Watson's Theorem When There Are Three Strongly Interacting Particles in the Final State

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Amado has recently conjectured the following extension of Watson's final-state-interaction theorem: The phase of an amplitude leading to a final state with three strongly interacting particles, when considered as a function of the relative energy of one pair, in a given-pair partial wave, all other variables being held constant, is the scattering phase of that pair, δ. This conjecture is investigated in two simple, complementary models. In the first, a final-state interaction is grafted on to a specific production amplitude; as expected, if the production amplitude is real, the phase of the total amplitude is δ. The question of the phase turns out to be intimately related to the question of whether triangle singularities are observable or not; Amado's results on this point are corrected. In the second model, the production mechanism is ignored and attention is focused on the possibility that there may be two ("overlapping") resonances in the final state. Here the theorem does *not* hold, in general (however it does turn out that the corrections are either small or easily calculable). In this case too, the phase question is linked with the observability of triangle singularities, and the rather negative conclusions reached by Schmid (who also treated this second model) are criticized. A basic tool is a theorem that the on- and off-shell parts of a rescattering graph contribute equally near a singularity which is near the physical region.

I. INTRODUCTION

LET $T_l(s)$ be the lth partial-wave amplitude for a reaction $A+B\rightarrow 2+3$, where the particles 2 and 3 are strongly interacting, with s the square of the 2-3 energy in the center-of-mass (c.m.) system. Let $t_{23}{}^l(s)$ be the lth partial wave amplitude for elastic 2-3 scattering, with phase δ. Then unitarity and time-reversal invariance imply Watson's theorem[1]: That part of the phase of $T_l(s)$ which arises from the 2-3 final-state interaction (f.s.i.) effects is δ, for s in the region of elastic 2-3 scattering. When combined with analyticity assumptions, this result implies[2] that $T_l(s)$ can be factored into a part which varies slowly with s, and a part (the "enhancement factor") which is rapidly varying, and which, to a good approximation, is proportional to $t_{23}{}^l(s)$. The $\gamma N\rightarrow \pi N$ reaction near the (3,3) elastic πN resonance is a well-known example of this effect.

What happens when there are *three* strongly interacting final-state particles? Phenomenologically it is certainly the case that two-body resonances do show up, essentially undistorted, in three-body final states (compare the N^* seen in $NN\rightarrow N^*N\rightarrow NN\pi$ with that in $\gamma N\rightarrow N^*\rightarrow N\pi$). The hope that this is always the case is indeed the basis of most measurements on unstable mesons, for example. But a detailed theoretical justification is still lacking. A full analysis is complicated, for a start, by the fact that a production process $A+B\rightarrow 1+2+3$ needs five kinematic variables to describe it. To make the problem as much like the $A+B\rightarrow 2+3$ case as possible, we can imagine fixing three of these variables (for instance, the total energy in the over-all c.m. system, and two invariant momentum transfers), and then performing, in the 2-3 c.m. system, a partial-wave analysis with respect to an angle variable, leaving ourselves with partial amplitudes $F_l(s)$ to consider.[3] We can now ask, what is the phase of $F_l(s)$?

Amado has recently conjectured[4] that, just as in the $A+B\rightarrow 2+3$ case, the phase of $F_l(s)$ arising from the f.s.i. is that of $t_{23}{}^l(s)$. At first sight this is a surprising thing to hope for. The work of Faddeev[5] showed the value of thinking of a three-particle amplitude as a sum of three pieces, each consisting of all possible rescatterings in which a given pair interact *last*. Surely it is only the piece $F_l{}^1(s)$, in which particles 2 and 3 interact last, that has a significant phase variation as a function of s, its phase being just δ. Amado shows that this is false: First, the multiple scattering formalism shows that although the final factor in $F_l{}^1$ is the 2-3 amplitude $t_{23}{}^l$, preceding this there is a factor G_0, the free three-particle Green's function, which does *not* have a slowly varying phase. Thus $G_0t_{23}{}^l$ does not have the phase δ. On the other hand, Amado shows that in the complete amplitude F_l, for every term ending in $G_0t_{23}{}^l$ there is another (which he calls the "feeding term"), identical except that it does not have the final

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[1] K. M. Watson, Phys. Rev. **88**, 1163 (1952); an excellent modern discussion is contained in Ref. 2.

[2] J. Gillespie, *Final State Interactions* (Holden-Day, Inc., San Francisco, Calif., 1964).

[3] It is true that this is somewhat artificial experimentally, since a projection on one axis of the Dalitz plot sums over all l; but if one 2-3 wave dominates (e.g., is resonant, or at low energies), our discussion is adequate.

[4] R. D. Amado, Phys. Rev. **158**, 1414 (1967).

[5] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1966) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

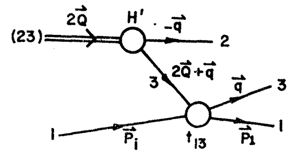

FIG. 1. The peripheral, or stripping, graph for the process $1+(23)\rightarrow 1+2+3$. The momenta are referred to in the text.

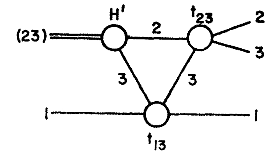

FIG. 2. The rescattering correction to Fig. 1.

factor $G_0t_{23}{}^l$. The *combination* therefore has a factor $(1+G_0t_{23}{}^l)$, and this, Amado shows, *does* have the phase δ. Amado's main point, in fact, is that it is the *coherent* combination of the "1" and the $G_0t_{23}{}^l$ which is essential to give the total amplitude a final factor with the phase δ; more specifically, it is the imaginary, or on-shell (δ-function) part of G_0 which combines with the "1" to produce the phase δ.

But this does not constitute a proof of the conjecture, as Amado points out, because the rest of the total amplitude (i.e., the bit preceding the $1+G_0t_{23}{}^l$ factor) is, in general, complex. The hope is that this part varies slowly with s. It is fairly clear that one cannot decide whether this is so or not by a general argument; it is a question of dynamics, i.e., of models. Our purpose here is to investigate Amado's conjecture in two simple, complementary models, to see how the presence of the third final-state particle may modify Watson's theorem. In both cases, as we shall see, the question of the phase of F_l is to some extent bound up with "triangle singularities"; we hope that our discussion will correct some confusions in the literature on this subject.

