S-Matrix Approach to Internal Symmetries. II*

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An S-matrix formulation of internal symmetries reported in a previous paper is extended to the case of broken symmetries. It was shown before that crossing and unitarity can determine permitted internal symmetries by determining what constant real orthogonal matrices can diagonalize an S matrix. It is shown here that a generalization of the S-matrix diagonalization postulate which corresponds to patterns of symmetry breaking is an S-matrix stationary principle. In a calculation retaining only first-order deviations from the limit of exact symmetry, it is shown that crossing and unitarity determine what linear relations between S-matrix elements can remain stationary under a perturbation from the symmetry limit. Amplitude relations of conventional broken symmetries are thus derived for the breaking of isotopic-spin symmetry and unitary symmetry in the scattering of pseudoscalar mesons from pseudoscalar mesons. The application of unitarity on these amplitude relations leads to elegant derivations of mass formulas.

I. INTRODUCTION

A CONSIDERABLE understanding of the essential assumptions which go into dynamical derivations of internal symmetries was achieved in a recent work by Blankenbecler, Coon, and Roy.[1] It was shown there that rigorous application of unitarity and crossing can determine what constant real orthogonal matrices can diagonalize an S matrix. The predictions obtained in this way coincide with those conventionally obtained by assuming invariance under certain Lie groups. It was also shown that in previous approximate dynamical derivations[2] predictions of results of internal symmetry are actually due to the above-mentioned S-matrix diagonalization postulate and further approximations and assumptions such as the bootstrap hypothesis are inessential. The next natural question to ask is: What postulate concerning the S-matrix elements would lead, by a rigorous application of unitarity and crossing, to the predictions of patterns of symmetry breaking in the same way as the S-matrix diagonalization postulate leads to the predictions of exact internal symmetries? Indeed, the power of the S-matrix approach would be very limited if it were not a natural way of understanding broken symmetries, since the agreement with experiment of broken-$SU(3)$ predictions such as mass and coupling-constant sum rules[3] constitutes a major aspect of the usefulness of the symmetry concept.

It is clear that a new type of postulate is needed. When the diagonalization postulate is relaxed, the internal-symmetry amplitude relations are no longer valid, and unitarity and crossing alone do not predicate any pattern of symmetry breaking. A possibility that suggested itself to us, on examination of the conventional approach to broken symmetries, is that the interaction may carry remnants of the symmetry in such a way that while each amplitude is perturbed from the internal-symmetry limit, certain linear combinations of scattering amplitudes remain stationary under the symmetry-breaking perturbation.[4] The present paper demonstrates, retaining only first-order deviations from the symmetry limit, that an S-matrix stationary principle leads via unitarity and crossing to amplitude relations usually derived from group-theoretical broken-symmetry hypotheses. We therefore propose that the S-matrix stationary principle may be taken to be the meaning of the existence of a broken internal symmetry in the S-matrix language in much the same way as the existence of constant real orthogonal transformations diagonalizing the S matrix may be taken to define the existence of exact internal symmetries.[1]

One special feature of the present treatment of broken symmetries is that, since kinematics is an essential part of the use of unitarity and crossing, we are obliged to specify the arguments of the scattering amplitudes that satisfy a certain linear relation, and we know what kinematic factors to use (reliable up to first order in the perturbation) in comparing these relations with experiments, for example, via the optical theorem or by means of polygonal inequalities for differential cross sections.[5] There has been much discussion in the literature on this point and empirical guidelines have been proposed.[5] Our discussion will clarify this point. We shall mention here two striking results which

* Most of this work was done while the author was at the University of California, San Diego, La Jolla, Calif.

[1] R. Blankenbecler, D. D. Coon, and S. M. Roy, Phys. Rev. 156, 1624 (1967), hereafter referred to as BCR.

[2] These are listed in BCR.

