Harmonic-Oscillator Model for Baryons*

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Attention is drawn to a possible band structure displayed by the known N^* resonances. The observed spectrum is similar to that predicted by a paraquark harmonic-oscillator shell model. Exact three-body Schrödinger wave functions are used to compute the $N\pi$ and $\Delta\pi$ partial decay widths, assuming the pion emission takes place via a one-quark de-excitation. The agreement with experiment is very good.

I. INTRODUCTION

INTENSIVE studies of pion-nucleon scattering in recent years by means of phase-shift analyses[1,2] have led to the discovery of a very large number of N^* resonances. The spectrum of the known resonances is displayed in Fig. 1, where the resonances have been labelled by the corresponding pion-nucleon partial wave amplitudes. One would hope that these particles are not all independent of one another; the observed spectrum, therefore, presents a challenge to physicists to find the underlying mechanism or structure whereby such a pattern is produced.

Several schemes have been suggested in the past to account for such a large number of particles, as well as to try and relate some of their properties, such as their partial decay widths. These fall into two main categories: one in which use is made of higher-symmetry groups[3] with the resonances allocated to various representations of the groups, and the other in which the resonances are generated through different kinds of dynamical models.[4] Neither of these approaches has been particularly successful. More recently, Barger and Cline[5] have observed that the resonances can possibly be assigned to several Regge families; however, Regge theory unfortunately does not at present provide a means of relating for example the widths of the different resonances, as one would obviously like to do, especially for members of the same Regge family.

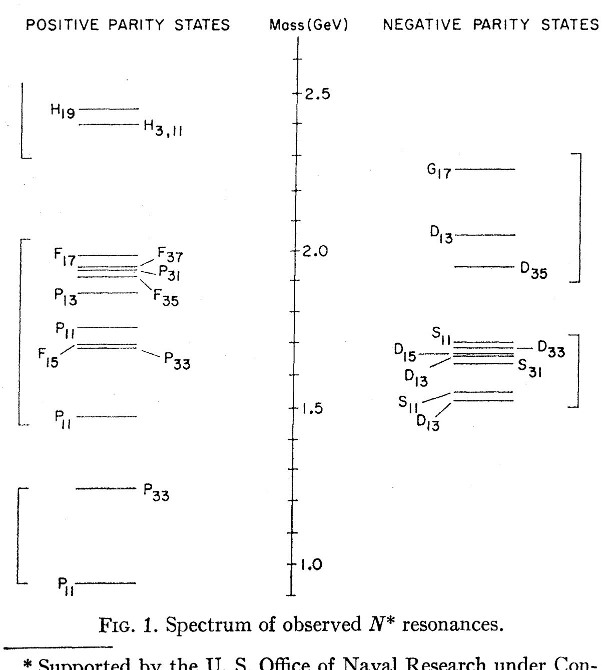

FIG. 1. Spectrum of observed N^* resonances.

Another proposal, propounded in particular by Dalitz,[6] has been to interpret these resonances as excitations of a basic three-quark system. Such an excitation scheme is strongly suggested by the spectrum in Fig. 1, where it is tempting to group the resonances into reasonably well-defined bands.

The first band contains two positive-parity states, the nucleon and the $\Delta(1236)$ resonance. These are members of a well-known $J^P=\frac{1}{2}^+$ octet and $J^P=\frac{3}{2}^+$ decuplet of baryons, respectively, which together form a symmetric 56-dimensional representation of $SU(6)$. In the quark model, they are taken to be in the ground-state level with total quark orbital angular momentum $L=0^+$. The $\Delta(1236)$ can, of course, decay into the nucleon by the emission of a pion or a photon. Assuming this emission takes place via a one-quark interaction, Becchi and Morpurgo[7] have calculated the partial widths for these decays (the parameters involved being previously determined by the pion-nucleon coupling constant and the proton magnetic moment) and found them to be in

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[5] V. Barger and D. Cline, Phys. Rev. Letters **20**, 298 (1968).

[6] R. H. Dalitz, in *Les Houches Lectures, 1965* (Gordon and Breach Science Publishers, Inc., New York, 1965).

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reasonable agreement with the experimental values. With this interaction, the radiative decay is predicted to be entirely a magnetic-dipole transition, the electric-quadrupole contribution being zero; this has also been verified[8] with experiment.

The second band of resonances is a set of odd-parity baryons. Dalitz[6] suggested in 1965 that this set corresponded to a mixed $SU(6)$ symmetry representation with dimensionality 70, the quarks having total orbital angular momentum $L=1^-$. A **70** representation contains an octet with quark spin $\frac{3}{2}$, and a decuplet, octet and singlet with quark spin $\frac{1}{2}$. On coupling the quark spin with the total orbital angular momentum, one requires N^*'s which would correspond to resonances in the pion-nucleon partial waves D_{15}, D_{13}, and S_{11} originating from the $\mathbf{8}^{3/2}$, and D_{33}, S_{31}, D_{13}, and S_{11} originating from the $\mathbf{10}^{1/2}$, $\mathbf{8}^{1/2}$. This is precisely what is observed. (In general, the two S_{11}'s and two D_{13}'s will be mixtures of $\mathbf{8}^{3/2}$, $\mathbf{8}^{1/2}$ states.) Many of the accompanying $SU(3)$ members are also well known. Again assuming a one-quark interaction, Mitra and Ross[9] calculated the partial widths for these particles in the **70**, $L=1^-$ decaying via meson emission, and on the whole their results are in reasonable agreement with the measured values.

Within the last year or so, many new resonances have been discovered,[1,2] and as seen from Fig. 1 these lend further credence to the idea of excitation bands. This poses the question as to which representations these further resonances belong and to what values of the quark orbital angular momentum. Also, one may wonder why the **70**, $L=1^-$ seems to be the first excited level rather than some other just as likely candidate. One therefore requires a scheme whereby the symmetric **56**, $L=0^+$ and mixed symmetry **70**, $L=1^-$ states occur in a natural way as the ground and first excited states, respectively. Such a scheme was in fact suggested some time ago by Greenberg,[10] and reemphasized recently by Dalitz,[11] and is based on the ideas of the shell model of nuclear physics. However, it makes the important assumption that the spin-$\frac{1}{2}$ quarks obey para-Fermi statistics of order 3 rather than Fermi statistics. With the latter kind of statistics, one is faced with several problems. For example, the spatial wave function associated with the nucleon and the Δ that belong to the symmetric **56** $SU(6)$ representation, has to be antisymmetric in the quark coordinates. From our experience in nuclear physics, it would be rather surprising to find an antisymmetric state lying lowest in energy, and strong exchange forces would be required to bring this about. Also, Majumdar and Mitra[12] have shown that typical antisymmetric wave functions lead to nodes in the nucleon form factors, contrary to observation. With the assumption of para-Fermi statistics, none of these problems arises and a spatially symmetric ground state is allowed.