II. GENERAL FORMULATION

We use dispersion theory. In these terms, the general problem is to determine the function $F(s)$ (we drop the l now, taking s waves for simplicity) from the integral equation[2,6]

$$F(s)=B(s)+\frac{1}{\pi}\int_R \frac{g^*(s')F(s')}{s'-s-i\epsilon}ds', \quad (\epsilon\rightarrow 0) \qquad (1)$$

where $g=e^{i\delta}\sin\delta$ (our previous $\rho t_{23}{}^0$), and where $B(s)$ is some "driving term"[7] which does *not* have the right-hand cut R: $s\geq(m_2+m_3)^2$ of $g(s)$. Equation (1) just says that there are terms in F, namely the part F^1, which have the elastic-scattering cut R, with a discontinuity given by unitarity, and other terms $B(s)$ which do not have the cut R although they may be complex. The role of G_0 is played by the dispersion denominatory $(s'-s-i\epsilon)^{-1}$.

The solution of Eq. (1) may be written in either of the two following forms[6] (assuming convergence):

$$F(s)=\frac{C}{D(s)}+B(s)+\frac{1}{\pi D(s)}\int_R \frac{g(s')D(s')B(s')}{s'-s-i\epsilon}ds' \qquad (2)$$

or

$$F(s)=e^{i\delta}\Bigg\{C\left[\exp\left(\frac{P}{\pi}\int_R\frac{\delta(s')}{s'-s}ds'\right)\right] +B(s)\cos\delta+\left[\exp\left(\frac{P}{\pi}\int_R\frac{\delta(s')}{s'-s}ds'\right)\right] \times\left[\frac{P}{\pi}\int_R\frac{g(s')D(s')B(s')}{s'-s}ds'\right]\Bigg\}, \qquad (3)$$

where

$$D(s)=\exp\left(-\frac{1}{\pi}\int_R\frac{\delta(s')}{s'-s-i\epsilon}ds'\right)$$

is the denominator function of ρ, and P stands for "principal part." C is an arbitrary constant not determined by Eq. (1). Equation (3) is obtained from Eq. (2) by separating out the δ function and principal-value parts of the integral in (2), using the identity

$$\frac{1}{x-i\epsilon}\equiv\frac{P}{x}+i\pi\delta(x).$$

From Eq. (3) we can already draw one simple, and indeed venerable,[8] conclusion: *If* $B(s)$ *is real throughout* R, $F(s)$ *has the phase* δ. Hence in any model in which the "driving term" is real, F will have the phase δ *exactly*. On the other hand, if $B(s)$ is complex, no immediate deduction about the phase of F can be made.

To proceed further, we must therefore specify a particular $B(s)$. We will consider two cases—one a real $B(s)$, the other a complex one. In the first case we shall of course recover the result that the phase of F is δ; but the calculation is instructive, and complements (and corrects) the similar calculation presented by Amado in Sec. III of his paper. In the second case, our choice for $B(s)$ will produce essentially the model considered by Schmid[9]; it will turn out that there are circumstances in which the phase of F is definitely not δ.

III. FIRST MODEL: PERIPHERAL GRAPH AS DRIVING TERM

Consider the process $1+(23)\rightarrow 1+2+3$ where (23) is a bound state of particles 2 and 3. A familiar approximation to this amplitude is the peripheral, or "stripping," term shown in Fig. 1. We consider the s-wave projection of this in the 2-3 c.m. system, as our driving term $B(s)$; it is the driving term for Fig. 2, which is the

[6] R. Omnès, Nuovo Cimento 8, 376 (1958).
[7] We prefer this to "feeding term"; the choice is a matter of taste.

[8] See, for instance, G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957).
[9] C. Schmid, Phys. Rev. 154, 1363 (1967).

same as Fig. 1, except that a final 2-3 scattering is tacked on, so that we call it the rescattering correction to Fig. 1. Let us call the amplitudes of Figs. 1 and 2, respectively, N_d and N_δ, and denote the s-wave projection of N_d by $\tilde{N}_d$. The approximation $F=\tilde{N}_d+N_\delta$ is then the model considered by Amado in Sec. III of his paper.

In our dispersion theory formulation of Sec. II above, the terms $\tilde{N}_d$ and N_δ are, respectively, the second and third terms on the right-hand side of Eq. (2); C is set equal to zero. To avoid mistakes, it is best to work with explicit expressions for these terms. For Fig. 1, this is straightforward; for Fig. 2, we can get an explicit expression if we use nonrelativistic kinematics and assume that the 2-3 amplitude g is such that $gD=N\rho$, where ρ is the 2-3 phase space factor and N is a constant.[10] We let all the particles have equal mass m, and *assume that* t_{13}, *the 1-3 amplitude appearing in Figs. 1 and 2 is constant* (though not necessarily real). This last assumption is the key one in this model; the second model we consider will go to the other extreme and ignore the single-particle-exchange pole of Fig. 1 but will consider a *resonant* amplitude t_{13}. Our driving term is then (taking the s-wave projection in the 2-3 c.m. system of the pole term indicated in Fig. 1)

$$\tilde{N}_d=\frac{A}{4Qq}\ln\left(\frac{(q+Q+i\alpha)(q+Q-i\alpha)}{(q-Q+i\alpha)(q-Q-i\alpha)}\right), \quad (4)$$

where A stands for the product of the vertex (23) $\to 2+3$ and t_{13}. Also, α^2/m is the (23) binding energy, and $Q=\frac{1}{2}|\mathbf{p}_i-\mathbf{p}_1|$, where $\mathbf{p}_i$, $\mathbf{p}_1$ are the 3-momenta of the incident particle 1 and of the final particle 1 in the 2-3 c.m. system; q is the magnitude of the 3-momentum of particle 2 or 3 in the 2-3 c.m. system.