[3] M. Gell-Mann, California Institute of Technology Report No. CTSL-20, 1961 (unpublished); S. Okubo, Progr. Theoret. Phys. (Kyoto) 27, 949 (1962); S. Coleman and S. L. Glashow, Phys. Rev. Letters 6, 423 (1961); E. C. G. Sudarshan, University of Rochester Report No. NYO-10268 (unpublished); M. Muraskin and S. L. Glashow, Phys. Rev. 132, 482 (1963); V. Gupta and V. Singh, *ibid.* 135, B1442 (1964).

[4] A group-theoretical stability principle for tensor operators under perturbations has been considered by E. C. G. Sudarshan and N. Mukunda, Phys. Rev. 158, 1424 (1967).

[5] C. A. Levinson, H. J. Lipkin, and S. Meshkov, Phys. Letters 1, 44 (1962); S. Meshkov, C. A. Levinson, and H. J. Lipkin, Phys. Rev. Letters 10, 100 (1963); H. J. Lipkin, C. A. Levinson, and S. Meshkov, Phys. Letters 7, 159 (1963); S. Meshkov, G. A. Snow, and G. B. Yodh, Phys. Rev. Letters 12, 87 (1964); H. Harari and H. J. Lipkin, *ibid.* 13, 208 (1964); P. G. O. Freund, H. Ruegg, D. Speiser, and A. Morales, Nuovo Cimento 25, 307 (1962); H. Harari and H. J. Lipkin, Phys. Rev. Letters 15, 983 (1966); S. Meshkov and G. B. Yodh, *ibid.* 18, 474 (1967); M. Konuma and K. Tomozawa, *ibid.* 12, 493 (1964).

illustrate that the questions of broken (or exact) symmetry predictions for amplitudes and for masses cannot be considered separately. First, in our approach mass formulas emerge as simple consequences of unitarity applied to the amplitude relations themselves. This is to be constrated with the usual derivation, which proceeds by assigning a group-theoretical transformation property [an $SU(3)$ transformation property in the case of the Gell-Mann-Okubo mass formula] to the mass (or mass-squared) operator occurring in the Lagrangian. We are of course not able to decide between a mass formula with masses and one with their squares appearing, because in our first-order calculation they are identical, and because Lagrangians and mass operators do not occur in the S-matrix approach. Second, the amplitude relations deduced from the stationary principle are valid to first order in the perturbation, both when all amplitudes are evaluated at the same c.m. energy and scattering angle, and when all amplitudes are evaluated at the same c.m. energy and momentum transfer, in spite of the fact that the individual terms in the amplitude relations suffer first-order changes when we switch from one to the other mode of evaluation. This happy accident is made possible by the masses being related according to the mass formula determined by unitarity. We regard this as fortunate because, while an amplitude relation at the same energy and scattering angle has the virute that it immediately yields the corresponding relation between the partial-wave amplitudes, amplitude relations evaluated at the same energy and momentum transfer have the virtue of reproducing such relations on crossing. We shall also demonstrate that a similar answer obtains to the much discussed question[5] of whether to evaluate an amplitude relation at a fixed c.m. energy, or at a fixed Q value, etc.

In Sec. II, we explain the origin of S-matrix stationary principles on the basis of conventional group-theoretical assumptions concerning symmetry breaking. In Sec. III, the form of the stationary principle abstracted from Sec. II is treated as a basic postulate in the S-matrix framework, and the methods of using unitarity to get mass formulas and of using crossing to get amplitude relations are explained. The special simplifications achieved because of retaining only first-order perturbations and because of choosing the particular form of the stationary principle suggested in Sec. II are discussed. In Sec. IV, we apply our stationary principle to discuss the breaking of isotopic-spin symmetry in the scattering of pions from pions. We get two solutions to our stationary problem. One solution has amplitude relations that would be obtained in the conventional approach by assuming that the symmetry-breaking Hamiltonian has no $I=2$ part, and the other solution has amplitude relations that would be obtained by assuming that the symmetry-breaking Hamiltonian has no $I=4$ part. In Sec. V, we discuss the more interesting but algebraically formidable problem of deriving the consequences of a stationary principle in the scattering of the pseudoscalar mesons (π,K,η) from pseudoscalar mesons, assuming isotopic-spin conservation and retaining only first-order deviations from the unitary-symmetry limit. Again, we find two solutions to the stationary problem. One solution has amplitude relations and a mass formula which would be derived in the usual approach by assuming absence of octet-parts in the Hamiltonian and in the mass matrix; the other solution has amplitude relations and a mass formula which would be derived by assuming absence of 27-type tensors in the Hamiltonian and in the mass matrix. The mass formula in the latter case is the Gell-Mann-Okubo[3] mass formula, and the amplitude relations are a subset of those derived by Itabashi[6] by assuming that the Hamiltonian has singlet and octet parts only. The former solution, although on an equal theoretical footing to the latter as a solution to the stationary problem, has a mass formula that is obviously violated experimentally and presumably for this reason it has not been discussed elsewhere. Concluding remarks including the discussion of avenues that need further exploration are made in Sec. VI.