In the present paper, we, first of all, examine the spectrum for the lower-mass baryons which one would expect from such a shell model (Greenberg[10] considered only quarks in their lowest radial state; here we also allow radial excitations). A well-known problem[13] in the shell model for an n-particle system is the separation of the center-of-mass motion; in the present case with three quarks interacting via simple harmonic forces, we eliminate the center-of-mass motion exactly and determine the correct wave functions which describe the various states. Secondly, assuming a one-quark interaction, we calculate the $N\pi$ and $\Delta\pi$ widths of the observed resonances decaying to the ground state. In these calculations, there are two basic parameters, namely the quark-pion coupling constant and the well parameter. Fortunately, in the spectrum, there are several resonances (some have quite different quark configurations) whose widths depend on only these two parameters, and these provide an effective test for this model. Most of the resonances that occur can, however, in principle, be mixtures of states, and this introduces further mixing angle parameters. In the case of the **70**, $L=1^-$ we estimate the mixing angles for the two S_{11}'s and the two D_{13}'s. Our formulas are derived with the masses of the resonances in each band taken to be the same. However, in evaluating the widths from these formulas, the physical masses have been used. Our results are in good agreement with the experimental data.

II. SHELL MODEL

The shell model we consider is the one based on harmonic-oscillator forces. The situation with a particle moving in a central three-dimensional harmonic-oscillator potential is a familiar one. The energy levels for the Hamiltonian $H=(1/2M)\mathbf{p}^2+\frac{1}{2}M\omega^2\mathbf{r}^2$ occur for $E=(l+2k+\frac{3}{2})\omega$, where l is the orbital angular momentum quantum number, and $k=0, 1, 2, \cdots$ is associated with the number of nodes in the radial wave function. The resulting spectrum is shown in Fig. 2. The corresponding spatial wave functions can be written[14] as

$$\Psi_{nlm}=N(\alpha r)^l L_k^{l+1/2}(\alpha^2r^2)\exp(-\tfrac{1}{2}\alpha^2r^2)Y_l^m(\theta,\varphi),$$

where $n=l+2k$, $\alpha^2=M\omega$, and L is a Laguerre polynomial. The normalization constant N is given by

$$|N|^2=\frac{2\alpha^3k!}{(\sqrt{\pi})(k+l+\frac{1}{2})(k+l-\frac{1}{2})\cdots\frac{3}{2}\times\frac{1}{2}}.$$

A few of these functions are listed in Appendix A.

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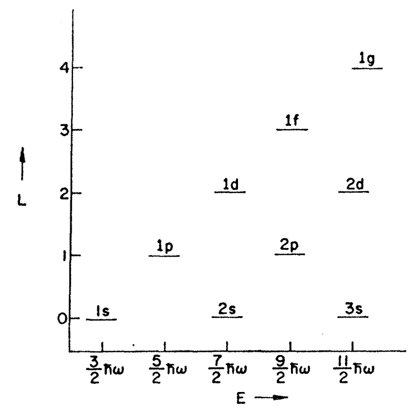

FIG. 2. Energy levels of a three-dimensional harmonic oscillator.

The Hamiltonian for three particles at positions $\mathbf{r}_j$ ($j=1, 2, 3$) interacting via harmonic-oscillator forces is

$$H=\sum_j \frac{\mathbf{p}_j^2}{2M}+\tfrac{1}{2}M\omega^2 \sum_{i<j} (\mathbf{r}_i-\mathbf{r}_j)^2,$$

whereas the shell-model Hamiltonian for the three particles is

$$H_{\text{sm}}=\sum_j \frac{\mathbf{p}_j^2}{2M}+\tfrac{1}{2}M\omega^2 \sum_j \mathbf{r}_j^2.$$

Defining the position vector $\mathbf{R}$ of the center of mass and two relative coordinates $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$ by

$$\mathbf{R}=\tfrac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3),$$

$$\boldsymbol{\lambda}=\frac{1}{\sqrt{6}}(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3),$$

$$\boldsymbol{\rho}=\frac{1}{\sqrt{2}}(\mathbf{r}_1-\mathbf{r}_2),$$

we can rewrite the above Hamiltonians as

$$H=\frac{\mathbf{P}^2}{2(3M)}+\left[\frac{1}{2M}(\mathbf{p}_\lambda^2+\mathbf{p}_\rho^2)+\tfrac{1}{2}M(\sqrt{3}\omega)^2(\boldsymbol{\lambda}^2+\boldsymbol{\rho}^2)\right],$$

$$H_{\text{sm}}=\frac{\mathbf{P}^2}{2(3M)}+\tfrac{1}{2}(3M)\omega^2\mathbf{R}^2+\left[\frac{1}{2M}(\mathbf{p}_\lambda^2+\mathbf{p}_\rho^2)+\tfrac{1}{2}M\omega^2(\boldsymbol{\lambda}^2+\boldsymbol{\rho}^2)\right],$$

where $\mathbf{P}$, $\mathbf{p}_\lambda$, $\mathbf{p}_\rho$ are the momenta canonically conjugate to $\mathbf{R}$, $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, respectively.

The eigenfunctions Ψ, Ψ_{sm} corresponding to these two Hamiltonians may be written in the form

$$\Psi=\exp(i\mathbf{P}\cdot\mathbf{R})\varphi(\boldsymbol{\lambda})\psi(\boldsymbol{\rho}),$$

$$\Psi_{\text{sm}}=\chi(\mathbf{R})\varphi(\boldsymbol{\lambda})\psi(\boldsymbol{\rho}),$$

where χ, φ, ψ are one-body harmonic-oscillator wave functions. Thus, as far as enumerating the states is concerned, one can equivalently use the eigenfunctions Ψ or Ψ_{sm}, provided that in the latter $\chi(\mathbf{R})$ is restricted to be in one specific state. In the following, we make use of the eigenfunctions Ψ_{sm} rather than Ψ, since it is in fact slightly easier to write down eigenfunctions of H_{sm} with the correct symmetry in the three quarks. We enumerate the states with the center-of-mass motion $\chi(\mathbf{R})$ held in the ground state ($1s$) (the extraneous set of states that correspond to higher oscillations of the center of mass are referred to as being "spurious"[13,15]).

The shell-model wave functions appropriate for the first few levels are set out in Appendix B. To determine the complete wave function, these spatial parts have to be combined with $SU(6)$ wave functions which involve the quark spin and isospin. Since we require the over-all wave function to be symmetric, a spatial wave function of symmetric, mixed symmetric, or antisymmetric type must be combined with the **56**, **70**, or **20** representation, respectively, of $SU(6)$. The rules for taking such combinations so that the resultant is symmetric are well known, and are listed for example in Bolsterli *et al.*[16] and in Blatt *et al.*[17]

The lowest level is the $(1s)^3$, out of which one can form only a symmetric spatial wave function. The ground state therefore is a **56**, $L=0^+$, as required by experiment.