For Fig. 2, i.e., the third term on the right-hand side of (2), we obtain[11]

$$N_\delta=\frac{igA}{2Qq}\ln\left(\frac{q+Q+i\alpha}{q-Q+i\alpha}\right). \quad (5)$$

Our model is now to let $F(s)=\tilde{N}_d+N_\delta$. As can be seen from (4) and (5), both $\tilde{N}_d$ and N_δ actually depend on Q as well as on q (i.e., s). However, we shall consider Q to be fixed, which is why only s has been written as the argument of F.

[10] A detailed discussion of the relation between rescattering corrections in dispersion theory and Feynman triangle graphs is given by I. J. R. Aitchison, Nuovo Cimento **35**, 434 (1965). The relevant nonrelativistic triangle graph for our present purposes is discussed in this reference, and in the works cited in Ref. 11.

[11] See, for example, R. Karplus and L. S. Rodberg, Phys. Rev. **115**, 1058 (1959); also V. V. Komarov and A. M. Popova, Zh. Eksperim. i Teor. Fiz. **45**, 214 (1963) [English transl.: Soviet Phys.—JETP **18**, 151 (1964)]; and A. M. Popova and V. V. Komarov, Nucl. Phys. **A90**, 625 (1967). The result is derived by evaluating the integral in Eq. (2) in the nonrelativistic limit.

Inspecting Eqs. (4) and (5), we see that we can rewrite F as

$$F=e^{i\delta}\left[N_d\cos\delta+\frac{iA}{4Qq}\ln\left(\frac{(q+Q+i\alpha)(q-Q-i\alpha)}{(q-Q+i\alpha)(q+Q-i\alpha)}\right)\right]. \quad (6)$$

In this form it is not immediately obvious that F has the phase δ. But the argument of the logarithm in (6) has modulus unity, while that of the logarithm in $\tilde{N}_d$ has zero phase, so that (6) can also be written as

$$F=\frac{Ae^{i\delta}}{4Qq}\left[\cos\delta\ln\left(\frac{(Q+q)^2+\alpha^2}{(Q-q)^2+\alpha^2}\right)+2\sin\delta\left(\tan^{-1}\frac{\alpha}{|q-Q|}-\tan^{-1}\frac{\alpha}{q+Q}\right)\right]. \quad (7)$$

It follows that only if A is real will the phase of F be δ, but in so far as A is approximately constant as a function of s, the phase variation of F will be given by that of δ.

We would now like to comment on Eq. (7), which is to be compared with Eq. (16) of Amado's paper, from the point of view of triangle singularities. The first term on the right-hand side of Eq. (7) comes from combining the driving term $\tilde{N}_d$ with the δ-function part of the $(s'-s-i\epsilon)$ denominator in Eq. (2): It is the "1+on-shell part of G_0" for Amado. This combination results in a factor $\cos\delta$ multiplying a term which has a logarithmic singularity ("triangle" singularity) close to the physical region at $q=Q-i\alpha$. Notice that this singularity is present in the complete expressions for both $\tilde{N}_d$ and N_δ. Amado claims that it follows that if the 2-3 amplitude resonates, the singularity will disappear, being multiplied by $\cos\delta\approx 0$. This is not true, however; for although *this* term is of course zero if $\cos\delta=0$,[12] the *other* term in Eq. (7) is *also* singular at $q=Q-i\alpha$, as can easily be verified. In other words, the *off-shell* [principal-value part of $(s'-s-i\epsilon)^{-1}$] contribution to N_δ is *also* singular at $q=Q-i\alpha$: The triangle singularity is *not* uniquely contained in the on-shell (δ-function) part. Further, since the off-shell part is multiplied by $\sin\delta$, it will certainly be present even if $\delta=\frac{1}{2}\pi$.

This seems to be a new, or at least little-known result. It is amusing to check explicitly what is going on. The on-shell part of N_δ is, from Eq. (2),

$$N_{\delta,\mathrm{on}}=ig\tilde{N}_d, \quad (8)$$

so that the total on-shell contribution to F is

$$F_{\mathrm{on}}=\tilde{N}_d+N_{\delta,\mathrm{on}}=e^{i\delta}\cos\delta\,\tilde{N}_d.$$

But subtracting (8) from (5) to obtain the remainder

[12] The fact that the on-shell part of the rescattering term exactly cancels the driving term at $\delta=\frac{1}{2}\pi$ leaving only the off-shell rescattering, is actually an old result; it was noted by P. A. Carruthers, Ann. Phys. (N. Y.) **14**, 227 (1961); and by C. J. Goebel and H. J. Schnitzer, Phys. Rev. **123**, 1021 (1961).

