattering[29] they are

$$2P_{\mu'}P_\mu\boldsymbol{Q}=m\{P,Q\}_{\mu'\mu}-m\nu g_{\mu'\mu}+P^2g_{\mu'\mu}\boldsymbol{Q}\,, \qquad (51)$$

$$\begin{aligned}
m\{P,Q\}_{\mu'\mu}\boldsymbol{Q} &= -2Q^2P_{\mu'}P_\mu+\tfrac{1}{2}tQ_{\mu'}Q_\mu+2\nu\{P,Q\}_{\mu'\mu}\\
&\quad +m\nu g_{\mu'\mu}\boldsymbol{Q}-(\nu^2+\tfrac{1}{4}tQ^2)g_{\mu'\mu}\,, \qquad (52)
\end{aligned}$$

while for NN scattering[28] they are

$$m(\gamma^{(2)}\cdot P+\gamma^{(1)}\cdot Q)=\nu(\mathcal{K}_S-\mathcal{K}_P)+m^2\mathcal{K}_V-\tfrac{1}{4}t\mathcal{K}_T\,, \qquad (53)$$

$$\gamma^{(2)}\cdot P\gamma^{(1)}\cdot Q=-m^2\mathcal{K}_P+\nu\mathcal{K}_V-\tfrac{1}{4}t\mathcal{K}_A\,, \qquad (54)$$

and

$$\begin{aligned}
\gamma_5^{(2)}\gamma^{(2)}\cdot P\gamma_5^{(1)}\gamma^{(1)}\cdot Q &= \nu(-\mathcal{K}_S+\mathcal{K}_P+\mathcal{K}_A)\\
&\quad +P^2(\mathcal{K}_V-\mathcal{K}_T)\,. \qquad (55)
\end{aligned}$$

Such relations, which we term "equivalence theorems," will always occur for high-spin processes. They arise essentially because of the redundancy of components of the DRS wave functions and are derived in a general way[30] for all M functions of the third kind in Appendix B.

The equivalence theorems not only provide the necessary relations between seemingly independent covariants; their form also indicates which covariants may be eliminated without introducing kinematic singularities. Namely, covariants which occur with numerical coefficients, such as those on the left-hand side of Eqs. (51)–(55), may be eliminated in favor of the others without dividing by factors such as ν and t and thereby introducing kinematic singularities.

Thus possible KSF sets of covariants are $(PP,QQ,g)_{\mu'\mu}\times\{P,Q\}_{\mu'\mu}$, $(QQ,g)_{\mu'\mu}\boldsymbol{Q}$ for $N^*\pi$ scattering and the so-called "β-decay" covariants $(\mathcal{K}_S,\mathcal{K}_V,\mathcal{K}_T,\mathcal{K}_A,\mathcal{K}_P)$ for NN scattering. However, other choices are possible. The covariant $P_{\mu'}P_\mu\boldsymbol{Q}$ may be used instead of $\{P,Q\}_{\mu'\mu}$ in the $N^*\pi$ case and $(\gamma^{(2)}\cdot P+\gamma^{(1)}\cdot Q)$ instead of $\mathcal{K}_V$, $\gamma^{(2)}\cdot P\gamma^{(1)}\cdot Q$ instead of $\mathcal{K}_P$ in the NN case.

Consider next the process $\frac{1}{2}+0\rightarrow\frac{3}{2}+1$ (e.g., $N\pi\rightarrow N^*\rho$) with 12 independent covariants. The 14 possible (normal) covariants $(PP,QQ,PQ,QP,g)_{\mu'\nu'}(1,\boldsymbol{Q})$, $(P,Q)_{\mu'}(\gamma,[\gamma,\boldsymbol{Q}])_{\nu'}$ can be reduced to a KSF set of 12 by removing, for example, $P_{\mu'}P_{\nu'}\boldsymbol{Q}$ and $Q_{\mu'}Q_{\nu'}\boldsymbol{Q}$ by means of the equivalence theorems

$$\begin{aligned}
P_{\mu'}P_{\nu'}\boldsymbol{Q} &= -P_{\mu'}Q_{\nu'}\boldsymbol{Q}+m_+Q_{\mu'}P_{\nu'}+\tfrac{1}{2}m'P_{\mu'}[\gamma_{\nu'},\boldsymbol{Q}]\\
&\quad -m_-Q_{\mu'}Q_{\nu'}-p\cdot Qg_{\mu'\nu'}+(m_+{}^2-\tfrac{1}{4}t)\\
&\quad \times(g_{\mu'\nu'}\boldsymbol{Q}-Q_{\mu'}\gamma_{\nu'})\,, \qquad (56)
\end{aligned}$$

$$\begin{aligned}
m_+Q_{\mu'}Q_{\nu'}\boldsymbol{Q} &= -m'P_{\mu'}Q_{\nu'}\boldsymbol{Q}+m_-Q_{\mu'}P_{\nu'}\boldsymbol{Q}+(\nu+\tfrac{1}{2}\Delta\cdot Q)\tfrac{1}{2}P_{\mu'}\\
&\quad \times[\gamma_{\nu'},\boldsymbol{Q}]+p'\cdot QQ_{\mu'}P_{\nu'}+(p'\cdot Q-Q^2)P_{\mu'}Q_{\nu'}\\
&\quad +[p\cdot Q-(m_-{}^2-\tfrac{1}{4}t)]Q_{\mu'}Q_{\nu'}-Q^2P_{\mu'}P_{\nu'}\\
&\quad -m'p\cdot QQ_{\mu'}\gamma_{\nu'}+m'Q^2P_\mu\gamma_{\nu'}-\tfrac{1}{2}(m_-{}^2-\tfrac{1}{4}t)Q_{\mu'}\\
&\quad \times[\gamma_{\nu'},\boldsymbol{Q}]+(m_+\nu-\tfrac{1}{2}m_-\Delta\cdot Q)g_{\mu'\nu'}\boldsymbol{Q}\\
&\quad -(\nu^2+\tfrac{1}{4}tQ^2-m_-{}^2Q^2-\tfrac{1}{4}(\Delta\cdot Q)^2)g_{\mu'\nu'}\,, \qquad (57)
\end{aligned}$$