At the next level, the $(1s)^2(1p)$, it is possible to form both symmetric and mixed symmetric spatial wave functions. However, we see that the symmetric one is proportional to the position vector $\mathbf{R}$ of the center of mass. It is therefore a spurious state[10] and must be discarded. [It corresponds to the internal quark motion being in its ground state but the center of mass moving in a ($1p$) state.] On the other hand, the mixed symmetric spatial wave function is entirely nonspurious since its basis functions involve only the relative coordinate vectors $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$. Thus, at this level one obtains only a **70**, $L=1^-$, again as called for by the experimental data.

The third level is more complicated, and one has to consider simultaneously $(1s)^2(2s)$, $(1s)^2(1d)$, and $(1s)(1p)^2$ since they all correspond to the same energy. In general, any shell-model wave function has a spurious component, indicated by its dependence on $\mathbf{R}$. However, since we have derived explicit expressions for the shell-model wave functions (Appendix B), the center-of-mass motion can be easily eliminated by taking the appropriate combinations of these wave functions to

[15] However, when doing calculations such as for decay widths, one must remember to use the eigenstates of H, replacing the center-of-mass oscillator by a plane wave; integration over $\mathbf{R}$ in the matrix element then leads to the statement of conservation of momentum in the process.

[16] M. Bolsterli and E. Jezak, Phys. Rev. **135**, B510 (1964).

[17] J. M. Blatt and G. Derrick, Nucl. Phys. **8**, 310 (1958).

TABLE I. Predicted spectrum.

$(1s)^3$	**56**, $L=0^+$	$8^{1/2}(P_{11})$	$10^{3/2}(P_{33})$
$(1s)^2(1p)$	**70**, $L=1^-$	$8^{3/2}(D_{15},D_{13},S_{11})$	
		$8^{1/2}(D_{13},S_{11})$	$10^{1/2}(D_{33},S_{31})$
	56, $L=1^-$ (spurious)		
$(1s)^2(2s)$, $(1s)^2(1d)$, $(1s)(1p)^2$	**56**, $L=0^+$	$8^{1/2}(P_{11})$	$10^{3/2}(P_{33})$
	70, $L=0^+$	$8^{3/2}(P_{13})$	
		$8^{1/2}(P_{11})$	$10^{1/2}(P_{31})$
	56, $L=2^+$	$8^{1/2}(F_{15},P_{13})$	$10^{3/2}(F_{37},F_{35},P_{33},P_{31})$
	70, $L=2^+$	$8^{3/2}(F_{17},F_{15},P_{13},P_{11})$	
		$8^{1/2}(F_{15},P_{13})$	$10^{1/2}(F_{35},P_{33})$
	20, $L=1^+$	$8^{1/2}(P_{13},P_{11})$	
	56, $L=0^+$ and 2^+; **70**,$L=0^+$, 1^+ and 2^+ (spurious)		

remove this dependence. The nonspurious states[18] that occur are found to be

$$\Psi(\mathbf{56},\, L=0^+)=(\sqrt{\tfrac{2}{3}})(1s)^2(2s)+(\sqrt{\tfrac{1}{3}})(1s)(1p)^2\,,$$

$$\Psi(\mathbf{70},\, L=0^+)=(\sqrt{\tfrac{1}{3}})(1s)^2(2s)+(\sqrt{\tfrac{2}{3}})(1s)(1p)^2\,,$$

$$\Psi\mathbf{56},\, L=2^+)=(\sqrt{\tfrac{2}{3}})(1s)^2(1d)-(\sqrt{\tfrac{1}{3}})(1s)(1p)^2\,,$$

$$\Psi(\mathbf{70},\, L=2^+)=(\sqrt{\tfrac{1}{3}})(1s)^2(1d)-(\sqrt{\tfrac{2}{3}})(1s)(1p)^2\,,$$

$$\Psi(\mathbf{20},\, L=1^+)=(1s)(1p)^2.$$

There is also a set of spurious states: **56** with $L=0^+$ and 2^+, **70** with $L=0^+$ and 2^+ (these are the orthogonal combinations to the above nonspurious states), and **70** with $L=1^+$. The latter possibility is manifestly spurious since its basis functions are proportional to $\mathbf{R}\times\boldsymbol{\lambda}$ and $\mathbf{R}\times\boldsymbol{\rho}$, in contrast to the **20**, $L=1^+$, which depends on $\boldsymbol{\lambda}\times\boldsymbol{\rho}$.

It is interesting to interpret the spurious states. The spurious **56**, $L=0^+$ corresponds to the internal motion in the ground state, with the center of mass in a $(2s)$ excitation. The spurious **56**, $L=2^+$ also has a ground-state internal motion but with the center of mass in a $(1d)$ state. The spurious **70**, $L=0^+$, 1^+ and 2^+ all originate from an internal $(1s)^2(1p)$ motion with the center of mass in a $(1p)$ state.

This scheme can in principle be extended to higher levels, but this seems somewhat academic at the moment, owing, firstly to the present lack of experimental data in these regions, and, secondly, to the large amount of configuration mixing that is *a priori* to be expected as the number of states increases.

III. PREDICTED SPECTRUM

The remarks made above are summarized by Table I. The various N^*'s predicted[19] after coupling the quark spin to the orbital angular momentum L would correspond to resonances in the pion-nucleon partial waves that are indicated.

As mentioned previously, this scheme has the merit that the **56**, $L=0^+$ and the **70**, $L=1^-$ arise in a natural way as the ground and first excited states. Experimentally, these are the configurations that do seem to occur and all the required N^*'s have been observed. The resonances in each level are evidently not degenerate. This means that there are additional forces present which cause perturbations from the harmonic-oscillator potential that is made use of here. We do not take these extra forces into account in the present investigations, but hope to return to this interesting topic at a later date.

As a third level, we expect to find a large number of positive-parity baryons. This again seems to be the case, though they occur in a fairly broad band. Their masses range from 1680 to 1985 MeV, apart from the Roper resonance $P_{11}(1450)$ that lies somewhat below the main group. Nevertheless, we include the Roper resonance in this band and attribute its low mass to the additional forces present. These would seem to be particularly attractive for the wave function that describes the physical $P_{11}(1450)$. A glance at Table I shows that more N^*'s are predicted by this model than have been observed to date. However, considering that several of these resonances were discovered[2] only within the past year, it would not be too surprising if a few more are still lurking undetected in this energy region. It is significant of course that no resonances have been discovered in this region whose quantum numbers cannot be accommodated by our scheme; for example the lowest mass $J^P=\frac{9}{2}^+$, $\frac{11}{2}^+$ particles known lie well above the band we are discussing and in fact appear to belong to a band of their own.

It would appear, therefore, that this harmonic-oscillator model predicts a spectrum of resonances similar to the one observed for baryons below 2 GeV. For higher masses, the small amount of information available indicates that a band structure persists, though the number of N^*'s for each succeeding band is expected to multiply rapidly.

IV. PARTIAL DECAY WIDTHS

It is not sufficient merely to list the spectrum expected from any model, but it is necessary to justify the model

[18] As we see, it is necessary to incorporate higher radial states to treat the center-of-mass problem correctly; these were not included in Greenberg's original scheme (Ref. 10). Also his **70**, $L=1^+$ is an entirely spurious state.