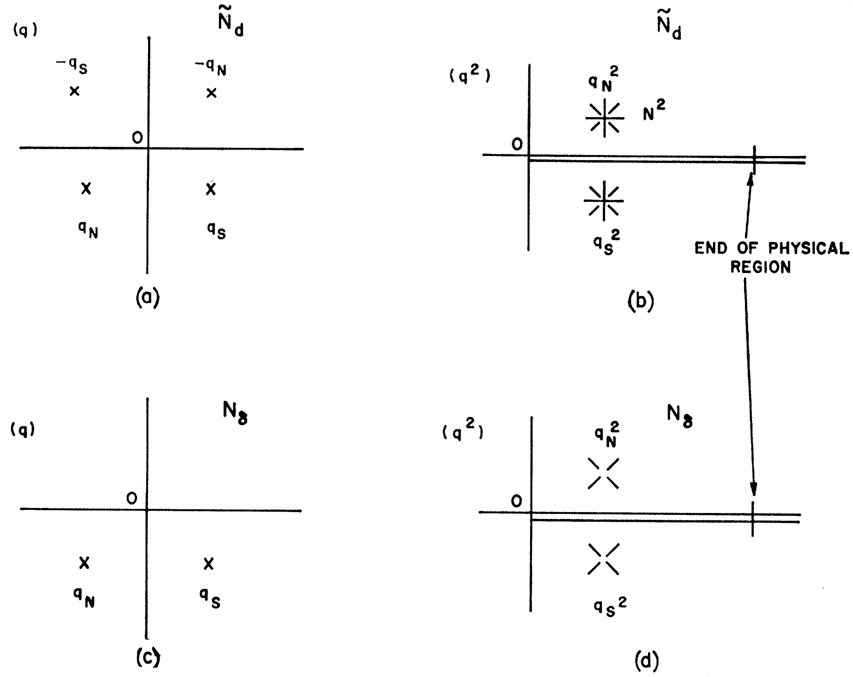

FIG. 3. The positions of the singularities of $\tilde{N}_d$ and N_δ. (a) Singularities of $\tilde{N}_d$ in the q plane; (b) singularities of $\tilde{N}_d$ in the q^2 plane; (c) singularities of N_δ in the q plane; (d) singularities of N_δ in the q^2 plane. In the q^2 (s) planes, broken diagonal crosses refer to the second sheet in s, Im$q<0$. Note that $q_S^*=-q_N$, and $q_N^*=-q_S$.

of N_δ, namely the off-shell part $N_{\delta,\text{off}}$, we find

$$N_{\delta,\text{off}}=\frac{igA}{4Qq}\ln\left(\frac{(q+Q+i\alpha)(q-Q-i\alpha)}{(q-Q+i\alpha)(q+Q-i\alpha)}\right),$$

which is also singular at $q=Q-i\alpha$. Furthermore, near the singular point $q=Q-i\alpha$, the contributions $N_{\delta,\text{on}}$ and $N_{\delta,\text{off}}$ are exactly *equal.*

In a "Note added in proof" at the end of Sec. III of his paper, Amado acknowledges that the principal-value part of the triangle-graph integral also has the triangle singularity. But he deduces from this, and the work of Schmid[9] (see Sec. IV below) that "near" the triangle singularity the behavior of the amplitude will be essentially $e^{2i\delta}\tilde{N}_d(s)$, and hence that since $|e^{2i\delta}\tilde{N}_d(s)|^2=|\tilde{N}_d(s)|^2$ no special effect of the singularity will be felt, whether $\delta=\frac{1}{2}\pi$ or not. This deduction is false. The fallacy lies in the word "near": In discussing the question of the observability of the triangle singularity, one has to restrict q to the real axis. Then for q real and near Q (i.e., as near $Q-i\alpha$ as we can get), we have

$$F\sim-\frac{A}{4Qq}\ln[(Q-q)^2+\alpha^2]-\frac{igA}{2Qq}\ln(q-Q+i\alpha). \quad (9)$$

In (9), the first term comes from $\tilde{N}_d$, the second from N_δ; both are singular at $Q-i\alpha$, but the first is *also* singular at the *equally near* point $Q+i\alpha$. Thus we cannot meaningfully isolate the point $Q-i\alpha$ as far as the complete amplitude F on the real q axis is concerned. As (9) shows, when we remember this we see that the true triangle singularity (that in N_δ) is in principle observable by distinguishing between the two terms in (9). This is true, despite the fact that, as (9) shows, for *complex* q near $Q-i\alpha$, the amplitude F is indeed $\sim e^{2i\delta}B(s)$.

In summary, then, our investigation of this model (which parallels Amado's Sec. III) shows that, as expected, *if* the driving term $B(s)$ is real the phase of F is δ, and that *if* A (i.e., t_{13}) is constant the phase variation of F is given by that of δ, as Amado claims. On the other hand, we do not accept his statements about the observability of triangle graphs.

We have seen that near the singularity $q=Q-i\alpha$ of N_δ the on- and off-shell parts of N_δ contribute equally. We shall now show that this is a general result, using the proper relativistic expression in Eq. (2); indeed the result is another form of a theorem first proved by Schmid.[9]

IV. THEOREM ABOUT TRIANGLE SINGULARITIES

We consider the third term on the right-hand side of Eq. (2); letting, as before, $gD=N\rho$ with N a constant (or a function which has singularities very far from the cut R). The singularities of this term are precisely those of the Feynman amplitude for the triangle graph pictured in Fig. 2.[10] Such a graph has two singularities in the variable q (in addition to the usual threshold branch point $q=0$), conventionally denoted by q_S and q_N. (In terms of our explicit nonrelativistic expression (5), $q_S=Q-i\alpha$, $q_N=-Q-i\alpha$.) In general, only one of these q_S may be singular near the physical region (i.e., in the lower-half q plane with Re$q>0$). Introducing $s_S=4m^2+4q_S{}^2$ and $s_N=4m^2+4q_N{}^2$, s_S and s_N are, when s_S is near the physical region, disposed as

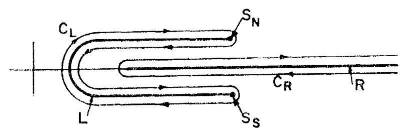

FIG. 4. The cuts R and L and the contours C_R and C_L for the function $\chi(s)$.

in Fig. 3. [The q^2 and s variables and planes are related by $s=4(m^2+q^2)$.] Now let $\chi=gDB$; where in this model still, $B(s)=N\theta(s)$. We prove the following theorem.

Theorem:

$$i\chi(s)\approx\frac{P}{\pi}\int_R\frac{\chi(s')}{s'-s},\quad \text{for}\quad s\approx s_S \tag{10}$$

when s_S is a singularity near the physical region (i.e., Im$q_S<0$, Re$q_S>0$).