[24] Remember that because of the subsidiary condition Δ_ν is equivalent to $-2Q_\nu$ and $\Delta_{\nu'}$ to $2Q_{\nu'}$.

[25] A. C. Hearn, Nuovo Cimento **21**, 333 (1961).

[26] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (N. Y.) **2**, 226 (1957).

[27] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[28] D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960). We use the notation $\mathcal{K}_S=1^{(2)}1^{(1)}$, $\mathcal{K}_P=\gamma_5^{(2)}\gamma_5^{(1)}$, $\mathcal{K}_V=\gamma^{(2)}\cdot\gamma^{(1)}$, $\mathcal{K}_A=\gamma_5^{(2)}\gamma_5^{(1)}\gamma^{(2)}\cdot\gamma^{(1)}$, and $\mathcal{K}_T=\frac{1}{2}\sigma_{\mu\nu}{}^{(2)}\sigma_{\mu\nu}{}^{(1)}$.

[29] H. F. Jones and M. D. Scadron, Nuovo Cimento **52A**, 62 (1968).

[30] From the present point of view the NN and $N^*\pi$ problems are on exactly the same footing; the first two NN equivalence theorems were originally derived (Ref. 28) by a different method which does not lend itself to generalization.

which are special cases of the general BF equivalence theorems (B1) and (B2).

The third BF reaction, $\frac{1}{2}+1\rightarrow\frac{1}{2}+1$, for elastic scattering (e.g., $N\rho\rightarrow N\rho$) has the set of 14 normal covariants $(PP,QQ,\{P,Q\},g)_{\nu'\nu}(1,Q)$, $\{\gamma,P\}_{\nu'\nu}$, $\{\gamma,Q\}_{\nu'\nu}$, $[P,[\gamma,Q]]_{\nu'\nu}$, $[Q,[\gamma,Q]]_{\nu'\nu}$, $[\gamma_{\nu'},\gamma_\nu]$, and $[\gamma Q\gamma]_{\nu'\nu}\equiv\gamma_{\nu'}Q\gamma_\nu-\gamma_\nu Q\gamma_{\nu'}$, from which the covariants $\{P,Q\}_{\nu'\nu}Q$ and $Q_{\nu'}Q_\nu Q$ can be removed via the equivalence theorems [again special cases of (B1) and (B2)]

$$2\{P,Q\}_{\nu'\nu}Q=-m\nu[\gamma_{\nu'},\gamma_\nu]-P^2[\gamma Q\gamma]_{\nu'\nu}-m[[P,\gamma]_{\nu'\nu},Q]+2\nu\{Q,\gamma\}_{\nu'\nu},\quad(58)$$

$$2mQ_{\nu'}Q_\nu Q=-(\nu^2+\tfrac{1}{4}tQ^2)[\gamma_{\nu'},\gamma_\nu]-m\nu[\gamma Q\gamma]_{\nu'\nu}-\nu[[P,\gamma]_{\nu'\nu},Q]+\tfrac{1}{4}t[[\gamma,Q]_{\nu'\nu},Q]+2mQ^2\{Q,\gamma\}_{\nu'\nu}-2Q^2\{P,Q\}_{\nu'\nu}+2\nu Q_{\nu'}Q_\nu.\quad(59)$$

The general-mass FF reaction $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$ has eight independent normal covariants. We can choose the set $\mathcal{K}_S\cdots\mathcal{K}_P$, $\mathcal{K}_6\equiv\gamma^{(2)}\cdot P-\gamma^{(1)}\cdot Q$, $\mathcal{K}_7\equiv\gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma^{(2)}\cdot P$, $\mathcal{K}_8\equiv\gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma^{(1)}\cdot Q$, without introducing any kinematic singularities, as can be seen from the general-mass FF equivalence theorems of Appendix B.

In the equal-mass elastic-scattering limit the equivalence of the t and u annihilation channels (or Fermi statistics in the s channel) can best be exploited by making a Fierz transformation to the five natural s-channel covariants.[28] In the unequal-mass case, the Fierz matrix is enlarged to an 8×8 matrix; its exact form is discussed elsewhere.[31]

Abnormal BF or FF covariants are constructed simply by inserting an over-all factor of γ_5, that is, $\mathcal{K}^-=\gamma_5\mathcal{K}^+$ for BF and either $\gamma_5{}^{(1)}\mathcal{K}^+$ or $\gamma_5{}^{(2)}\mathcal{K}^+$ for FF reactions. The corresponding abnormal equivalence theorems can immediately be obtained from the normal equivalence theorems of Appendix B by the substitutions $m'\leftrightarrow-m'$ $(m_+\leftrightarrow-m_-)$ for BF reactions and either $m'\leftrightarrow-m'$ $(m_+\leftrightarrow-m_-)$ or $\mu'\leftrightarrow-\mu'$ $(\mu_+\leftrightarrow-\mu_-)$, respectively, for FF reactions.