[19] After this work was completed, we received a paper by R. H. Dalitz (to be published), in which the predicted spectrum of this model is also discussed.

further by relating some of the properties of the various particles. Our second objective, therefore, is to pursue this aspect and to compare the rates of decay of the resonances into the ground state by pion emission, that is we calculate $\Gamma(N^*\to N\pi)$ and $\Gamma(N^*\to \Delta\pi)$.

Following Becchi and Morpurgo,[7] we assume that the emission takes place via a one-quark de-excitation. The nonrelativistic form of the interaction is

$$\sum_j \frac{f_q}{\mu}(\boldsymbol{\sigma}_j\cdot\mathbf{k})(\boldsymbol{\tau}_j\cdot\boldsymbol{\pi})\exp(-i\mathbf{k}\cdot\mathbf{r}_j)\frac{1}{(2E_\pi)^{1/2}},$$

where the summation is over each constituent quark; f_q is the quark-pion coupling constant, μ is any standard mass that we take to be the pion mass for definiteness, E_π is the energy of the emitted pion and $\mathbf{k}$ is its momentum. Since the wave functions for the particles are known, the amplitude M_{fi} for a decay process can be obtained to lowest order in f_q by taking the matrix element of this interaction operator between the relevant N^* and the N or Δ wave functions, and integrating over all space. The decay rate then follows from

$$dw=2\pi|M_{fi}|^2\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{dE_f},$$

where E_f is the total final-state energy. All of the integrations involved can be done exactly by using the identity[20]

$$\int_0^\infty dr\, r^{\mu-1}J_\nu(kr)\exp(-\alpha^2r^2)=\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)(k/2\alpha)^\nu}{2\alpha^\mu\Gamma(\nu+1)}\times\exp\left(-\frac{k^2}{4\alpha^2}\right)F\left(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+1,\nu+1;\frac{k^2}{4\alpha^2}\right),$$

where the Bessel functions originate from the usual angular momentum expansion for $\exp(-i\mathbf{k}\cdot\mathbf{r})$, and F is the hypergeometric function. The series expansion for F terminates if $\frac{1}{2}(\nu-\mu)$ is a negative integer.

The resulting formulas are listed in Table II for the ground state **56**, $L=0^+$ and the **70**, $L=1^-$, and in Table III for some of the N^*'s in the third band. These formulas are derived assuming degeneracy of the particles within each band. However, when computing the widths, the observed masses of the resonances have been used. This introduces to a certain extent some effect due to mass splitting.

It may be noted that the rates of decay for the particles in the various bands go as k^3, k^5, and k^7 respectively, independent of the final-state orbital angular momentum L_f. The origin of this lies in our assumption of a one-quark interaction—it is the type of one-quark de-excitation that is important in deriving the formulas for the widths, not the final state L_f. Conserva-

TABLE II. Decay rates for **56**, $L=0^+$ and **70**, $L=1^-$. Here $A=(1/\mu^2)(f_q^2/4\pi)\exp(-k^2/3\alpha^2)E/M^*$, where E, k are the total energy, momentum of the final state baryon (N or Δ) of mass M^*.

Resonance	$\Gamma(N^*\to N\pi)$	$\Gamma(N^*\to\Delta\pi)$
P_{33}	$\frac{16}{3}k^3A$	
D_{15}	$\frac{4}{45}\frac{k^5}{\alpha^2}A$	$\frac{56}{45}\frac{k^5}{\alpha^2}A$
D_{33}	$\frac{2}{27}\frac{k^5}{\alpha^2}A$	$\frac{16}{27}\frac{k^5}{\alpha^2}A$
S_{31}	$\frac{2}{27}\frac{k^5}{\alpha^2}A$	$\frac{40}{27}\frac{k^5}{\alpha^2}A$
$S_{11}{}^a$	$\frac{4}{27}\frac{k^5}{\alpha^2}(\cos\theta_s+2\sin\theta_s)^2A$	$\frac{8}{27}\frac{k^5}{\alpha^2}(\cos\theta_s+2\sin\theta_s)^2A$
$S_{11}{}^b$	$\frac{4}{27}\frac{k^5}{\alpha^2}(-\sin\theta_s+2\cos\theta_s)^2A$	$\frac{8}{27}\frac{k^5}{\alpha^2}(-\sin\theta_s+2\cos\theta_s)^2A$
$D_{13}{}^a$	$\frac{2}{135}\frac{k^5}{\alpha^2}[\cos\theta_d+2(\sqrt{10})\sin\theta_d]^2A$	$\frac{8}{135}\frac{k^5}{\alpha^2}[20+21\cos^2\theta_d+(\sqrt{10})\sin2\theta_d]A$
$D_{13}{}^b$	$\frac{2}{135}\frac{k^5}{\alpha^2}[-\sin\theta_d+2(\sqrt{10})\cos\theta_d]^2A$	$\frac{8}{135}\frac{k^5}{\alpha^2}[20+21\sin^2\theta_d-(\sqrt{10})\sin2\theta_d]A$

[20] G. N. Watson, in *Theory of Bessel Functions* (Cambridge University Press, New York, 1952).

TABLE III. Decay rates for unmixed states in the second positive parity band of N^*'s. Here $A=(1/\mu^2)(f_q^2/4\pi)\exp(-k^2/3\alpha^2)E/M^*$, where E, k are the total energy, momentum of the final state baryon (N or Δ).

Resonance	$\Gamma(N^*\to N\pi)$	$\Gamma(N^*\to\Delta\pi)$
$F_{37}(\mathbf{56},L=2^+)$	$\frac{8}{315}\frac{k^7}{\alpha^4}A$	$\frac{45}{315}\frac{k^7}{\alpha^4}A$
$F_{17}(\mathbf{70},L=2^+)$	$\frac{1}{315}\frac{k^7}{\alpha^4}A$	$\frac{12}{315}\frac{k^7}{\alpha^4}A$
$F_{15}(\mathbf{56},L=2^+)$	$\frac{25}{405}\frac{k^7}{\alpha^4}A$	$\frac{32}{405}\frac{k^7}{\alpha^4}A$
$F_{15}(\mathbf{70},8^{1/2},L=2^+)$	$\frac{8}{405}\frac{k^7}{\alpha^4}A$	$\frac{16}{405}\frac{k^7}{\alpha^4}A$
$F_{15}(\mathbf{70},8^{3/2},L=2^+)$	$\frac{2}{2835}\frac{k^7}{\alpha^4}A$	$\frac{220}{2835}\frac{k^7}{\alpha^4}A$
$P_{11}(\mathbf{56},L=0^+)$	$\frac{25}{162}\frac{k^7}{\alpha^4}A$	$\frac{32}{162}\frac{k^7}{\alpha^4}A$
$P_{11}(\mathbf{70},L=0^+)$	$\frac{4}{81}\frac{k^7}{\alpha^4}A$	$\frac{8}{81}\frac{k^7}{\alpha^4}A$
$P_{11}(\mathbf{70},L=2^+)$	$\frac{2}{405}\frac{k^7}{\alpha^4}A$	$\frac{4}{405}\frac{k^7}{\alpha^4}A$
$P_{11}(\mathbf{20},L=1^+)$	0	0
$P_{33}(\mathbf{56},L=0^+)$	$\frac{8}{162}\frac{k^7}{\alpha^4}A$	$\frac{25}{162}\frac{k^7}{\alpha^4}A$
$P_{33}(\mathbf{56},L=2^+)$	$\frac{8}{405}\frac{k^7}{\alpha^4}A$	$\frac{25}{405}\frac{k^7}{\alpha^4}A$
$P_{33}(\mathbf{70},L=2^+)$	$\frac{1}{405}\frac{k^7}{\alpha^4}A$	$\frac{20}{405}\frac{k^7}{\alpha^4}A$