Proof: The analytic properties of $\chi(s)$ are as follows: It has the cut R (due to gD, not $\tilde{N}_d$) and it has all the singularities of $\tilde{N}_d$, namely logarithmic singularities at q_S, $q_S{}^*$, q_N, and $q_N{}^*$ [c.f. Eq. (4), it is not hard to convince oneself of this in general] as shown in Fig. 3. The left-hand singularities of gD, namely those of N, are neglected, in accordance with our assumption that they are far from R. Since $\tilde{N}_d$ has singularities at s_S, s_N, we define it by means of a branch cut joining s_S to s_N, called L, as shown in Fig. 4. Then by Cauchy's theroem

$$\chi(s)=\frac{1}{2\pi i}\left\{\int_{C_R}+\int_{C_L}\right\}\frac{\chi(s')ds'}{s'-s},$$

where C_R, C_R are shown in Fig. 4. Now the cut R is square root in nature (from the phase-space factor q), so that

$$\frac{1}{2\pi i}\int_{C_R}\frac{\chi(s')ds'}{s'-s}=\frac{1}{\pi i}\int_R\frac{\chi(s')ds'}{s'-s-i\epsilon},$$

letting s approach the real axis from above, to obtain the physical amplitude. Thus to prove the theorem we have to show that, near s_S,

$$\chi(s)\approx\frac{-1}{2\pi i}\int_{C_L}\frac{\chi(s')ds'}{s'-s-i\epsilon}. \tag{11}$$

But near s_S, the integral over C_L picks out that part of the integrand which itself is singular at s_S, viz. $\tilde{N}_d$. Hence near s_S we can take the remaining part gD outside the integral, remembering, however, that it is to be evaluated *below* the cut R, and hence must carry a minus sign.

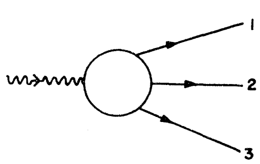

FIG. 5. A production process in which the momentum-transfer dependence is suppressed.

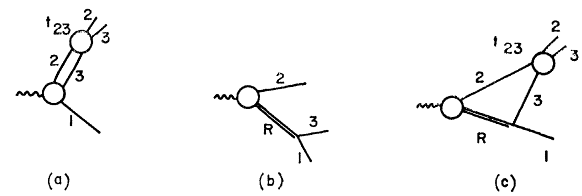

FIG. 6. Contributions to the process of Fig. 3. (a) A 2-3 final state interaction; (b) a resonant 1-3 final-state interaction; (c) rescattering correction to (b).

Thus, near s_S,

$$\frac{1}{2\pi i}\int_{C_L}\frac{\chi(s')ds'}{s'-s}\approx\frac{1}{2\pi i}[-g(s)D(s)]\int_{C_L}\frac{B(s')ds'}{s'-s}$$
$$=gDB \text{ by Cauchy's theorem for } C_L$$
$$=\chi(s)$$

proving (11) and hence (10).|

Since the on-shell part of the integral

$$I=\frac{1}{\pi}\int_R\frac{\chi(s')ds'}{s'-s-i\epsilon}$$

is exactly $i\chi(s)$, we obtain a simple corollary.

Corollary: The on- and off-shell parts of a triangle graph contribute equally at a triangle singularity when the latter is near the physical region. This result will hold whenever a triangle graph has a singularity q_S with Re $q_S>0$ and Im$q_S<0$. In particular, it is clearly not at all necessary that s_S should equal $s_N{}^*$.|

From this it follows that the total contribution of the integral I near s_S is just $2i\chi(s)$, so that near s_S the complete amplitude $F\sim B+2i\chi/D=B+2igB=e^{2i\delta}B$ [cf. Eq. (2)]. This result has been derived previously by Schmid, who emphasized the essential point that the rescattering correction to B contains B as its discontinuity.[13]

Since $2i\chi(s)$ is just the discontinuity of I across R, we can also restate the corollary in the form in which Schmid himself presented the result: Near s_S, I is equal to its own discontinuity. In this form the physical interpretation is clearer. The rescattering correction N_δ is given by

$$N_\delta=\frac{1}{D}\frac{1}{\pi}\int_R\frac{N\rho B\, ds'}{s'-s-i\epsilon}\approx 2i\frac{N}{D}N_\delta{}',\quad N_\delta{}'=\frac{1}{\pi}\int_R\frac{(\rho B/2i)ds'}{s'-s-i\epsilon}$$

for slowly varying N. Now $N_\delta{}'$ is exactly the Feynman graph of Fig. 2, in which the 2-3 amplitude is merely ρ, as follows from Cutkosky's rules. But near the singularity s_S of Fig. 2, the intermediate particles propagate freely, i.e., over arbitrarily large space-time distances. Hence we expect that a measurement of the process described by $N_\delta{}'$ near s_S, would yield the result that

[13] Schmid's own derivation of this result was within the framework of the model we shall describe in Sec. V.

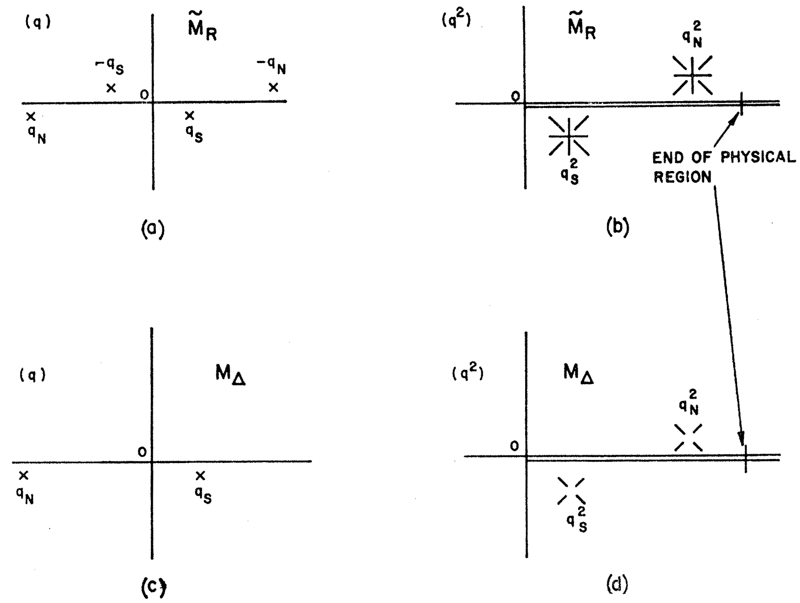

FIG. 7. The positions of the singularities of $\tilde{M}_R$ and of M_Δ for the case in which q_S is near the physical region (rescattering kinematically allowed). (a) Singularities of $\tilde{M}_R$ in the q plane; (b) singularities of $\tilde{M}_R$ in the q^2 plane; (c) singularities of M_Δ in the q plane; (d) singularities of M_Δ in the q^2 plane. In the q^2 planes, broken diagonal crosses refer to the second sheet in s, Im$q<0$.

the amplitude *factorizes* into two parts: the first, B, describing the knock-out of particle 3, and the second, ρ, describing the final 2-3 scattering. That is, at s_S we expect the physical sequential process to have an amplitude $N_\delta'=B\rho$. But at the singularity we measure the singular part of the amplitude; hence on physical grounds we expect the singular part of the complete amplitude to be $2i(N/D)B\rho=2igB$, exactly the result obtained. The essential qualitative point is that near s_S the graph is determined by its discontinuity, which, by Cutkosky's rules, is related to products of physical amplitudes.