BB reactions of the third kind have two general equivalence theorems (cf. Appendix B). However, when a spin-2 particle is present, the symmetry of its wave function reduces the equivalence theorems to one.

Thus for $0+2\rightarrow0+2$ elastic scattering, the nine KSF covariants are $(PPPP,\ PQPQ,\ PQQQ+QQQP,\ QPPP+PPPQ,\ QQQQ)_{\nu_1'\nu_2'\nu_1\nu_2}$ and $g_{\nu_1'\nu_1}(PP,\{P,Q\},QQ,g)_{\nu_2'\nu_2}$. The covariant $(PPQQ+QQPP)_{\nu_1'\nu_2'\nu_1\nu_2}$ can be related to the others by the equivalence theorem [cf. Eq. (B8)]

$$\mu^2(PPQQ+QQPP)_{\nu_1'\nu_2'\nu_1\nu_2}=-4P^2Q_{\nu_1'}Q_{\nu_2'}Q_{\nu_1}Q_{\nu_2}-2(Q^2-\tfrac{1}{4}t)P_{\nu_1'}Q_{\nu_2'}P_{\nu_1}Q_{\nu_2}+4\nu(PQQQ+QQQP)_{\nu_1'\nu_2'\nu_1\nu_2}+\tfrac{1}{4}t(P^2Q^2-\nu^2)g_{\nu_1'\nu_1}g_{\nu_2'\nu_2}+2[(Q^2-\tfrac{1}{4}t)P^2-\nu^2]\times g_{\nu_1'\nu_1}Q_{\nu_2'}Q_{\nu_2}-\tfrac{1}{2}tQ^2g_{\nu_1'\nu_1}P_{\nu_2'}P_{\nu_2}+\tfrac{1}{2}t\nu g_{\nu_1'\nu_1}\{P,Q\}_{\nu_2'\nu_2},\quad(60)$$

[31] B. Kellett, Nuovo Cimento (to be published).

the result of Rivers and of Papastamatiou and Pakvasa.[32]

For the normal reaction $0+1\rightarrow1+2$, there are 24 possible covariants, of which 23 are independent. The covariant $P_{\mu'}Q_{\nu_1'}K_{\nu_2'}Q_\nu$, for example, can be removed by the general-mass form of Eq. (B8).

For $1+1\rightarrow1+1$ elastic scattering there are two equivalence theorems relating the 27 possible covariants to 25 independent ones. Specializing further to the scattering of identical particles (e.g., $\rho\rho\rightarrow\rho\rho$), these numbers are reduced by Eq. (39) to 19 possible covariants, of which 17 are independent. The problem here is to write the equivalence theorems in such a way that two covariants appear multiplied only by mass coefficients and may therefore be eliminated without introducing kinematic singularities.

With $\mathcal{K}_1\cdots\mathcal{K}_{19}$ as defined in Appendix B, we evaluate Eq. (B8) and then again with μ' and μ interchanged. Next we add and subtract the resulting equations and group terms to obtain the equivalence theorems

$$4m^2(\mathcal{K}_3-\mathcal{K}_4)=4P^2(2\mathcal{K}_5+\mathcal{K}_3-\mathcal{K}_4)-4\nu(4\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4)+\tfrac{1}{4}stu(\mathcal{K}_{19}-\mathcal{K}_{17})+tP^2(\mathcal{K}_{11}-\mathcal{K}_{12})+t\nu(\mathcal{K}_{16}-\mathcal{K}_{14})+(su+t\nu)(\mathcal{K}_{15}-\mathcal{K}_{13}),\quad(61)$$

$$8m^2(\mathcal{K}_1+2\mathcal{K}_2)=4\nu(2\mathcal{K}_5+\mathcal{K}_3-\mathcal{K}_4)+2t(\mathcal{K}_1+5\mathcal{K}_2)+2t\mathcal{K}_7+\tfrac{1}{4}stu(2\mathcal{K}_{18}-\mathcal{K}_{17}-\mathcal{K}_{19})+2(su-tP^2)\mathcal{K}_8-(su+t\nu)(\mathcal{K}_{13}+\mathcal{K}_{15})-tP^2(2\mathcal{K}_9+\mathcal{K}_{11}+\mathcal{K}_{12})+t\nu(2\mathcal{K}_{10}-\mathcal{K}_{14}-\mathcal{K}_{16}).\quad(62)$$

From these forms it is clear that of the covariants $\mathcal{K}_1\cdots\mathcal{K}_5$, we can eliminate $\mathcal{K}_3-\mathcal{K}_4$, and $\mathcal{K}_1+2\mathcal{K}_2$ in favor of $2\mathcal{K}_5+\mathcal{K}_3-\mathcal{K}_4$, $4\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4$, and $\mathcal{K}_1+5\mathcal{K}_2$ without introducing kinematic singularities.

Having singled out all the relatively simple equivalence theorems, we classify all remaining high-spin reactions as having M functions of the fourth kind. The difficulties encountered in the analysis of these reactions are of two types.

First, the previous basic equivalence theorems may occur with permuted indices, as in normal BB reactions, where, for example, in the processes $0+2\rightarrow0+s_B$, $0+1\rightarrow1+s_B$ the number of equivalence theorems is 1, 2, 3, $\cdots$, for $s_B=2, 3, 4, \cdots$, and in the process $1+1\rightarrow1+s_B$ the number is 2, 4, 6, $\cdots$, for $s_B=1, 2, 3, \cdots$.

