Regge Behavior and the Cerulus-Martin Lower Bound of High-Energy Scattering

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From the Regge hypothesis for the high-energy scattering amplitude and the unitarity condition in the s channel, the functional equations for the Regge parameters $\alpha(t)$ and $\beta(t)$ are obtained. By examining these equations, the lower bound due to Cerulus and Martin for the high-energy scattering amplitude is derived. An additional assumption, that the scattering amplitude does not increase exponentially as s goes to infinity for fixed $\cos\theta$, enables us to show that the high-energy behavior of the scattering amplitude for fixed $\cos\theta$ is something between $\exp[-a_1(\cos\theta)(\sqrt{s})\ln(s/s_0)]$ and $\exp[-a_2(\cos\theta)\sqrt{s}]$. The possibility of the appearance of narrow dips in the high-energy scattering at relatively large momentum transfer is also pointed out.

I. INTRODUCTION

SEVERAL years ago Cerulus and Martin[1] derived the lower bound of the elastic scattering amplitude $A(s,t)$ for fixed z in the region $-1<z<1$:

$$|A(s,t)| \geqq C \exp[-a(\sqrt{s})\ln(s/s_0)] \tag{1}$$

for large s. It is remarkable that this lower bound was obtained from the very general assumption that $A(s,t)$ is bounded by s^N when z is fixed in the cut z plane, in addition to the assumption (which is directly obtained from experiment) that the total cross section $\sigma_{\rm tot}(s)$ is almost constant for large s. Later, Kinoshita[2] proposed a hypothesis of minimum amplitude, which states that the scattering amplitude $A(s,t)$ realized in nature is the minimum amplitude consistent with the lower bound of Cerulus and Martin given in Eq. (1). This hypothesis agrees with the measurement of high-energy proton-proton scattering at relatively large momentum transfer except in the neighborhood of $\theta=90°$. It is worthwhile to point out that the gross features of the elastic p-p scattering at high energy (except for the forward cone) can be reproduced by Orear's formula,[3]

$$s d\sigma/d\Omega = A \exp[-(p/p_1)\sin\theta], \tag{2}$$

which is consistent with the lower bound of Eq. (1). Precise measurement of the differential cross section for p-p scattering at $\theta=90°$ by Akerlof *et al.*[4] revealed some deviation from Orear's fit of Eq. (2). The deviation of these data from Orear's formula is shown in Fig. 1. On the other hand, usually high-energy p-p scattering data are fitted[5] by two straight lines on a plot of $\ln(d\sigma/dt)$ against $s\sin\theta$, with a break at $s\sin\theta=18$ $(\text{GeV}/c)^2$:

$$d\sigma/dt = B\exp[-(s/g)\sin\theta], \tag{3}$$

with

$$B=(134.6\pm11.7)\ \text{mb}(\text{GeV}/c)^{-2},$$
$$g=(1.24\pm0.01)\ \text{GeV}^2, \quad \text{for} \quad s\sin\theta \leqq 16.0\ \text{GeV}^2$$

and

$$B=(56.4\pm3.4)\ \mu\text{b}(\text{GeV}/c)^{-2},$$
$$g=(2.77\pm0.02)\ \text{GeV}, \quad \text{for} \quad s\sin\theta \geqq 20.0\ \text{GeV}^2.$$

Equation (3) violates the lower bound due to Cerulus and Martin, as pointed out by Akerlof *et al.*[4] Therefore, if we take the lower bound of Eq. (1) seriously, it is more desirable to fit $\ln(s d\sigma/d\Omega)$ with a linear function of p plus possible dips.