tion of angular momentum is brought about through the projection of the appropriate term in the expansion of $\exp(-i\mathbf{k}\cdot\mathbf{r})$. Moreover, the contributions from the $(1s)(1p)^2$ components of the wave functions are zero, since this requires a two-quark de-excitation to reach the ground state $(1s)^3$.

V. COMPARISON WITH EXPERIMENT

As may be seen from Table I, of all the states predicted in the first three bands, only seven can be formed in a unique way, namely, N, Δ; D_{15}, D_{33}, S_{31}; F_{37} and F_{17}. (We assume no mixing between the different bands.) Any two of these states may therefore in principle be used to determine the values of the two basic parameters (the quark-pion coupling constant and the well parameter α). From these, the widths of the remaining five states may be predicted. All the other resonances that occur are mixtures of harmonic-oscillator states, and so their decay rates will depend in addition upon mixing angles.

The calculation for the $N\pi$ width of $\Delta(1236)$ is essentially the same as that done by Becchi and Morpurgo,[7] except for the extra multiplicative factor $\exp(-k^2/3\alpha^2)$ that arises from the harmonic-oscillator radial integral [Becchi and Morpurgo[7] approximated $\exp(-i\mathbf{k}\cdot\mathbf{r})$ by unity]. This factor is particularly important for the higher N^*'s where it helps to counteract the increasing powers of the momentum that enter the formulas. Its effect is very similar to having an extra finite-interaction radius term. The calculation of the widths for the decay of the **70**, $L=1^-$ has some resemblance to the work of Mitra and Ross.[9,21] However, since they have no specific potential for both the **56**, $L=0^+$ and the **70**, $L=1^-$, all their integrations over radial wave functions have to be treated as parameters. Further parameters are introduced when a decay can take place through more than one orbital angular momentum L_f in the final two-body state. With the present model, none of these additional parameters is necessary—the spatial wave functions are known explicitly, and where a decay can occur through more than one value of L_f, this model gives the proportion of each L_f.

The parameters $f_q^2/(4\pi)$ and α^2 were determined so as to give $\Gamma(\Delta\to N\pi)$ exactly and simultaneously to yield a best fit to the $N\pi$ decays of the other unmixed resonances. The corresponding widths are shown in Table IV, the values of the parameters being $f_q^2/(4\pi)=0.055$ and $\alpha^2=0.10$ $(\text{GeV}/c)^2$. All of the calculated widths are close to the widths estimated by phase-shift analyses; considering that some of the wave functions that describe these resonances are completely different, we feel that these results are very encouraging.

It is also interesting to compare the $\Delta\pi$ widths with the $N\pi$ widths. Their ratios compare very favorably with experiment if the estimated Γ_{tot} is mainly made up of contributions from the $N\pi$ and $\Delta\pi$ modes.

For some of the $\Delta\pi$ decays, two values of the final-state orbital angular momentum L_f are possible. For $D_{33}\to\Delta\pi$, both $L_f=0$ and 2 are possible; with the one-

TABLE IV. Decay widths of unmixed states with $f_q^2/4\pi=0.055$ and $\alpha^2=0.10$ $(\text{GeV}/c)^2$. All widths are in MeV.

Resonance	Mass (MeV)	$\Gamma_{N\pi}$	$\Gamma_{\Delta\pi}$	$\Gamma_{N\pi}$(expt)	Γ_{tot}(expt)
$\Delta(\mathbf{56},L=0^+)$	1236	120	···	120	120
$D_{15}(\mathbf{70},L=1^-)$	1680	33	112	~68	~170
$D_{33}(\mathbf{70},L=1^-)$	1691	28	55	~38	~269
$S_{31}(\mathbf{70},L=1^-)$	1640	24	121	~54	~180
$F_{37}(\mathbf{56},L=2^+)$	1950	86	142	~86	~221
$F_{17}(\mathbf{70},L=2^+)$	1983	12	53	~29	~225

[21] Mitra and Ross (Ref. 9) have an additional recoil term in their one-quark interaction. This is important for decays with little phase space available, for example, in the $S_{11}(1550)$ decaying to a nucleon by the emission of an η meson. This term is not important for our $N\pi$ widths, and has therefore been omitted.

quark interaction, this partial width gets equal contributions from these two angular momenta. Likewise for $D_{15}\to\Delta\pi$, only $L_f=2$ is allowed ($L_f=4$ is also possible); and for F_{37}, $F_{17}\to\Delta\pi$ there is a contribution only from $L_f=3$. However, it is unlikely that these results can be checked with experiment.

Some comments on the values of the parameters should be made. The value $f_q^2/(4\pi)=0.055$ for the quark-pion coupling constant is close to the value of the pion-nucleon coupling constant $f^2/(4\pi)=0.08$. By considering the matrix element of the one-quark interaction between nucleon states, Becchi and Morpurgo[7] obtained the relation $f_q^2/(4\pi)=(9/25)f^2/(4\pi)\approx 0.03$. However, we chose to leave f_q as an adjustable parameter in our calculations, since all of the decays considered are physical and do not involve an extrapolation off the mass shell. The value of $\alpha^2=0.10$ (GeV/c)^2 corresponds to an interaction radius of about 0.8 F. This seems a reasonable range for a nucleonic process. However, if the separation of the mean masses of different bands is taken to be about 400 MeV, this value of α^2 corresponds to a quark mass of about 300 MeV since $\alpha^2=M\omega$. This can only be interpreted as an "effective mass." To increase the effective quark mass significantly would require a larger value of α^2 with a corresponding radius of interaction much smaller than 0.8 F.

VI. MIXED STATES IN THE 70, $L=1^-$

There are two pairs of mixed states in the **70**, $L=1^-$, namely, the two S_{11}'s and the two D_{13}'s. With the physical states defined as

$$S_{11}{}^a=\cos\theta_s\, S_{11}(8^{3/2})+\sin\theta_s\, S_{11}(8^{1/2}),$$
$$S_{11}{}^b=-\sin\theta_s\, S_{11}(8^{3/2})+\cos\theta_s\, S_{11}(8^{1/2}),$$

and similarly for $D_{13}{}^a$, $D_{13}{}^b$ with mixing angle θ_d, the formulas for the $N\pi$ and $\Delta\pi$ widths can be derived, and these are written down in Table II.