Second, new equivalence theorems exist for higher-spin BF, FF,[31] and abnormal[33] BB scattering. In the BF case two "3-index" equivalence theorems can be obtained from the forms $\epsilon_{\mu'\nu'}(p'Q)\epsilon_\mu(\gamma Qp)$ and $\epsilon_{\mu'\nu'}(p'Q)\times\epsilon_\mu(p'\gamma p)$. The processes $\frac{1}{2}+1\rightarrow\frac{1}{2}+s_B$ or $\frac{1}{2}+0\rightarrow\frac{3}{2}+s_B$

[32] R. J. Rivers, Phys. Rev. **161**, 1687 (1967); N. J. Papastamatiou and S. Pakvasa, *ibid.* **161**, 1554 (1967). There are, however, some errors in the equivalence theorem as given in these references.

[33] B. Kellett, Nuovo Cimento **53**, 625 (1968).

have 2, 6, 10, $\cdots$, equivalence theorems for $s_2=1$, 2, 3, $\cdots$, and the process $\frac{3}{2}+0\rightarrow\frac{3}{2}+s_B$ has 2, 14, $\cdots$, for $s_B=0$, 1. For example, in $\frac{3}{2}+0\rightarrow\frac{3}{2}+1$ there are 38 possible but only 24 independent covariants. Of the 14 equivalence theorems, two are new, four are the two fundamental relations (51) and (52) with permuted indices, and the remaining eight are the fundamental relations (56) and (57) with permuted indices.

We stress that in these latter cases the complications that arise are of the same fundamental nature as in previous cases. M functions of the third kind include equivalence theorems of the 4-index BB, the 2-index BF, and the 0-index FF varieties, whereas M functions of the fourth kind also include equivalence theorems of the 3- and 4-index BF and 1-, 2-, 3-, and 4-index FF types. Once these relations are tabulated, in a manner analogous to that of Appendix B, no new relations occur for arbitrarily high values of the spins. Permutation symmetry for high spins does not alter the KSF properties of the covariants. Furthermore, we think it reasonable to conclude that the basic equivalence theorems of the fourth kind (finite in number) always contain terms with with constant (mass) factors, as is consistently the case for equivalence theorems of the third kind. In this sense we have demonstrated the covariant KSF development of reactions involving particles of arbitrary spin.

6. KINEMATIC SINGULARITIES

The basis of our claim that the amplitudes in the expansion that we have chosen are KSF is the method of Hearn.[25] Namely, one takes the point of view that the *complete* set of covariants which would arise from perturbation theory have KSF amplitudes and that an *independent* set of amplitudes will be KSF if in reducing to them from the complete set one does not introduce kinematic singularities. These would arise from eliminating covariants which appeared in the equivalence theorems multiplied by kinematic factors such as ν, t, etc. We have demonstrated explicitly that the equivalence theorems for the M functions of the first–third kind can be cast into a form where the covariants to be removed appear with *constant* (mass) coefficients.

Another possible way of proceeding is to relate the kinematic structure of our invariant amplitudes to that of helicity amplitudes and in particular to the fact that according to Ref. 3 the singularities of $T_{\lambda'\lambda}{}^{(t)}$ in s are entirely contained in the factors

$$(\cos\tfrac{1}{2}\theta_t)^{|\lambda'+\lambda|}(\sin\tfrac{1}{2}\theta_t)^{|\lambda'-\lambda|},$$

so that the reduced helicity amplitude

$$\bar{T}_{\lambda'\lambda}{}^{(t)}\equiv(\cos\tfrac{1}{2}\theta_t)^{-|\lambda'+\lambda|}(\sin\tfrac{1}{2}\theta_t)^{-|\lambda'-\lambda|}T_{\lambda'\lambda}{}^{(t)}$$

is KSF in s. Expanding the M function in terms of t-channel natural momenta, we have

$$T_{\lambda'\lambda}{}^{(t)}(s,t)=\sum_\kappa A_\kappa{}^{(t)}(s,t)\langle\lambda'|\mathcal{K}_\kappa(P,Q)|\lambda\rangle \quad (63)$$

in an obvious notation.

The crux of the proof is that because

$$\epsilon^{(\pm)}(q')\cdot Q=\epsilon^{(\pm)}(q)\cdot Q=\epsilon^{(\pm)}(p')\cdot P=\epsilon^{(\pm)}(p)\cdot P=0,$$
$$\bar{v}^{(\pm)}(p')u^{(\pm)}(p)=0$$

in the t-channel c.m. frame, the matrix $\langle\lambda'|\mathcal{K}_\kappa(P,Q)|\lambda\rangle$ is "triangular" [and also contains the correct angular factors $(\cos\frac{1}{2}\theta_t)^{|\lambda'+\lambda|}(\sin\frac{1}{2}\theta_t)^{|\lambda'-\lambda|}$] for the M-function expansions of Secs. 4 and 5. Thus the maximum helicity-flip reduced amplitude $\bar{T}_{(\Delta\lambda)_{\max}}$ will be proportional to just one invariant amplitude,[34] which is therefore KSF. The next reduced helicity amplitude will be proportional to a linear combination of the first invariant amplitude and a second one, which is therefore KSF, and so on. Such a "triangularization" always occurs with our choice of invariant amplitudes, so that the proof that they are KSF in s is fairly straightforward.