In a previous paper[6] we obtained the functional equations for Regge parameters $\alpha(t)$ and $\beta(t)$ from the Regge behavior of scattering amplitudes and the unitarity condition in the s channel. The purpose of the present article is to derive the lower bound due to Cerulus and Martin by solving the functional equations for $\alpha(t)$ and $\beta(t)$. In Sec. II we shall briefly review the assumptions that were used in order to obtain the equations for Regge parameters as well as these equations. In Sec. II the solutions of these equations are examined and the upper bound, in addition to the lower bound due to Cerulus and Martin, is derived. In Sec. IV the general form of the high-energy scattering amplitude at relatively large momentum transfer is given, and the possible appearance of the narrow dips in this region of scattering is pointed out.

II. EQUATIONS FOR THE REGGE PARAMETERS

We start from the following two assumptions:

(1) The high-energy behavior of the quasi-two-body scattering amplitude $A_{f,i}(s,t)$ is given by

$$A_{f,i}(s,t)=[\ln(s/s_0)]^N \beta_{f,i}(t)(s/s_0)^{\alpha(t)}+\cdots, \tag{4}$$

where N is a non-negative integer to be determined. In Eq. (4), i and f designate the quasi-two-body (including two-body) channels.

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[1] F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).
[2] T. Kinoshita, Phys. Rev. Letters **12**, 257 (1964).
[3] J. Orear, Phys. Rev. Letters **12**, 112 (1964); Phys. Letters **13**, 190 (1964).
[4] C. W. Akerlof *et al.*, Phys. Rev. Letters **17**, 1105 (1966).
[5] J. V. Allaby *et al.*, Phys. Letters **23**, 389 (1966); **25B**, 156 (1967).
[6] T. Sawada, Phys. Rev. **156**, 1848 (1968).

(2) When we write the unitarity condition by separating the intermediate states into two parts,

$$\langle f|T|i\rangle-\langle f|T|i\rangle^{*}=2i\sum_{|n'\rangle}\langle f|T|n'\rangle\langle i|T|n'\rangle^{*}+2i\sum_{|n''\rangle}\langle f|T|n''\rangle\langle i|T|n''\rangle^{*}, \quad (5)$$

where the $|n'\rangle$'s are the quasi-two-body intermediate states and the $|n''\rangle$'s are the uncorrelated intermediate states with more than two particles, then the second term of the right-hand side of Eq. (5) is negligible for relatively large momentum transfer, say $|t|$ larger than some critical value $|t_c|$.

We may justify assumption (2) in the following way. At $t=0$ the two terms of the right-hand side of Eq. (5) have the same order of magnitude for large s, since experimentally $\sigma_{\rm el}(s)/\sigma_{\rm tot}(s)\approx 0.2$ for $P_L>10$ (GeV/c). Therefore we cannot neglect the second term in Eq. (5). However, when $-t$ increases from zero, the second term of the right-hand side of Eq. (5) decreases faster than the first. Van Hove[7] has computed the t dependence of $\langle f|T|n''\rangle\langle i|T|n''\rangle^{*}$ to be $\exp(-a|t|)$ for $|t|\ll|s|$, when we take the intermediate state $|n''\rangle$ as the uncorrelated jet state. On the other hand, as we shall see later, $\langle f|T|n'\rangle\langle i|T|n'\rangle^{*}$ decreases as $\exp(-a\sqrt{-t})$ if $|n'\rangle$ is a quasi-two-body intermediate state. Concerning the intermediate states with a relatively large number of particles, we can regard the states $|n''\rangle$ as approximately uncorrelated jet states and $\langle f|T|n''\rangle\langle i|T|n''\rangle^{*}$ decreases as $\exp(-a|t|)$. We can expect that there is a tendency for $\langle f|T|n''\rangle\langle i|T|n''\rangle^{*}$ to decrease faster with $|t|$ as the number of particles of the uncorrelated intermediate states $|n''\rangle$ increases. Thus for $|t|>|t_c|$ we can neglect the second term of the right-hand side of Eq. (5) and retain only the terms with quasi-two-particle intermediate states. Equation (5) becomes