II. ORIGIN OF S-MATRIX STATIONARY PRINCIPLES

This section is devoted to demonstrating how S-matrix stationary principles arise from conventional group-theoretical broken-symmetry hypotheses. The S matrix for a multichannel scattering process of the type

$$p_1+p_2\rightarrow p_3+p_4 \tag{2.1}$$

for particles with masses m_1, m_2, m_3, and m_4, will be written in the form

$$S=1+2i(\sqrt{\rho})M(\sqrt{\rho}), \tag{2.2}$$

where M is the invariant-amplitude matrix and ρ is a diagonal phase-space matrix with diagonal elements

$$\rho_i=q_i/(32\pi^2\sqrt{s}); \tag{2.3}$$

we define

$$s=-(p_1+p_2)^2,\quad t=-(p_1-p_3)^2,\quad u=-(p_1-p_4)^2, \tag{2.4}$$

and q_i is the c.m. momentum in the ith channel. We will consider first-order perturbations of the S-matrix elements from an internal-symmetry limit such as isotopic spin or unitary symmetry, assuming that certain more sacred conservation laws such as charge, baryon-number, and strangeness conservation split the perturbed as well as unperturbed scattering matrices into reducible forms consisting of various sub-blocks (r) such that matrix elements between channels i_r, j_s of different sub-blocks r, s are zero. The unitary operator relating the perturbed and unperturbed free-particle states of two-particle channels with the same c.m.

[6] K. Itabashi, Phys. Rev. 137, B1312 (1965).

energy, and the same direction of the relative momentum vector will be defined as

$$|\phi_{i_r}\rangle = e^{-i\epsilon}|\phi_{i_r}{}^0\rangle, \tag{2.5}$$

where ϵ is a Hermitian operator which goes to zero in the unperturbed limit. The unperturbed and perturbed scattering amplitudes will be denoted as

$$\langle\phi_{i_r}{}^0|M^0|\phi_{j_s}{}^0\rangle \equiv M^0{}_{i_r,j_s} \tag{2.6}$$

and

$$\langle\phi_{i_r}|M|\phi_{j_s}\rangle \equiv \langle\phi_{i_r}{}^0|\hat{M}|\phi_{j_s}{}^0\rangle \equiv M_{i_r,j_s}, \tag{2.7}$$

where

$$\hat{M} = e^{i\epsilon}Me^{-i\epsilon}. \tag{2.8}$$

In our matrix notation the indices characterize those properties of the particles in the scattering channels, like charge, hypercharge, etc., which are the same in the perturbed and unperturbed situations. The matrix elements in (2.6) and (2.7) vanish for $r\neq s$. The real orthogonal matrix independent of energy and scattering angle diagonalizing M^0, will be called U, and is of a reducible form like M^0. We shall denote

$$\Lambda^0 = UM^0U^T, \quad \Lambda = UMU^T, \tag{2.9}$$

where Λ^0 is a diagonal matrix, and Λ has nonzero off-diagonal elements on account of the breaking of internal symmetry. To be specific, we shall consider the breaking of $SU(3)$ symmetry, but the applicability of the arguments to other cases will be obvious. Then, the matrix elements of Λ^0 and Λ are just the matrix elements of M^0 and $\hat{M}$ in the basis of states belonging to irreducible representations (IR) of $SU(3)$. Thus