To show that the invariant amplitudes are also KSF in t, we switch to the expansion of the M function in terms of the s-channel natural momenta. Again, since $\epsilon^{(\pm)}(p')\cdot\Lambda'=\epsilon^{(\pm)}(q')\cdot\Lambda'=0$, etc., in the s-channel c.m. frame, the relation between the invariant amplitudes $A_\kappa{}^{(s)}$ and the s-channel reduced helicity amplitudes will be triangular, showing that the $A_\kappa{}^{(s)}$ are KSF in t.

Finally, we can relate the $A_\kappa{}^{(s)}$ to the $A_\kappa{}^{(t)}$ by crossing, writing $\Lambda=\frac{1}{2}(P-Q)-\Delta$, etc. For BB scattering the crossing matrix[35] is a purely numerical matrix; hence in this case either set $A_\kappa{}^{(s)}$ or $A_\kappa{}^{(t)}$ will be KSF in both s and t.

For BF scattering the natural s-channel expansion of the M function is of the form $A_+(\boldsymbol{K}+\sqrt{s})+A_-(\boldsymbol{K}-\sqrt{s})$. Again, the relation between $A_\pm$ and the s-channel reduced helicity amplitudes will be triangular, so that $A_\pm$ are KSF in t; similarly, A and B, occurring in the expansion $A+B\boldsymbol{Q}$, are KSF in s. Now, however, the crossing matrix is not purely numerical:

$$B=A_++A_-,$$
$$A=m_+(A_++A_-)+(\sqrt{s})(A_+-A_-), \quad (64)$$

Thus A and B are KSF in both s and t but $A_\pm$ will have a kinematic pole in $\sqrt{s}$.

[34] For example, in πN^* scattering, Ref. 28, $\bar{T}^{(t)}\propto B_{QQ}$.

[35] Strictly speaking, every invariant (scalar) amplitude crosses into itself under a crossing operation. A "crossing matrix" in the covariant formalism simply relates one complete set of covariants (say, the natural t-channel covariants) to another complete set of covariants (say, the natural s-channel covariants), as, for example, the Fierz matrix of FF scattering.

In FF scattering we must make sure that the Fierz crossing matrix has only constant coefficients. This is indeed the case for the GGMW[27] choice of amplitudes, which are therefore KSF in s and t. However, the ALV[36] set $\mathcal{K}_S$, $\mathcal{K}_V$, $\mathcal{K}_P$, $\gamma^{(2)}\cdot P+\gamma^{(1)}\cdot Q$, $\gamma^{(2)}\cdot P\gamma^{(1)}\cdot Q$ will have kinematic singularities in t. This is also clear from the equivalence theorems (53) and (54), where the covariants to be eliminated, $\mathcal{K}_T$ and $\mathcal{K}_A$, appear multiplied by a factor of t.

Finally, we mention briefly the method used by GGMW[27] for NN scattering. Here, with the M function written as $\mathfrak{M}_{\beta\alpha}=\sum_\kappa A_\kappa(s,t)\mathcal{K}_{\beta\alpha}{}^\kappa$, one constructs the objects

$$\mathfrak{N}_\kappa=\mathrm{Tr}\bar{\mathcal{K}}_{\alpha'\beta'}{}^\kappa \mathcal{P}_{\beta'\beta}{}^f\, \mathfrak{M}_{\beta\alpha}\mathcal{P}_{\alpha\alpha'}{}^i$$

$$=\sum_{\kappa'} L_{\kappa\kappa'}A_{\kappa'}(s,t)\,, \qquad (65)$$

where

$$L_{\kappa\kappa'}=\mathrm{Tr}\bar{\mathcal{K}}_{\alpha'\beta'}{}^\kappa \mathcal{P}_{\beta'\beta}{}^f \mathcal{K}_{\beta\alpha}{}^{\kappa'}\mathcal{P}_{\alpha\alpha'}{}^i.$$

By the Hall-Wightman theorem the $\mathfrak{N}_\kappa$ are free of kinematic singularities. Therefore, by inverting (65), so are the A_κ, except possibly where $\det L=0$, which turns out always to be the boundary of the physical region. This last possibility is rather difficult to eliminate. One could hope to do so by demanding that $d\sigma/dt$, given by

$$d\sigma/dt \propto \sum A_\kappa{}^* L_{\kappa\kappa'}A_{\kappa'}\,, \qquad (66)$$

and all its polarization moments be finite, but for higher spins the procedure becomes quite impracticable.

Finally, we note that the invariant amplitudes are also free of kinematic zeros and, in constrast to helicity amplitudes, are independent at thresholds and pseudothresholds. Insofar as we have here displayed the equivalence theorems and have previously[10,11] indicated the method of covariant partial-wave expansions, we believe that invariant amplitudes, with their simple analytic and crossing properties, will continue to play an important role in elementary-particle physics.