The mixing angle θ_s may be evaluated from the known $N\pi$ width of $S_{11}(1550)$, which we take to be $S_{11}{}^b$. There are two solutions, $\theta_s\approx 35°$ and $\theta_s\approx 90°$. The $N\pi$ width of $S_{11}{}^a(1710)$ can then be predicted, as well as the two $\Delta\pi$ widths. These are presented in Table V. The agreement with experiment is good [the major inelastic contribution to $S_{11}(1550)$ is known[22] to be the $N\eta$ mode (about 70% of the total width), not the $\Delta\pi$ mode]. The higher $S_{11}(1710)$ is predicted to be very broad but substantially elastic, and this seems to be the case from the phase-shift analyses.[1,2]

The solution $\theta_s\approx 90°$ corresponds to $S_{11}(1710)$ being mainly $8^{1/2}$, with $S_{11}(1550)$ mainly $8^{3/2}$. This is the solution of Mitra and Ross.[9] However, it contradicts a selection rule deduced by Moorhouse,[23] who showed that the N^* states originating from the $8^{3/2}$ cannot be photoexcited from a proton. Experimentally,[24] the $S_{11}(1550)$ is strongly photoproduced, implying that it is not primarily $8^{3/2}$, There is no such contradiction with the solution $\theta_s\approx 35°$ that we therefore favor.

Likewise θ_d can be estimated from the measured $\Gamma[D_{13}{}^b(1525)\to N\pi]$.[25] The higher $D_{13}{}^a$ is then predicted (Table V) to be somewhat broader than $D_{13}{}^b$ (1525). Again two values of the mixing angle are possible, $\theta_d\approx 35°$ and $\theta_d\approx 127°$. Even although $D_{13}(1525)$ is strongly photoexcited from a proton, the latter possibility $\theta_d\approx 127°$ cannot be ruled out immediately and a more detailed examination is necessary. However, it is interesting that the S_{11} and D_{13} widths can be accounted for by a similar amount of mixing, namely with $\theta_m\approx 35°$, and possibly indicates that the mixing in these states is due to a common mechanism.

VII. HIGHER MIXED STATES

With the exception of the F_{37} and F_{17} in the third band, all the other resonances are in principle mixtures of states. There is not enough information available to untangle the situation completely. However, a few comments may be made.

(1) Most of the resonances in this band can reach the ground state either by direct decay or by several intermediate steps, such as through transitions to lower-mass members of this band or through transitions to the **70**, $L=1^-$ states. One might therefore expect these resonances to be quite broad, and this is the case experimentally. In order to try and shed some light on the observed members of this band, we have set out in Table VI the $N\pi$ widths for several unmixed configurations, calculated from the formulas in Table III for typical masses throughout the band, with $f_q^2/4\pi=0.055$ and $\alpha^2=0.10$ (GeV/c)^2.

(2) Three F_{15} resonances are anticipated, only one of which has been seen. From Table VI, it would seem likely that the observed $F_{15}(1687)$ is primarily a **56**, $L=2^+$ configuration. Since the **70**$(8^{1/2})$, $L=2^+$ configuration also has an appreciable coupling to $N\pi$, we should expect a second F_{15} to be experimentally observable in this band. If the third F_{15} is substantially a **70**$(8^{3/2})$, $L=2^+$ configuration, its $N\pi$ width would be very small and so would have easily escaped detection.

(3) A similar situation holds for the three P_{33} resonances. The calculations of Table VI suggest that $P_{33}(1690)$, the only one observed so far, is some mixture of **56**, $L=0^+$ and 2^+. We anticipate a second P_{33} in this band with a sizeable $N\pi$ width, but the third P_{33} will

[22] A. T. Davies and R. G. Moorhouse, Nuovo Cimento **52A**, 1112 (1967).

[23] R. G. Moorhouse, Phys. Rev. Letters **16**, 771 (1966).

[24] C. Bacci, C. Mencuccini, G. Penso, G. Salvini, and V. Silvestrini, Nuovo Cimento **45**, 983 (1966); C. A. Heusch, C. Y. Prescott, E. D. Bloom, and L. S. Rochester, Phys. Rev. Letters **17**, 573 (1966); R. Prepost, D. Lindquist, and D. Quinn, *ibid.* **18**, 82 (1967); S. R. Deans and W. G. Holladay, Phys. Rev. **161**, 1466 (1967).

[25] The formulas for $D_{13}\to\Delta\pi$ are somewhat complicated. This is because $D_{13}(8^{1/2})$ decays through final-state orbital angular momentum $L_f=0$ and 2 with equal proportions, whereas the $D_{13}(8^{3/2})$ decays with the proportions 25:16, respectively.

TABLE V. The S_{11}, D_{13} decays in the **70**, $L=1^-$ with $f_q^2/4\pi=0.055$ and $\alpha^2=0.10$ (GeV/c)^2. The widths with superscript x were used as input to estimate the mixing angles. For y, we note that the main inelastic decay of $S_{11}^b(1550)$ is through the $N\eta$ mode (70% of the total width) and not the $\Delta\pi$ mode. All widths are in MeV.

Resonance	Mass (MeV)	θ_m	$\Gamma_{N\pi}$	$\Gamma_{\Delta\pi}$	$\Gamma_{N\pi}$(expt)	Γ_{tot}(expt)
S_{11}^a	1710	{35°, 89°}	231	132	∼240	∼300
S_{11}^b	1550	{35°, 89°}	37^x	6	∼39	$\sim130^y$
D_{13}^a	[1690]	{35°, 127°}	109	{197, 133}		
D_{13}^b	1525	{35°, 127°}	60^x	{16, 23}	∼63	∼115

TABLE VI. Decay widths for P_{11}, P_{33}, F_{15} as unmixed states with typical masses $M^*=1500$, 1800, and 2100 MeV appropriate to this band evaluated with $f_q^2/4\pi=0.055$ and $\alpha^2=0.10$ (GeV/c)^2. All widths are in MeV.

Resonance	Configuration	$\Gamma_{N\pi}(M^*=1500)$	$\Gamma_{N\pi}(M^*=1800)$	$\Gamma_{N\pi}(M^*=2100)$
P_{11}	**56**, $L=0^+$	51	321	716
	70, $L=0^+$	16	103	230
	70, $L=2^+$	2	10	23
P_{33}	**56**, $L=0^+$	16	103	230
	56, $L=2^+$	7	41	92
	70, $L=2^+$	1	5	12
F_{15}	**56**, $L=2^+$	20	125	288
	70$(8^{1/2})$, $L=2^+$	7	41	92
	70$(8^{3/2})$, $L=2^+$	0.2	1	3

Resonance	$\Gamma_{N\pi}$(expt)	Γ_{tot}(expt)
$P_{11}(1450)$	∼139	∼221
$P_{11}(1750)$	∼105	∼327
$P_{33}(1690)$	∼28	∼281
$F_{15}(1687)$	∼99	∼177

probably be rather difficult to detect owing to the expected smallness of its $N\pi$ width.