$$\Lambda_{ij} = \sum_{kl} U_{ik}M_{kl}U_{jl} = \sum_{kl}\langle U_{ik}\phi_k{}^0|\hat{M}|U_{jl}\phi_l{}^0\rangle, \tag{2.10}$$

where we have omitted the suffixes, r, s, etc., of the indices i, j, k, and l for simplicity of writing. The matrix elements of a general tensor operator in such a basis may be expressed, using the Wigner-Eckart theorem,[7] as

$$\langle\phi_{\nu_2}{}^{(\mu_2)}|\sum_{\mu\nu} T_\nu{}^{(\mu)}|\phi_{\nu_1}{}^{(\mu_1)}\rangle = \sum_{\mu\nu\gamma}\begin{pmatrix}\mu_1 & \mu & \mu_{2\gamma}\\ \nu_1 & \nu & \nu_2\end{pmatrix}(\mu_2\|T^{(\mu)}\|\mu_1)_\gamma. \tag{2.11}$$

This equation can be inverted, with the help of the orthogonality relations for the Clebsch-Gordan coefficients to solve for the reduced matrix elements. If a tensor operator $T=\sum_{\mu\nu}T_\nu{}^{(\mu)}$ does not contain a particular IR μ, the corresponding reduced matrix element must vanish, and we obtain

$$\sum_{\nu_1}(-)^{I_{1z}+\frac{1}{2}Y_1}\begin{pmatrix}\mu_1{}^* & \mu_2 & \mu_{\gamma'}\\ -\nu_1 & \nu_1 & \nu\end{pmatrix} \times\langle\phi_{\nu_1}{}^{(\mu_2)}|T|\phi_{\nu_1}{}^{(\mu_1)}\rangle = 0. \tag{2.12}$$

In writing (2.12) we have assumed that the magnetic quantum number ν_1, designating isotopic spin and hypercharge, is conserved. Such equations have been written down by Itabashi[6] from a spurion method. We remark that the relations (2.12) are linear relations between transition elements between a fixed pair of representations μ_1, μ_2 but various magnetic quantum numbers ν_1. If T were an $SU(3)$ scalar, all the matrix elements of T in (2.12) with different values of ν_1 would be equal and nonzero if $\mu_1=\mu_2$, and equal to zero if $\mu_1\neq\mu_2$. Applying (2.12) to the scattering-matrix elements we conclude that if a certain IR is absent from the operators M^0 and $\hat{M}$, we obtain linear relations between the scattering amplitudes of the form

$$\sum_r c_r\Lambda^0{}_{i_r,j_r}(s,\cos\theta) = \sum_r c_r\Lambda_{i_r,j_r}(s,\cos\theta) = 0, \tag{2.13}$$

where the subscripts r exhibit the conservation of the magnetic quantum numbers, hypercharge and isospin in the absence as well as in the presence of the perturbation. The fact that all the terms in (2.13) are the matrix elements of M^0 and $\hat{M}$ between a given pair of representations μ_1, μ_2 implies the condition that:

To all terms Λ_{i_r,j_r} in (2.13) *correspond a fixed pair of values, independent of r, for the corresponding unperturbed diagonal elements $\Lambda^0{}_{i_r,i_r}$ and $\Lambda^0{}_{j_r,j_r}$.* (2.14)

For $\mu_1\neq\mu_2$, the $\Lambda^0{}_{i_r,j_r}$ in (2.13) are all zero and further subject to (2.14). For $\mu_1=\mu_2$, Eq. (2.13) involves diagonal elements of Λ^0 and Λ, all the diagonal elements of Λ^0 occurring being equal according to (2.14). Equation (2.13) subject to (2.14) states that *certain linear combinations of the amplitudes Λ_{i_r,j_r} remain stationary under the perturbation, the channels involved in each linear combination being such that the unperturbed eigenamplitudes $\Lambda^0{}_{i_r,i_r}$ are the same for all r, and the $\Lambda^0{}_{j_r,j_r}$ are the same for all r.*