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APPENDIX A: COVARIANT IDENTITIES

All possible relations derivable among covariant vectors and spinors follow from the three fundamental identities:

$$\epsilon_{\alpha'\beta'\gamma'\delta'}\epsilon_{\alpha\beta\gamma\delta}=-\begin{vmatrix} g_{\alpha'\alpha} & g_{\alpha'\beta} & g_{\alpha'\gamma} & g_{\alpha'\delta} \\ g_{\beta'\alpha} & g_{\beta'\beta} & g_{\beta'\gamma} & g_{\beta'\delta} \\ g_{\gamma'\alpha} & g_{\gamma'\beta} & g_{\gamma'\gamma} & g_{\gamma'\delta} \\ g_{\delta'\alpha} & g_{\delta'\beta} & g_{\delta'\gamma} & g_{\delta'\delta} \end{vmatrix}, \qquad (\mathrm{A1})$$

$$\epsilon_{\alpha\beta\gamma\delta}g_{\sigma\tau}=\epsilon_{\sigma\beta\gamma\delta}g_{\alpha\tau}+\epsilon_{\alpha\sigma\gamma\delta}g_{\beta\tau} +\epsilon_{\alpha\beta\sigma\delta}g_{\gamma\tau}+\epsilon_{\alpha\beta\gamma\sigma}g_{\delta\tau}\,, \qquad (\mathrm{A2})$$

$$\gamma_5\epsilon_{\alpha\beta\gamma\delta}=\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta-g_{\alpha\beta}\gamma_\gamma\gamma_\delta-g_{\gamma\delta}\gamma_\alpha\gamma_\beta-g_{\alpha\delta}\gamma_\beta\gamma_\gamma -g_{\beta\gamma}\gamma_\alpha\gamma_\delta+g_{\alpha\gamma}\gamma_\beta\gamma_\delta+g_{\beta\delta}\gamma_\alpha\gamma_\gamma +g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}. \qquad (\mathrm{A3})$$

A special case of the latter is also useful:

$$\gamma_5\epsilon_{\alpha\beta\gamma\delta}\gamma^\delta=\gamma_\alpha\gamma_\beta\gamma_\gamma-g_{\alpha\beta}\gamma_\gamma+g_{\alpha\gamma}\gamma_\beta-g_{\beta\gamma}\gamma_\alpha. \qquad (\mathrm{A4})$$

Our metric is $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. The γ matrices are defined by $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$, $\gamma_0\gamma_\mu\gamma_0=\gamma_\mu{}^\dagger$. γ_5 is defined as $\gamma_0\gamma_1\gamma_2\gamma_3$, so that $\gamma_5{}^2=-1$.

APPENDIX B

In this Appendix we derive the general BF, FF, and BB^+ equivalence theorems of the third kind.

(i) BF equivalence theorems: Consider the double epsilon form $\epsilon_\beta(p'Q\gamma)\epsilon_\alpha(p'\gamma p)$ between spinors $\bar{u}(p')$ and $u(p)$. First expand it by determinants, Eq. (A1), and then by spinors, Eq. (A4). Equating the results gives

$$0=[p',p]_{\beta\alpha}\mathbf{Q}+\tfrac{1}{2}[(mp'+m'p),[\gamma,\mathbf{Q}]]_{\beta\alpha} +\tfrac{1}{2}(p'\cdot p+m'm)[\gamma\mathbf{Q}\gamma]_{\beta\alpha}+\tfrac{1}{2}(mp'+m'p)\cdot Q[\gamma_\beta,\gamma_\alpha] +p'\cdot Q[p,\gamma]_{\beta\alpha}+p\cdot Q[\gamma,p']_{\beta\alpha}\,, \qquad (\mathrm{B1})$$

where

$$[\gamma\mathbf{Q}\gamma]_{\beta\alpha}\equiv\gamma_\beta\mathbf{Q}\gamma_\alpha-\gamma_\alpha\mathbf{Q}\gamma_\beta=2\gamma_5\epsilon_{\mu\nu}(\gamma\mathbf{Q})\,,$$
$$[p,\gamma]_{\beta\alpha}\equiv p_\beta\gamma_\alpha-\gamma_\beta p_\alpha\,,$$
$$[p',p]_{\beta\alpha}\equiv p_\beta{}'p_\alpha-p_\beta p_\alpha{}'\,,$$

and

$$[A[\gamma,\mathbf{Q}]]_{\beta\alpha}=[[A,\gamma]_{\beta\alpha},\mathbf{Q}].$$

In a similar manner, the form $\epsilon_\beta(p'Q\gamma)\epsilon_\alpha(Q\gamma p)$ yields

$$0=[(mp'-m'p),Q]_{\beta\alpha}\mathbf{Q}+p'\cdot Q[p,\gamma]_{\beta\alpha}\mathbf{Q} +p\cdot Q\mathbf{Q}[\gamma,p']_{\beta\alpha}+\tfrac{1}{2}(mp'+m'p)\cdot Q[\gamma\mathbf{Q}\gamma]_{\beta\alpha} +\tfrac{1}{2}[2p'\cdot Qp\cdot Q-Q^2(p'\cdot p-m'm)][\gamma_\beta,\gamma_\alpha] -\tfrac{1}{2}(p'\cdot p-m'm)[Q,[\gamma,\mathbf{Q}]]_{\beta\alpha}+(mp'-m'p)\cdot Q[Q,\gamma]_{\beta\alpha} -Q^2[(mp'-m'p),\gamma]_{\beta\alpha}+Q^2[p',p]_{\beta\alpha}. \qquad (\mathrm{B2})$$

These equations are valid for any masses and spins, i.e., subsidiary conditions on the β and α labels. We could convert p' and p to momenta P and Δ using $p'=P+\tfrac{1}{2}\Delta$, $p=P-\tfrac{1}{2}\Delta$, obtaining relations such as $\tfrac{1}{2}(p'\cdot p+m'm)=P^2-m_-{}^2$ $\tfrac{1}{2}(p'\cdot p-m'm)=m_-{}^2-\tfrac{1}{4}t$, etc. It is clear from these equations that either of the first two terms of Eq. (B1) but only the first term of Eq. (B2) can be eliminated without introducing kinematic singularities.