$$\mathrm{Im}A_{f,i}(s,t)=\frac{p}{\sqrt{s}}\sum_k\int\frac{d\Omega'}{4\pi}A_{f,k}(s,t')A_{i,k}^{*}(s,t''),\quad \text{for } |t|>|t_c| \quad (6)$$

where the summation k extends to all the quasi-two-body states. In Eq. (16), t, t', and t'' are connected by

$$\cos\theta''=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi', \quad (7)$$

where θ, θ', and θ'' are related to t, t', and t'' by

$$t=-2p^2(1-\cos\theta), \quad \text{etc.} \quad (8)$$

For large values of s, we can replace the scattering amplitudes in Eq. (6) by the asymptotic form given in

[7] L. Van Hove, Rev. Mod. Phys. 36, 655 (1964).

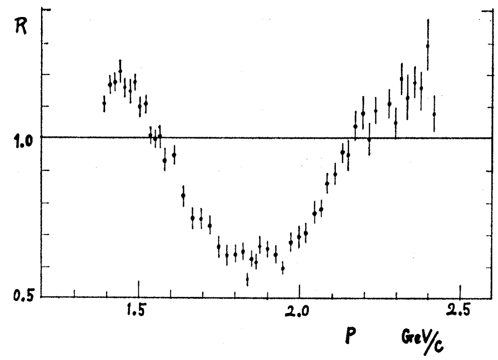

FIG. 1. Deviation of the elastic proton-proton cross section from Orear's formula: $R=[(sd\sigma/d\Omega)_{\rm exp}/(sd\sigma/d\Omega)_{\rm Orear}]$.

Eq. (4):

$$[\ln(s/s_0)]^N\,\mathrm{Im}\beta_{f,i}(t)(s/s_0)^{\alpha(t)}=\tfrac{1}{2}[\ln(s/s_0)]^{2N}\sum_k\int\frac{d\Omega'}{4\pi}\beta_{f,k}(t')\beta_{i,k}^{*}(t'')\times(s/s_0)^{\alpha(t')+\alpha(t'')}. \quad (9)$$

By comparing the power of s/s_0 and its coefficient on the two sides of Eq. (9), we obtain[6]

$$N=2N-1, \quad (10)$$

$$\bar\alpha(\eta)=2\bar\alpha(\tfrac{1}{2}\eta)-1, \quad (11)$$

$$\mathrm{Im}\bar\beta_{f,i}(\eta)=\tfrac{1}{2}f(\tfrac{1}{2}\eta)\sum_k\bar\beta_{f,k}(\tfrac{1}{2}\eta)\bar\beta_{i,k}^{*}(\tfrac{1}{2}\eta), \quad (12)$$

where we define

$$\eta=\sqrt{(-t)},\quad \bar\alpha(\eta)=\alpha(t),\quad \bar\beta_{f,i}(\eta)=\beta_{f,i}(t), \quad (13)$$

and

$$f(\tfrac{1}{2}\eta)=\left[\frac{2}{\eta}\bar\alpha'(\tfrac{1}{2}\eta)\bar\alpha''(\tfrac{1}{2}\eta)\right]^{-1/2}. \quad (14)$$

III. SOLUTIONS OF THE EQUATIONS AND THE LOWER BOUND DUE TO CERULUS AND MARTIN

Let us first find the solutions of Eqs. (10), (11), and (12). Equation (10) implies that the leading Regge trajectory is a Regge dipole[8] rather than a Regge pole, namely, $N=1$ in Eq. (4). The general solution of Eq. (11) is

$$\bar\alpha(\eta)=1-\eta T((\ln\eta)/(\ln 2)), \quad (15)$$

where $T(x)$ is an arbitrary periodic function with $T(x+1)=T(x)$. Therefore, the Regge trajecotory $\alpha(t)$ is essentially a linear function of $\sqrt{(-t)}$. If we separate