III. S-MATRIX APPROACH TO BROKEN SYMMETRIES

In S-matrix theory, the exact internal-symmetry limit can be defined by the existence of an energy- and angle-independent real orthogonal transformation U diagonalizing the S matrix, and then U can be determined following the procedure in BCR. Knowing U, we can construct the $\Lambda^0{}_{i_r,j_r}$. These can be divided into various sets in each of which all the $\Lambda^0{}_{i_r,j_r}$ are equal to each other, because the eigenamplitudes are independent of the magnetic quantum numbers (r). Thus eigenamplitudes for a given $SU(3)$ representation are independent of the hypercharge and isotopic spin, and transition amplitudes between eigenstates of different $SU(3)$ representations are zero for all values of hyper-

[7] C. Eckart, Rev. Mod. Phys. 2, 302 (1930); E. P. Wigner, *Gruppentheorie* (Friedrich Vieweg und Sohn, Braunschweig, Germany, 1931). We use the $SU(3)$ notation of J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

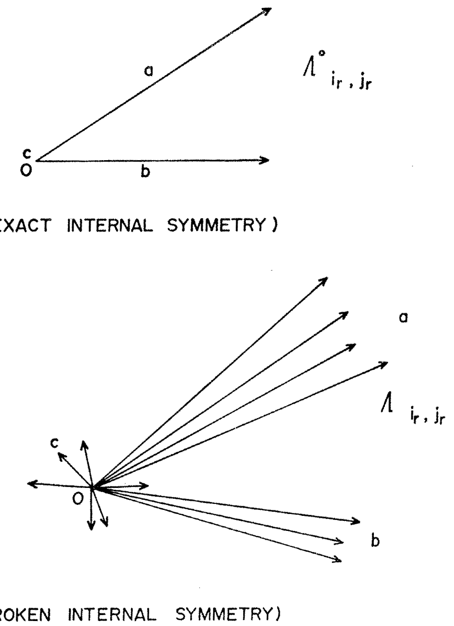

FIG. 1. Diagrams showing that three sets *a*, *b*, *c* of coinciding vectors $\Lambda^0_{i_r,j_r}$ in the symmetry limit become bunches of closely spaced vectors Λ_{i_r,j_r} when the symmetry is broken; *c* denotes the set of vectors which have length zero in the symmetry limit.

charge and isotopic spin. If we represent the $\Lambda^0_{j_r,j_r}$ as vectors in the Argand diagram (Fig. 1), we have several sets of coinciding vectors including a set in which all vectors have zero length. The set with $\Lambda^0_{i_r,j_r}=0$ is further divided into subsets such that all vectors $\Lambda^0_{i_r,j_r}$ of a subset are characterized by a given pair of values for the corresponding diagonal elements $\Lambda^0_{i_r,j_r}$ and $\Lambda^0_{i_r,i_r}$.

When a small symmetry breaking is introduced, the vectors in each set (subset) separate out into a bunch of closely spaced vectors Λ_{i_r,j_r}. The existence of a broken symmetry will be taken to mean that:

There exist certain linear combinations of the vectors Λ_{i_r,j_r} *in the bunch corresponding to each set (or subset) which remain stationary under the symmetry-breaking perturbation at the value zero.*

Thus one gets an equation of the form (2.13) subject to the condition (2.14) with unknown coefficients c_r. We prove now some general consequences of this stationary principle using unitarity and crossing. The results proved here will show several simplifying features of the particular form of the stationary principle chosen, and hence provide reasons stated in purely S-matrix language why this form rather than some other should be taken to characterize a broken symmetry in S-matrix theory.