[36] We remark that ALV only claimed this set to be the perturbative set, and not necessarily KSF.

(ii) *FF* equivalence theorems: In a similar manner to the *BF* case, we consider the forms $\epsilon_{\alpha\beta}(Q\gamma^{(2)})\epsilon_{\alpha\beta}(P\gamma^{(1)})$, $\epsilon_{\alpha\beta\gamma\delta}\Delta_\gamma\sigma_{\rho\delta}{}^{(2)}\epsilon_{\alpha\beta\zeta\kappa}\Delta_\zeta\sigma_{\rho\kappa}{}^{(1)}$, etc., between the natural t-channel spinors. The latter gives

$$\mu_+\gamma^{(2)}\cdot P+m_+\gamma^{(1)}\cdot Q=\nu(\mathcal{K}_S-\mathcal{K}_P)+\mu_+m_+\mathcal{K}_V-\tfrac{1}{4}t\mathcal{K}_T -\mu_-m_-\mathcal{K}_A-\mu_-\mathcal{K}_7-m_-\mathcal{K}_8 \quad \text{(B3)}$$

and the former yields

$$\gamma^{(2)}\cdot P\gamma^{(1)}\cdot Q=-m_+\mu_+\mathcal{K}_P+\nu\mathcal{K}_V+(m_-^2+\mu_-^2-\tfrac{1}{4}t)\mathcal{K}_A +m_-\mu_-\mathcal{K}_T+m_-\mathcal{K}_7+\mu_-\mathcal{K}_8, \quad \text{(B4)}$$

where $\mathcal{K}_7\equiv\gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma\cdot{}^{(2)}P$ and $\mathcal{K}_8\equiv\gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma\cdot{}^{(1)}Q$. We can obtain the third equivalence theorem by a chiral transformation on Eq. (B4) with $m_+\leftrightarrow -m_-$, $\mu_+\leftrightarrow -\mu_-$ [or consider forms such as $\epsilon_{\alpha\beta}(\Delta\gamma^{(2)})\epsilon_{\alpha\beta}\times(\Delta\gamma^{(1)})$],

$$\gamma_5{}^{(2)}\gamma\cdot{}^{(2)}P\gamma_5{}^{(1)}\gamma\cdot{}^{(1)}Q=\nu\mathcal{K}_A-m_+\mu_+\mathcal{K}_T +(m_+{}^2+\mu_+{}^2-\tfrac{1}{4}t)\mathcal{K}_V-\mu_-m_-\mathcal{K}_S -(m_+\gamma^{(2)}\cdot P+\mu_+\gamma^{(1)}\cdot Q). \quad \text{(B5)}$$

A similar analysis also yields the "spinor identities"

$$\begin{aligned}\gamma^{(2)}\cdot\gamma^{(1)}\gamma^{(1)}\cdot Q&=\mu_+\mathcal{K}_S+\mu_-\mathcal{K}_T+m_-\mathcal{K}_A+\mathcal{K}_7,\\ \gamma^{(2)}\cdot\gamma^{(1)}\gamma^{(2)}\cdot P&=m_+\mathcal{K}_S+m_-\mathcal{K}_T+\mu_-\mathcal{K}_A+\mathcal{K}_8,\end{aligned} \quad \text{(B6)}$$

along with their "chiral equivalents"

$$\begin{aligned}\gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma^{(2)}\cdot\gamma^{(1)}\gamma^{(1)}\cdot Q &=-m_+\mathcal{K}_V+\mu_+\mathcal{K}_T-\mu_-\mathcal{K}_P+\gamma^{(2)}\cdot P,\\ \gamma_5{}^{(2)}\gamma_5{}^{(1)}\gamma^{(2)}\cdot\gamma^{(1)}\gamma^{(2)}\cdot P &=-\mu_+\mathcal{K}_V+m_+\mathcal{K}_T-m_-\mathcal{K}_V+\gamma^{(1)}\cdot Q,\end{aligned} \quad \text{(B7)}$$

which we differentiate from equivalence theorems in that the covariants appear only with numerical (mass) coefficients.

We note that from the general-mass *BF* or *FF* equivalence theorems it is always best to eliminate covariants with mass sums (m_+,μ_+) rather than mass differences (m_-,μ_-), as the latter vanish in the equal-mass limit. In Eq. (B3) it is also better to write $\mu_+\gamma^{(2)}\cdot P+m_+\gamma^{(1)}\cdot Q$ as $\frac{1}{2}(m_++\mu_+)(\gamma^{(2)}\cdot P+\gamma^{(1)}\cdot Q)-\frac{1}{2}(m_+-\mu_+)\times(\gamma^{(2)}\cdot P-\gamma^{(1)}\cdot Q)$ and then eliminate the sum as in the equal-mass case, Eq. (53).