It was Schmid's result $F\sim e^{2i\delta}B$ to which we referred in Sec. III above, and which led Amado, in his "Note added in proof," to suggest that there would be no observable effect of the triangle singularity in Fig. 2. As we have seen, this is false: Indeed, if it were not we should have the paradoxical result that on the real axis "near" s_S the phase of F is not δ but 2δ, if B is real, whereas we have already proved that it is always δ in that case. As explained above, the error lies in forgetting the singularity at $Q+i\alpha$ of $B(s)=\tilde{N}_d$, which is as near the real q axis as is $Q-i\alpha$.

It is, however, quite easy to construct a model where Schmid's form of the theorem does play an important role in the question of the phase of F. We simply start from a B which does *not* have two singularities equally near the real axis. We turn therefore to our second model, which is of this kind.

V. SECOND MODEL [PARALLEL FINAL-STATE RESONANCE FOR $B(s)$], AND APPLICATION OF THE THEOREM

The essential feature of this model is that in it we pay attention to the fact that the scattering amplitude of a pair of particles in the final state *other* than 2-3 may be strongly energy-dependent, while we ignore production mechanisms entirely. In other words, we are now going to let the amplitude t_{13} in Fig. 1 vary rapidly as a function of the 1-3 c.m. energy, ignoring the momentum-transfer pole in the figure. This model therefore complements the first one (for simplicity only, we are assuming that the pair 1-2 do not interact). The complete process is now represented pictorially as Fig. 5, since all momentum-transfer dependence is suppressed.

Returning again to our basic solution [Eq. (2)] the various terms on the right-hand side are now reinterpreted as follows. The first represents the production of particles 1, 2, and 3 into the final state ($C\neq 0$) with final-state interaction (D^{-1}) between 2 and 3; this is shown in Fig. 6(a). The second term, $B(s)$ represents the s-wave projection, in the 2-3 c.m. system of the "parallel channel resonance" shown in Fig. 6(b); i.e., here the three particles are produced and particles 1 and 3 scatter resonantly in the final state. The third term, shown in Fig. 6(c), represents the rescattering correction to Fig. 6(b) and is precisely the triangle Feynamn graph indicated, with an internal resonance.[14]

Once again we can, using nonrelativistic kinematics, obtain explicit expressions for all these terms. Figure 6(a) is already known, and calling the s-wave projection of Fig. 6(b) $\tilde{M}_R$, we find

$$\tilde{M}_R=\frac{C_2'\gamma}{4pq}\ln\left(\frac{[q-q_N+\xi(p-p_N)][q+q_S+\xi(p-p_S)]}{[q-q_S-\xi(p-p_S)][q+q_N-\xi(p-p_N)]}\right), \tag{12}$$

where we are following the notation of Ref. 14—

[14] For all the details associated with such graphs we refer to I. J. R. Aitchison and C. Kacser, Phys. Rev. **142**, 1104 (1966); see also *ibid.*, **152**, 1518(E) (1966).

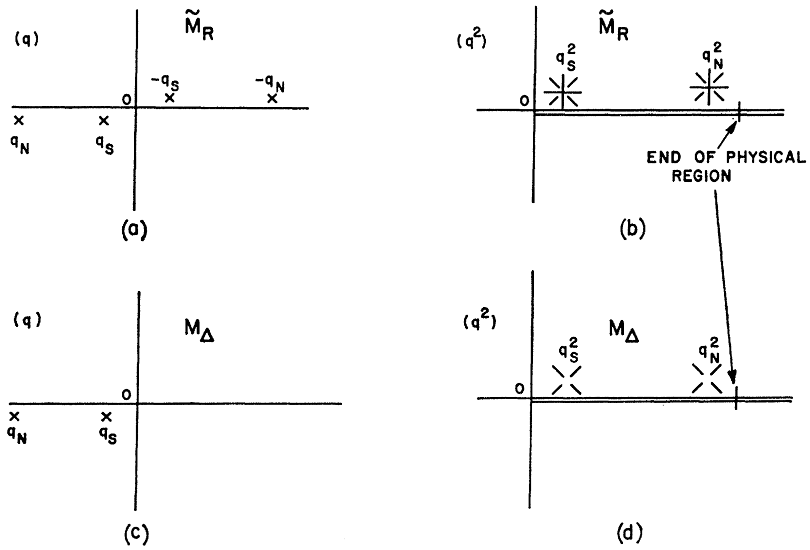

FIG. 8. As in Fig. 7, but for the case in which q_S moves away from the vicinity of the physical region (rescattering kinematically forbidden). (a), (b), (c), and (d) as in Fig. 7.

that is, C_2' is a constant representing the production amplitude for producing the resonance in t_{13} in the final state, γ is related to the width of the resonance, p is the magnitude of the momentum of particle 1 in the 2-3 c.m. system, and $\xi=m_1/(m_1+m_2+m_3)$. In this formula, $p_N=p(q_N)$, $p_S=p(q_S)$, $p(-q)=p(q)$, and the positions of the singularities q_S and q_N are complicated functions of the masses and total energy[14]; they are the singularities of the triangle graph of Fig. 6(c), and that is why we have kept the same labels for them as we used for the singularities of Fig. 2—the algebraic details being of course different in the two cases.