[8] T. Sawada, Nouvo Cimento 48, 534 (1967); 51, 208 (1967). If $N=1$ and $\alpha(0)=1$, the total cross section increases as $\sigma_{\rm tot}\sim(\ln s)$. It is interesting to assume that some of the meson trajectories are also Regge dipoles; in this case the dipole form of the nucleon form factor can easily be understood.

the signature factor from the residue functions by

$$\bar{\beta}_{f,i}(\eta)=-\frac{\exp[-i\pi\bar{\alpha}(\eta)]+1}{\sin\pi\bar{\alpha}(\eta)}\bar{b}_{f,i}(\eta)\,, \qquad (16)$$

where $\bar{b}_{f,i}(\eta)$ are real at least for some range of η, then Eq. (12) becomes

$$\bar{b}_{f,i}(\eta)=\tfrac{1}{2}f(\tfrac{1}{2}\eta)\frac{1}{[\sin(\frac{1}{2}\pi\bar{\alpha}(\frac{1}{2}\eta))]^2}\times\sum_k \bar{b}_{f,k}(\tfrac{1}{2}\eta)\bar{b}_{i,k}{}^*(\tfrac{1}{2}\eta)\,. \qquad (17)$$

In order to eliminate the two factors in front of the summation on the right-hand side of Eq. (17), we introduce $B_{f,i}(\eta)$ by

$$\bar{b}_{f,i}(\eta)=f^{-1}(\eta)F(\eta)[\sin^2(\tfrac{1}{2}\pi\bar{\alpha}(\eta))]^{1/2}B_{f,i}(\eta)\,, \qquad (18)$$

where

$$F(\eta)=\prod_{n=1}^{\infty}[\sin^2(\tfrac{1}{2}\pi\bar{\alpha}(2^n\eta))]^{1/2n+1}. \qquad (19)$$

Equation (17) is reduced to

$$B_{f,i}(\eta)=\sum_k B_{f,k}(\tfrac{1}{2}\eta)B_{i,k}{}^*(\tfrac{1}{2}\eta)\,. \qquad (20)$$

In particular, for the elastic scattering, i.e., for $f=i$, Eq. (20) becomes

$$B_{i,i}(\eta)=\sum_k |B_{i,k}(\tfrac{1}{2}\eta)|^2. \qquad (21)$$

Equation (21) implies

$$B(\eta)\geqq|B(\tfrac{1}{2}\eta)|^2, \qquad (22)$$

where we have written merely $B(\eta)$ for $B_{i,i}(\eta)$. If we introduce $G(\eta)$ by

$$G_1(\eta)=-(1/\eta)\ln B(\eta)\,, \qquad (23)$$

then from Eq. (22) $G(\eta)$ satisfies $G(\eta)\leqq G(\frac{1}{2}\eta)$, or more generally

$$G(2^n\eta)\leqq G(\eta)\,,\quad n=1,2,3,\cdots. \qquad (24)$$

When the functional value of $G(\eta)$ is known in the basic region,

$$D\equiv[\surd(-t_c)\,,\quad 2\surd(-t_c)]\,, \qquad (25)$$

then Eq. (24) gives an upper bound of $G(\eta)$ in the region of $\eta>2\surd(-t_c)$. Therefore, the lower bound of $B(\eta)$ is

$$B(\eta)\geqq\exp[-G(\eta_1)\eta]\,, \qquad (26)$$

where η_1 is a point in the region D and connected to η by $\eta=2^n\eta_1$ with $n=1,2,3,\cdots$. From Eqs. (4), (16), and (18), the elastic scattering amplitude is

$$A(s,t)=\ln(s/s_0)\eta^{-1}C(\eta)B(\eta)(s/s_0)^{\bar{\alpha}(\eta)}, \qquad (27)$$

where

$$C(\eta)=-\frac{\exp[-i\pi\bar{\alpha}(\eta)]+1}{\sin\pi\bar{\alpha}(\eta)}|\sin\tfrac{1}{2}\pi\bar{\alpha}(\eta)|\times F(\eta)\{\eta f^{-1}(\eta)\} \qquad (28)$$

and $C(\eta)$ is a finitely oscillating function. If we combine Eqs. (15), (26), and (27), the lower bound of the elastic scattering amplitude is obtained:

$$\ln\{[\eta/\ln(s/s_0)]|A(s,t)|\}\geqq\ln|C(\eta)|-G_1(\eta_1)\eta+[1-\eta T((\ln\eta)/(\ln 2))]\ln(s/s_0)\,. \qquad (29)$$