(iii) BB^+ equivalence theorems: Such equivalence theorems first arise when the reaction in question has at least four covariant labels. Hence we consider the form $\epsilon_{\mu'\alpha}(PQ)\epsilon_{\nu'\alpha\beta}(\Delta)\epsilon_{\mu\gamma}(PQ)\epsilon_{\nu\gamma\beta}(\Delta)$, where μ', ν', μ, and ν do not yet refer to specific particles with spin. We then equate two of the possible determinantal expansions of this form. Assuming $P\cdot\Delta=Q\cdot\Delta=0$, this leads to

$$\begin{aligned}0=&\,t(PPQQ+QQPP-PQPQ-QPQP)_{\mu'\nu'\mu\nu}\\ &+Q^2(PP\Delta\Delta+\Delta\Delta PP-P\Delta P\Delta-\Delta P\Delta P)_{\mu'\nu'\mu\nu}\\ &+P^2(QQ\Delta\Delta+\Delta\Delta QQ-Q\Delta Q\Delta-\Delta Q\Delta Q)_{\mu'\nu'\mu\nu}\\ &+\nu(P\Delta Q\Delta+Q\Delta P\Delta+\Delta P\Delta Q+\Delta Q\Delta P-\Delta\Delta PQ\\ &-\Delta\Delta QP-PQ\Delta\Delta-QP\Delta\Delta)_{\mu'\nu'\mu\nu}+t(P^2Q^2-\nu^2)\\ &\times(g_{\mu'\nu'}g_{\mu\nu}-g_{\mu'\mu}g_{\nu'\nu})+(P^2Q^2-\nu^2)\\ &\times(g_{\mu'\mu}\Delta_{\nu'}\Delta_\nu+g_{\nu'\nu}\Delta_{\mu'\mu}-g_{\mu'\nu'}\Delta_\mu\Delta_\nu-g_{\mu\nu}\Delta_{\mu'}\Delta_{\nu'})\\ &+tQ^2(g_{\mu'\mu}P_{\nu'}P_\nu+g_{\nu'\nu}P_{\mu'}P_\mu-g_{\mu'\nu'}P_\mu P_\nu-g_{\mu\nu}P_{\mu'}P_{\nu'})\\ &+tP^2(g_{\mu'\mu}Q_{\nu'}Q_\nu+g_{\nu'\nu}Q_{\mu'}Q_\mu-g_{\mu'\nu'}Q_\mu Q_\nu-g_{\mu\nu}Q_{\mu'}Q_{\nu'})\\ &-t\nu(g_{\mu'\mu}\{P,Q\}_{\nu'\nu}+g_{\nu'\nu}\{P,Q\}_{\mu'\mu}-g_{\mu'\nu'}\{P,Q\}_{\mu\nu}\\ &-g_{\mu\nu}\{P,Q\}_{\mu'\nu'}).\end{aligned} \quad \text{(B8)}$$

Since our choice of the 4-epsilon form in no way associates any particular label with any one momentum, the above identity is also valid with $\mu'\leftrightarrow\mu$. All other permutations of the labels are equivalent to these two independent forms.

Various equivalence theorems can be obtained from Eq. (B8) and its permutation by converting Δ to either P or Q, according to the subsidiary conditions.

For the case of $1+1\rightarrow 1+1$ elastic scattering, we define the 19 possible identical particle $(m=\mu)$ covariants as

$$\begin{aligned}&\mathcal{K}_1=(PPPP+QQQQ)_{\mu'\nu'\mu\nu}, \quad \mathcal{K}_2=P_{\mu'}Q_{\nu'}P_\mu Q_\nu,\\ &\mathcal{K}_3=(PPQQ+QQPP)_{\mu'\nu'\mu\nu},\\ &\mathcal{K}_4=(PQQP+QPPQ)_{\mu'\nu'\mu\nu},\\ &\mathcal{K}_5=(PPPQ+PQPP+PQQQ+QQPQ)_{\mu'\nu'\mu\nu},\\ &\mathcal{K}_6=(QPPP+PPQP+QQQP+QPQQ)_{\mu'\nu'\mu\nu},\\ &\mathcal{K}_7=Q_{\mu'}P_{\nu'}Q_\mu P_\nu, \quad \mathcal{K}_8=g_{\mu'\mu}Q_{\nu'}Q_\nu+g_{\nu'\nu}P_{\mu'}P_\mu,\\ &\mathcal{K}_9=g_{\mu'\mu}P_{\nu'}P_\nu+g_{\nu'\nu}Q_{\mu'}Q_\mu,\\ &\mathcal{K}_{10}=g_{\mu'\mu}\{P,Q\}_{\nu'\nu}+g_{\nu'\nu}\{P,Q\}_{\mu'\mu},\\ &\mathcal{K}_{11}=g_{\mu'\nu'}(PP+QQ)_{\mu\nu}+g_{\mu\nu}(PP+QQ)_{\mu'\nu'},\\ &\mathcal{K}_{12}=g_{\mu\nu'}(PP+QQ)_{\mu'\nu}+g_{\mu'\nu}(PP+QQ)_{\mu\nu'},\\ &\mathcal{K}_{13}=g_{\mu'\nu'}P_\mu Q_\nu+g_{\mu\nu}P_{\mu'}Q_{\nu'},\\ &\mathcal{K}_{14}=g_{\mu'\nu'}Q_\mu P_\nu+g_{\mu\nu}Q_{\mu'}P_{\nu'},\\ &\mathcal{K}_{15}=g_{\mu\nu'}P_{\mu'}Q_\nu+g_{\mu'\nu}P_\mu Q_{\nu'},\\ &\mathcal{K}_{16}=g_{\mu\nu'}Q_{\mu'}P_\nu+g_{\mu'\nu}Q_\mu P_{\nu'},\\ &\mathcal{K}_{17}=g_{\mu'\nu'}g_{\mu\nu}, \quad \mathcal{K}_{18}=g_{\mu'\mu}g_{\nu'\nu}, \quad \mathcal{K}_{19}=g_{\mu\nu'}g_{\mu'\nu}.\end{aligned}$$

Since the *BF* and *BB* identities are valid for bosons of any mass, they will be applicable in particular to reactions involving photons.