For the amplitude represented by Fig. 6(c), call it M_Δ, we have given the expression[14]

$$M_\Delta=\frac{igC_2'\gamma}{2pq}\ln\left(\frac{q-q_N+\xi(p-p_N)}{q-q_S-\xi(p-p_S)}\right). \qquad (13)$$

Thus for the complete amplitude in this model we have

$$F(s)=C/D(s)+\tilde{M}_R+M_\Delta\,,$$

with $\tilde{M}_R$, M_Δ given by (12) and (13). (In practice, an "over-all background" constant C_0 might have to be added also.)

What is the phase of F in this model? The essential point is that now the driving term is *not* real. This is because the location of the branch points of $\tilde{M}_R$, $\pm q_S$, $\pm q_N$ are such that, unlike the previous case of $\tilde{N}_d$, $q_S^*\neq q_N$. Thus even without the rescattering correction M_Δ, the phase of F will not be δ. Of course, this is not serious; indeed it corresponds to the well-known phenomenon of "reflected resonances"—namely, the projections on one channel of the Dalitz plot of true resonances in other channels; this is just what $\tilde{M}_R$ is. In all isobar-model analyses such terms are always included; but it is important to realize that these terms already violate the simple extension of Watson's theorem proposed by Amado.

The rescattering correction does, however, produce one definite modification to this simple picture. The quantity M_Δ has singularities at q_S and q_N, of which only q_S can ever be near the physical region ($\mathrm{Re}q_S>0$), $\mathrm{Im}q_S<0$), and that only for a limited range of total c.m. energies. The prescription [14,15] is that if the total energy is such that Fig. 6(c) can occur as a real process, q_S is near the physical region, while q_N is far away; for larger values of the total energy q_S leaves the vicinity of the physical region.

The positions of the singularities of $\tilde{M}_R$ and of M_Δ are shown in Figs. 7 and 8. It is helpful to consider the situation on the Dalitz plot also; this is shown in Fig. 9. The low- (high-) energy intersection of the resonance band R with the Dalitz plot boundary is essentially $\mathrm{Re}s_S$ ($\mathrm{Re}s_N$); only if the band R cuts the plot on the top left-hand arc will s_S be near the physical region.[14,15]

Since $q_S^*\neq q_N$, it does make sense, now, to ask about the behavior of $F(s)$ for real s near s_S. Applying the theorem of Sec. IV (which is equally applicable in this case; cf. the corollary), we see that near s_S, when s_S is near the physical region, the sum of $\tilde{M}_R$ and M_Δ must be $e^{2i\delta}\tilde{M}_R$. This is exactly Schmid's result. The total amplitude is then

$$F\sim\frac{C}{D(s)}+e^{2i\delta}\tilde{M}_R \qquad (14)$$

near s_S, when s_S is an (isolated) nearby singularity. On the other hand, without M_Δ we should have simply

$$F\sim\frac{C}{D(s)}\tilde{M}_R. \qquad (15)$$

We see that the triangle graph modifies the *interference*

[15] S. Coleman and R. E. Norton, Nuovo Cimento 38, 438 (1965); the result for the triangle graph is in the paper by J. B. Bronzan, Phys. Rev. 134, B689 (1964).

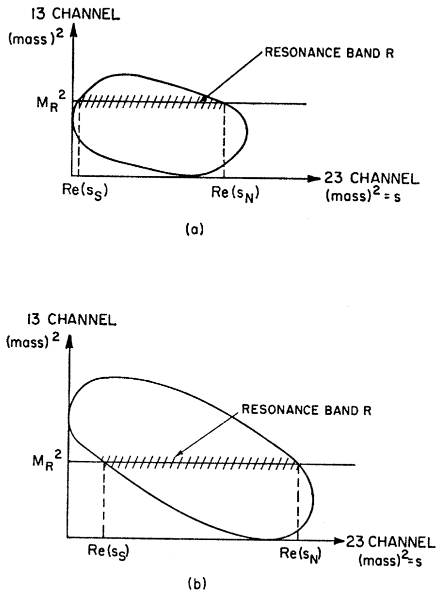

FIG. 9. The singularities on the Dalitz plot; the intersections of the resonance band R with the boundary of the plot are, approximately, the real parts of s_S and s_N. (a) The case in which the total c.m. energy is such that s_S is near the physical region (c.f. Fig. 7); (b) a larger c.m. energy, such that s_S has moved away from the vicinity of the physical region (cf. Fig. 8).

between the 2-3 and 1-3 contributions in the vicinity of s_S. We feel that Schmid was too negative when he implied that *no* effect peculiar to M_Δ would be observed in $|F|^2$ (as would indeed be the case if $C=0$: $|e^{2i\delta}\tilde{M}_R|^2=|\tilde{M}_R|^2$).

We may check explicitly, as before, that near s_S the on- and off-shell parts of M_Δ contribute equally.

In summary then, the fact that the projection of a parallel-channel resonance is complex (and energy-dependent) is one, well-known, reason why Watson's theorem will not hold. An additional reason is the presence of the rescattering term M_Δ; this, though complex, is slowly varying in the physical region, except when s_S is nearby. When it is, Eq. (14) shows that M_Δ modifies the phase of the (projected) parallel-channel resonance, and hence affects the interference between the two channels. In the interference region, no simple statement can be made about the over-all phase, but Eq. (14) provides a formula to describe the situation. One might wonder when this modification is likely to be most important. From (14) we see that the greatest effect will come when the factor $e^{2i\delta}$ varies rapidly near s_S. But, referring to Fig. 9(a), s_S is always towards the low-energy end of the spectrum. Thus to see the M_Δ effect best, we need a strong low-energy 2-3 interaction coupled with a 1-3 resonance cutting the Dalitz plot as in Fig. 9(a).[16] Quite possibly such circumstances arose in the ABC experiment.[17]

[16] I. J. R. Aitchison, Nuovo Cimento **51A**, 249 (1967); **51A**, 272 (1967).
[17] V. V. Anisovich and L. G. Dakhno, Phys. Letters **10**, 221 (1964).

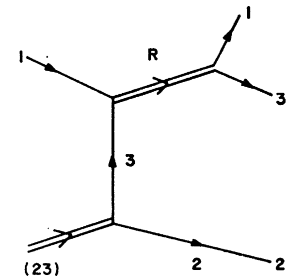

FIG. 10. A more realistic driving term.