Since $T(x)$ is a periodic function and $\eta=\sqrt{2}p(1-\cos\theta)^{1/2}$, Eq. (29) agrees with the lower bound of Eq. (1) due to Cerulus and Martin when we fix $\cos\theta$ in $-1<\cos\theta<1$, as long as we consider the exponential dependence of the scattering amplitudes.

Let us introduce an additional assumption that the scattering amplitude $A(s,t)$ does not increase exponentially for large s when $\cos\theta$ is fixed in $-1<\cos\theta<1$. If we define $H(s,\cos\theta)$ by

$$B(\eta)(s/s_0)^{\bar{\alpha}(\eta)-1}=\exp[-H(s,\cos\theta)\eta]\,, \qquad (30)$$

then, in order to meet the assumption, $H(s,\cos\theta)$ must be non-negative for large value of s, or more precisely

$$\inf\lim_{s\to\infty}H(s,\cos\theta)\geqq 0\,. \qquad (31)$$

Since, from Eqs. (15) and (23),

$$H(s,\cos\theta)=G(\eta)+T((\ln\eta)/(\ln 2))\ln(s/s_0)\geqq 0\,, \qquad (32)$$

we can estimate the upper and lower bound of $H(s,\cos\theta)$ by using Eq. (24) and Eq. (31), respectively,

$$T((\ln\eta)/(\ln 2))\ln(s/s_0)+c_1\geqq H(s,\text{vos}\theta)\geqq-\ln[\tfrac{1}{2}(1-\cos\theta)]T((\ln\eta)/(\ln 2))+c_2\geqq 0 \qquad (33)$$

for large values of s, where c_1 and c_2 are constants. Therefore, from Eqs. (27), (30), and (33), if we remember that $T(x)$ is a periodic function, the high-energy behavior of the scattering amplitude $A(s,t)$ for fixed values of $\cos\theta$ is something between

$$\exp[-a_1(\cos\theta)(\surd s)\ln(s/s_0)]\quad\text{and}\quad\exp[-a_2(\cos\theta)\surd s]\,,$$

as long as we consider the exponential dependence. This behavior agrees with the hypothesis of minimum amplitude of Kinoshita.[2]

IV. NARROW DIPS OF $d\sigma/d\Omega$ AT RELATIVELY LARGE MOMENTUM TRANSFER

If we introduce a phenomenological parameter $\rho^{-1}(\eta)$, which has the physical meaning of the inverse of elasticity, Eq. (21) is then reduced to

$$B(\eta)=B^2(\tfrac{1}{2}\eta)\rho^{-1}(\tfrac{1}{2}\eta). \qquad (34)$$

The general solution of Eq. (34) is

$$\ln B(\eta)=\eta R((\ln\eta)/(\ln 2))-\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\ln\rho^{-1}(2^n\eta), \qquad (35)$$

where $R(x)$ is an arbitrary periodic function with $R(x+1)=R(x)$. In particular, if the reciprocal of the elasticity, $\rho^{-1}(\eta)$, is almost constant and is taken equal to $\langle\rho^{-1}\rangle$, then the summation of the right-hand side of Eq. (35) becomes $\ln\langle\rho^{-1}\rangle$. Since

$$s\,d\sigma/d\Omega=[\ln(s/s_0)]^{2N}|\beta(t)|^2(s/s_0)^{2\alpha(t)}, \qquad (36)$$