(4) Four P_{11}'s occur in this band. Two have been seen, at 1450 and 1750 MeV, and Table VI would indicate that both of them contain appreciable **56**, $L=0^+$ components. It could happen however that $P_{11}(1450)$ is primarily **56**, $L=0^+$ and $P_{11}(1750)$ is primarily **70**, $L=0^+$. One might indeed expect, *a priori*, a resonance that is predominantly **56**, $L=0^+$ to lie low in mass because of its similarity to the ground state. It would then be natural to associate together the $P_{11}(1450)$ and the $P_{33}(1690)$, both being primarily **56**, $L=0^+$ with small admixtures of other configurations.

Actually there are indications from other sources that the $P_{11}(1450)$ has a large **56** component. For example, the cross sections for the processes $\pi^\pm p\to\pi^\pm P_{11}(1450)$ and $pp\to pP_{11}(1450)$ are found[26] to be constant over a wide range of energies, from 10 to 30 GeV/c, and likewise for the slopes of the forward diffraction peaks. These characteristics are strongly indicative of vacuum exchange, and suggest that the $P_{11}(1450)$ has a similar structure to that of the nucleon.[27]

The contributions of the configurations **70**, $L=2^+$ and **20**, $L=1^+$ to the P_{11} $N\pi$ widths are small. Should the two missing P_{11} resonances belong predominantly to these configurations, they will be difficult to find.

VIII. REGGE FAMILIES

Since the number of states in subsequent bands is expected to increase rapidly, the separation of the center-of-mass motion obviously becomes very laborious and in any case, the whole situation becomes confused by configuration mixing. Therefore in this model, the description of the higher members of a Regge trajectory (should the latter remain as an appropriate model for

[26] K. J. Foley *et al.*, Phys. Rev. Letters **19**, 387 (1967); R. B. Bell *et al.*, *ibid.* **20**, 164 (1968); I. M. Blair *et al.*, *ibid.* **17**, 789 (1966); E. W. Anderson *et al.*, *ibid.* **16**, 855 (1966).

[27] These experiments tend to rule out the $P_{11}(1450)$ being in an $SU(3)$ $\overline{\mathbf{10}}$ representation which is sometimes mentioned in the literature [C. Lovelace, CERN Report No. Th. 628, 1965 (unpublished); J. J. Brehm and G. L. Kane, Phys. Rev. Letters **17**, 764 (1966); A. Donnachie, Phys. Letters **24B**, 420 (1967]. This $\overline{\mathbf{10}}$ allocation also conflicts with the observation [Y. C. Chau, N. Dombey, and R. G. Moorhouse, Phys. Rev. **163**, 1632 (1967)] that $P_{11}^+(1450)$ gives an important contribution to pion photoproduction; moreover the $P_{11}^+(1450)$ is seen [Cambridge Bubble Chamber Group, Phys. Rev. **163**, 1511 (1967)] as a distinct shoulder in the reaction $\gamma p\to\pi^-\Delta^{++}$. If P_{11}^+ belonged to a $\overline{\mathbf{10}}$, it does not couple [H. Lipkin, Phys. Letters **12**, 154 (1964)] to either $\gamma+p$ (by U-spin conservation) or to $\pi+\Delta$ [by SU(3), since $\mathbf{8}\otimes\mathbf{10}$ does not contain a $\overline{\mathbf{10}}$], so that these photoproduction results can be explained only by assuming an unusally large amount of SU(3) breaking.

grouping some of the states) is in principle a complicated one.[28] Nevertheless, we can show that the $N\pi$ widths, which are obtained in this model for the higher Regge recurrences, are comparable to those observed. Consider, for example, the Δ trajectory. We have already seen that Δ is pure $(1s)^3$, while its first Regge recurrence F_{37} is a mixture of $(1s)^2(1d)$ and $(1s)(1p)^2$. The next Regge recurrence is an $H_{3,11}$ which is in principle a mixture of $(1s)^2(1g)$, $(1s)(1d)^2$, $(1p)^2(1d)$ and $(1s)(1p)$-$(1f)$. However, only the $(1s)^2(1g)$ component contributes to the $N\pi$ width via our one-quark interaction; therefore it is possible to calculate an upper limit for the $N\pi$ width of this $H_{3,11}$ by assuming it to be pure $(1s)^2(1g)$. The result is 85 MeV, compared to the experimental value of $\Gamma_{N\pi}\approx 34$ MeV for the observed $H_{3,1}(2420)$.

Recently, Barger and Cline[5] have suggested that there is parity doubling among the various Regge trajectories. Apart from the embarrassment of important missing states (such as the parity partner for the nucleon), the evidence for doubling is not particularly convincing and is based somewhat on the degeneracy of the $\frac{5}{2}^-$ $D_{15}(1680)$ and $\frac{5}{2}^+$ $F_{15}(1687)$. In the present model, this degeneracy appears to be accidental, the D_{15} originating from a configuration with quark spin $\frac{3}{2}$ in the **70**, $L=1^-$ and the F_{15} primarily from a configuration with quark spin $\frac{1}{2}$ in the **56**, $L=2^+$. On the other hand, one should perhaps expect to find several approximate (but accidental) degeneracies to occur in a spectrum that is so rich in positive- and negative-parity baryons.

In this model, parity doublets for N, Δ are not allowed. They would presumably belong to a similar configuration but with opposite parity, namely a **56**, $L=0^-$; but it is impossible to construct any kind of $L=0^-$ wave function out of the two vectors $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$ that are available.

IX. CONCLUSIONS

We now briefly summarize the successes and failures of the model under consideration.

In the first place, this harmonic oscillator model[10] predicts a spectrum of baryons that is certainly consistent with that observed. The data for higher-mass baryons suggest a continuance of the band structure that seems to characterize the N^*'s with a mass below 2 GeV. A true nonrelativistic harmonic-oscillator potential should of course yield energy levels (which become bands when some additional perturbing potential is present) that are equally spaced. Because of the mass splitting, it is difficult to say whether the data (Fig. 1) follow this general trend; in any case, it is desirable to extend this model to incorporate mass-splitting potentials in addition to the basic harmonic-oscillator forces. Hopefully, one can also treat the problem more relativistically.

Secondly, we have calculated the decay widths into $N\pi$ and $\Delta\pi$ for many of the baryons on the basis of a single mechanism, the de-excitation of a single quark by means of pion emission. The use of harmonic-oscillator wave functions renders the problem highly tractable as regards elimination of spurious states and ease of computation. The calculated widths are in remarkable agreement with the values estimated from phase-shift analyses for a number of states whose quark configurations are completely different. The authors are sufficiently encouraged by these results (and by some preliminary results on photoexcitation, not reported here) to feel that the shell model may well become as useful a tool for hadron spectroscopy as it is in nuclear physics.