VI. SUMMARY

We began by reiterating the old result that a final-state rescattering correction to a real driving term produces an amplitude with the scattering phase. We then examined how this came about explicitly in a simple model, which concentrated on the production mechanism whereby the particles were produced into the final state, ignoring the phase of the parallel-channel amplitude (or assuming it to be slowly varying, at least). The corresponding driving term was then real, so that the previous result held (or held as regards the variation in phase). We confirmed that the on-shell part of the $1+G_0t_{23}$ factor of Amado did produce the term $\cos\delta\times$(triangle singularity), as he claimed, but we also found that the singularity was present equally in the off-shell part (which went with $\sin\delta$), and therefore was observable, in principle, in this model even if $\delta=\frac{1}{2}\pi$.

We then proved that, quite generally, on- and off-shell parts of triangle graphs contribute equally at the triangle singularity when it is near the physical region. This turned out to be a restatement of an earlier result of Schmid, and we used it, as Schmid did, to examine the phase question in a second model, which concentrated, in contradistinction to the first model, precisely on the phase variation of the parallel-channel amplitude assuming, in fact, that it was resonant. We found that here the simple generalization of Watson's theorem does not hold. Partly this was because of the (trivial) fact that the projection of Fig. 6(b) has a varying phase. A nontrivial correction comes from Fig. 6(c), which, though generally small, may contribute significantly in certain circumstances, which we described.

We should like, in conclusion, to stress that the differences between the two models are entirely due to the different aspects of Fig. 1 that each treats. The first model contracts the t_{13} blob to a point and retains only the single-particle pole (the production mechanism); the second contracts the single-particle line to a point and replaces the t_{13} blob by a resonance. Doubtless a better model would start from a B such as Fig. 10. This is not the only possibility, however, as

Deck[18] and others have pointed out, the assumption of extreme forward peaking of the t_{13} amplitude leads the 2-3 c.m. projections of Fig. 1 to be peaked in the low-s region, producing, or quite possibly distorting, low-mass enhancements in the 2-3 channel. Further work on rescattering corrections to the Deck mechanism is desirable.

[18] R. T. Deck, Phys. Rev. Letters **13**, 169 (1964).

ACKNOWLEDGMENT

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Threshold Electropion Production from Current Algebra and Partially Conserved Axial-Vector Current

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Threshold electropion production on nucleons, $e+N \rightarrow e+N+\pi$, is studied by current-algebra techniques using the hypothesis of partially conserved axial-vector current, which have proved useful in describing low-energy meson-baryon elastic scattering and photopion production on nucleons. The electric and longitudinal multipole moments E_{0+} and L_{0+} are calculated at threshold in terms of the form factors of the electromagnetic and weak axial-vector currents. The experimental upper bounds on the slope of the differential cross section as a function of $|\mathbf{q}|$, the momentum in the πN c.m. system, i.e., $(1/|\mathbf{q}|)(d^2\sigma/d\Omega dS_{20}{}^L)$, where $S_{20}{}^L$ is the laboratory energy of the final electron, are sufficiently strong to relate the form factors for various values of $-k^2$, the momentum transfer squared of the electrons. More precisely, in this way one can relate the neutron charge form factor $G_e{}^n(k^2)$ to normalized axial-vector form factor $F_A(k^2)$. If one takes $F_A(k^2)$ to have the dipole form $F_A(k^2)=(1+k^2/M_A{}^2)^{-2}$ with $M_A{}^2=1.42$ BeV^2, which is given by arguments based on chiral $SU(2)\times SU(2)$ and consistent with recent neutrino experiments, then the resulting values of $G_e{}^n(k^2)$ in the range considered, $0.2\leq k^2\leq 0.6$ BeV^2, are consistent with information about $G_e{}^n(k^2)$ from electron-deuteron and thermal-neutron–electron scattering.

I. INTRODUCTION

In the past few years much activity in elementary-particle physics has been devoted to the complete exploitation of the principle that the equal-time commutators of the weak and electromagnetic currents of the strongly interacting particles form a chiral $SU(2)\times SU(2)$ algebra.[1] One of the most fruitful branches of these researches has been the investigation of low-energy processes involving these currents. The current algebra together with the hypothesis of partially conserved axial-vector current (PCAC) leads to simple models which compare remarkably well with the presently available data on meson-baryon scattering.[2] These methods have also been applied to the case of pion photoproduction.[3] Here we consider the extension to pion electroproduction.

Electroproduction provides an interesting problem both theoretically and experimentally, and has received considerable attention. Earlier analysis utilized the static model and stressed the importance of the electroproduction process in describing the nucleon form factors.[4] Fubini, Nambu, and Wataghin[4] (FNW) noted

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‡ Supported in part by the National Science Foundation.

[1] For a complete review of the references on current algebras, see S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968); and B. Renner, *Current Algebras and Their Applications* (Pergamon Press, Inc., New York, 1968).

[2] A. P. Balachandran, M. G. Gundzik, and F. Nicodemi, Nuovo Cimento **44**, 1257 (1966); Y. Tomazawa, *ibid.* **46**, 707 (1966); K. Raman and E. C. G. Sudarshan, Phys. Letters **21**, 450 (1966); Phys. Rev. **154**, 1499 (1967); S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[3] A. P. Balachandran, M. G. Gundzik, P. Narayanaswami, and F. Nicodemi, Ann. Phys. (N.Y.) **45**, 339 (1967); M. S. Bhatia and P. Narayanaswami, Phys. Rev. (to be published).

[4] G. F. Chew, F. Low, M. L. Goldberger, and Y. Nambu, Phys. Rev. **106**, 1345 (1957); S. Fubini, Y. Nambu, and A. Wataghin, *ibid.* **111**, 329 (1958); R. Blankenbecler, S. Gartenhaus, R. Huff, and Y. Nambu, Nuovo Cimento **42**, 775 (1960); P. Dennery, Phys. Rev. **124**, 2000 (1961).