then using Eqs. (10), (15), (18), and (35), we can determine the differential cross section up to two arbitrary periodic functions $T(x)$ and $R(x)$:

$$\ln\left[\frac{\eta^2}{[\ln(s/s_0)]^2}\frac{s_0}{s}\frac{d\sigma}{d\Omega}\right]$$
$$=2\eta[R((\ln\eta)/(\ln 2))-\ln(s/s_0)T((\ln\eta)/(\ln 2))]$$
$$+\{\ln[\eta^2 f^{-2}(\eta)]+\ln[|F(\eta)|^2]\}-2\ln\langle\rho^{-1}\rangle. \qquad (37)$$

The gross feature of the elastic scattering is given by the first term on the right-hand side of Eq. (37), $2\sqrt{(-t)}[R+\ln(s/s_0)T]$; and the second term gives the local structure—the dips. There are two types of dips: periodic dips and signature dips. The periodic dips occur when $\eta^2 f^{-2}(\eta)$ vanishes. Since

$$\eta^2 f^{-2}(\eta)=\frac{1}{\ln 2}\frac{d}{dx}\left[T(x)+\frac{1}{\ln 2}T'(x)\right]\Bigg|_{x=(\ln\eta)/(\ln 2)}, \qquad (38)$$

$\eta^2 f^{-2}(\eta)$ is a periodic function of $(\ln\eta)/(\ln 2)$; this type of dip appears in a periodic way and its width is relatively broad. The signature dips occur when $|F(\eta)|^2$ vanishes:

$$|F(\eta)|^2=\sum_{n=1}^{\infty}|\sin^2[\tfrac{1}{2}\pi\bar\alpha(2^n\eta)]|1/2^n=0; \qquad (39)$$

and the nth factor vanishes at

$$\bar\alpha(2^n\eta)=-2m \quad \text{with} \quad m=0,1,2,\cdots. \qquad (40)$$

Since $\bar\alpha(\eta)$ is a monotonous decreasing function, the zeros of $|F(\eta)|^2$ can be specified by two indices n and m. The widths of the dips decrease very rapidly with n; therefore, only the first few dips can be observed, which correspond to small values of n.

It is well known that the scattering amplitude $A(s,t)$ is an analytic function of t on the cut plane if s is fixed at a finite value. However, when we make the asymptotic expansion at $s=\infty$, in general, the coefficient of the leading term as a function of t goes not necessarily have the same analyticity; very often it is not even an analytic function.[9] In fact, our solution of $\beta(t)$ has infinitely many dips and is not an analytic function. For large but finite values of s, the very narrow dips must be smeared out, since $A(s,t)$ is an analytic function of t. Therefore, in the actual measurements we can observe only the dips with appreciable width.

Finally, it is worthwhile to point out that our derivation is true only in the region $s\gg|t|>|t_c|$, namely, at relatively large momentum transfer but at small scattering angle θ. At large scattering angle θ, we have to add a background term to the contribution from the leading Regge trajectory, since in this case the Regge term $\beta(t)(s/s_0)^{\alpha(t)}$ must be changed to $\beta(t)(-z_t)^{\alpha(t)}$, where

$$-z_t=2s/[2p^2(1-\cos\theta)+4M^2]-1, \qquad (41)$$

and this term does not dominate the high-energy scattering amplitude any more. However, we can still observe the narrow dips or bumps (which depend on the relative phase of two terms) at the points where $\beta(t)$ vanishes, if the background term is a smooth function. We shall comment about the regularity of the positions of these dips and their shapes elsewhere.[10]

ACKNOWLEDGMENT

I wish to thank Professor A. O. Barut for helpful discussions.

[9] For example, consider the entire function $f(x,y)=(\sin xy)/xy$.
[10] T. Sawada, Nuovo Cimento **55**, 342 (1968).