Since the simplest way of justifying any model with experiment is through a comparison of the $N\pi$ and $\Delta\pi$ widths, it would be highly desirable to have more information on the $\Delta\pi$ widths, particularly since almost none of them is known. We would therefore suggest a more detailed examination of the reaction $\pi N\rightarrow\pi\Delta$ over a wide range of energies. Though neither the experiments nor the subsequent analysis would be easy, the information gleaned would prove to be extremely useful. It is also possible that some of the resonances that have been predicted but not yet detected in the usual type of phase-shift analyses (which rely to a large extent on elastic πN scattering data) may show up on a close examination of this inelastic reaction.

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APPENDIX A: HARMONIC-OSCILLATOR WAVE FUNCTIONS

The following harmonic-oscillator wave functions were used in the text.

$$(1s)=\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}\exp(-\tfrac{1}{2}\alpha^2r^2)Y_0(\Omega)\,,$$

$$(1p)=(\sqrt{\tfrac{2}{3}})\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}\alpha r\exp(-\tfrac{1}{2}\alpha^2r^2)Y_1(\Omega)\,,$$

$$(1d)=\left(\sqrt{\frac{4}{15}}\right)\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}\alpha^2r^2\exp(-\tfrac{1}{2}\alpha^2r^2)Y_2(\Omega)\,,$$

$$(2s)=(\sqrt{\tfrac{2}{3}})\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}(\tfrac{3}{2}-\alpha^2r^2)\exp(-\tfrac{1}{2}\alpha^2r^2)Y_0(\Omega)\,,$$

where $\alpha^2=M\omega$.

[28] The succeeding members of a Regge family are not expected to correspond simply to successive single-quark excitations, for example $(1s)^3$, $(1s)^2(1d)$, $(1s)^2(1g)$, $\cdots$ for the Δ and N trajectories. We have already shown that F_{37}, the first Regge recurrence of the Δ, probably has a high proportion of $(1s)(1p)^2$, namely $0.82(1s)^2(1d)$-$0.58(1s)(1p)^2$ before mass splitting.

APPENDIX B: SHELL-MODEL WAVE FUNCTIONS

$(1s)^3$: Symmetric Representation, $L=0^+$

$$\psi=\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(1p)$: Symmetric Representation, $L=1^-$

$$\psi=\tfrac{1}{3}\sqrt{2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(1p)$: Mixed Representation, $L=1^-$

$$\psi_a=\tfrac{1}{3}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\frac{1}{\sqrt{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha(\mathbf{r}_1-\mathbf{r}_2)\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(2s)$: Symmetric Representation, $L=0^+$

$$\psi=\tfrac{1}{3}\sqrt{2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}[\tfrac{9}{2}-\alpha^2(r_1^2+r_2^2+r_3^2)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(2s)$: Mixed Representation, $L=0^+$

$$\psi_a=\tfrac{1}{3}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2(r_1^2+r_2^2-2r_3^2)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\frac{1}{\sqrt{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2(r_1^2-r_2^2)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(1d)$: Symmetric Representation, $L=2^+$

$$\psi=\frac{2}{\sqrt{45}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha^2[r_1^2Y_2(\Omega_1)+r_2^2Y_2(\Omega_2)+r_3^2Y_2(\Omega_3)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)^2(1d)$: Mixed Representation, $L=2^+$

$$\psi_a=\left(\sqrt{\frac{2}{45}}\right)\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha^2[r_1^2Y_2(\Omega_1)+r_2^2Y_2(\Omega_2)-2r_3^2Y_2(\Omega_3)]\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\left(\sqrt{\frac{2}{15}}\right)\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{4\pi}\alpha^2[r_1^2Y_2(\Omega_1)-r_2^2Y_2(\Omega_2)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)(1p)^2$: Symmetric Representation, $L=0^+$

$$\psi=\tfrac{2}{3}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2(\mathbf{r}_1\cdot\mathbf{r}_2+\mathbf{r}_2\cdot\mathbf{r}_3+\mathbf{r}_3\cdot\mathbf{r}_1)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2\ ].$$

$(1s)(1p)^2$: Mixed Representation, $L=0^+$

$$\psi_a=\tfrac{1}{3}\sqrt{2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2(\mathbf{r}_2\cdot\mathbf{r}_3+\mathbf{r}_3\cdot\mathbf{r}_1-2\mathbf{r}_1\cdot\mathbf{r}_2)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\sqrt{\tfrac{2}{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2(\mathbf{r}_2\cdot\mathbf{r}_3-\mathbf{r}_3\cdot\mathbf{r}_1)\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)(1p)^2$: Mixed Representation, $L=1^+$

$$\psi_a=\frac{1}{\sqrt{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2[(\mathbf{r}_3\times\mathbf{r}_1)-(\mathbf{r}_2\times\mathbf{r}_3)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\tfrac{1}{3}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2[(\mathbf{r}_3\times\mathbf{r}_1)+(\mathbf{r}_2\times\mathbf{r}_3)-2(\mathbf{r}_1\times\mathbf{r}_2)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)(1p)^2$: Antisymmetric Representation, $L=1^+$

$$\psi=\tfrac{1}{3}\sqrt{2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{3/2}}\alpha^2[(\mathbf{r}_2\times\mathbf{r}_3)+(\mathbf{r}_3\times\mathbf{r}_1)+(\mathbf{r}_1\times\mathbf{r}_2)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

$(1s)(1p)^2$: Symmetric Representation, $L=2^+$

$$\psi=\frac{2}{\sqrt{27}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{1/2}}\alpha^2[r_1r_2Y_2(\Omega_1\Omega_2)+r_3r_2Y_2(\Omega_2\Omega_3)+r_3r_1Y_2(\Omega_3\Omega_1)]\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

where $Y_2(\Omega_i\Omega_j)$ is the angular part obtained by combining $\mathbf{r}_i$, $\mathbf{r}_j$ to form $l=2$, and normalized to unity over Ω_i, Ω_j.

$(1s)(1p)^2$: Mixed Representation, $L=2^+$

$$\psi_a=\left(\frac{2}{27}\right)^{1/2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{1/2}}\alpha^2\times[r_2r_3Y_2(\Omega_2\Omega_3)+r_3r_1Y_2(\Omega_3\Omega_1)-2r_1r_2Y_2(\Omega_1\Omega_2)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)],$$

$$\psi_b=\tfrac{1}{3}\sqrt{2}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{3/2}\frac{1}{(4\pi)^{1/2}}\alpha^2\times[r_2r_3Y_2(\Omega_2\Omega_3)-r_3r_1Y_2(\Omega_3\Omega_1)]\times\exp[-\tfrac{1}{2}\alpha^2(r_1^2+r_2^2+r_3^2)].$$

These wave functions may equivalently be written in terms of $\mathbf{R}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\rho}$.