A. Mass Formulas

If the perturbed and unperturbed scattering amplitudes satisfy the stationary principle (2.13) subject to (2.14), and if there is an energy region (large compared with the perturbations in the masses from the symmetry limit) where the channels considered in formulating the stationary principle are the only open channels with the appropriate quantum numbers, then unitarity requires the following relation between the first-order perturbations in the phase-space factors:

$$\sum_r c_r(U\delta\rho(s)U^T)_{i_r,j_r}=0\,, \tag{3.1}$$

where

$$\delta\rho(s)\equiv\rho(s)-\rho^0(s)\,, \tag{3.2}$$

and ρ *and* ρ^0 *are the perturbed and unperturbed phase-space matrices defined as in (2.2) and (2.3),*

For proof, we start from the partial-wave unitarity equation. Suppressing the angular-momentum label we have

$$\mathrm{Im}M(s)=M^\dagger(s)\rho(s)M(s)\,. \tag{3.3}$$

Transforming this with the matrix U, we have

$$\mathrm{Im}\Lambda(s)=\Lambda^\dagger(s)U\rho(s)U^T\Lambda(s)\,. \tag{3.4}$$

Remembering that Λ^0 is a diagonal matrix and satisfies a unitarity equation analogous to (3.4), using the symmetry of Λ_{i_r,j_r}, and retaining only first-order terms in the perturbations, we deduce

$$\begin{aligned}\mathrm{Im}\delta\Lambda_{i_r,j_r}(s)&\\ =\{1+i\rho_r{}^0(s)[\Lambda^0{}_{j_r,j_r}(s)+\Lambda^0{}_{i_r,i_r}(s)]\}^{-1}&\\ \times\{(U\delta\rho(s)U^T)_{i_r,j_r}\,\mathrm{Re}[\Lambda^{0*}{}_{i_r,i_r}(s)\Lambda^0{}_{j_r,j_r}(s)]&\\ +\rho_r{}^0(s)\delta\Lambda_{i_r,j_r}(s)\,\mathrm{Re}[\Lambda^0{}_{i_r,i_r}(s)+\Lambda^0{}_{j_r,j_r}(s)]\}\,,&\end{aligned} \tag{3.5}$$

where $\rho_r{}^0(s)$ is the value of the diagonal elements of $\rho^0(s)$ for the channels with magnetic quantum number r, and

$$\delta\Lambda(s)\equiv\Lambda(s)-\Lambda^0(s)\,. \tag{3.6}$$

The stationary principle (2.13) yields for the partial wave amplitudes

$$\sum_r c_r\delta\Lambda_{i_r,j_r}(s)=0\,. \tag{3.7}$$

The coefficients c_r must be real on account of the real analyticity of the $\Lambda_{i_r,j_r}(s)$. Substituting (3.5) into the imaginary part of (3.7), noting that the conditions (2.14) enable us to bring the terms involving $\Lambda^0{}_{i_r,i_r}(s)$ and $\Lambda^0{}_{j_r,j_r}(s)$ outside the summation over r, and that the diagonal elements of Λ^0 are nonvanishing, we deduce Eq. (3.1).

The first point to notice is that the Eq. (3.1) in fact implies two linear relations between the perturbations in the masses of the particles in the various channels occurring in (3.1), or two mass formulas. The perturbation in the diagonal element of the phase-space matrix corresponding to a channel with particles of masses m_A and m_B, at a fixed s, is given by

$$\delta\rho_{AB}(s)=\alpha_1(s)(\delta m_A-\delta m_B)+\alpha_2(s)(\delta m_A+\delta m_B)\,, \tag{3.8}$$

where

$$\alpha_1(s)=-\frac{1}{64\pi^2 s}\frac{[s-(m_A{}^0+m_B{}^0)^2]^{1/2}}{[s-(m_A{}^0-m_B{}^0)^2]^{1/2}}(m_A{}^0-m_B{}^0) \tag{3.9}$$

and

$$\alpha_2(s) = -\frac{1}{64\pi^2 s}\frac{[s-(m_A{}^0-m_B{}^0)^2]^{1/2}}{[s-(m_A{}^0+m_B{}^0)^2]^{1/2}}(m_A{}^0+m_B{}^0). \quad (3.10)$$

Substituting (3.8) into (3.1) and noting that $\alpha_1(s)$ and $\alpha_2(s)$ are linearly independent functions of s, we obtain

$$(m_A{}^0-m_B{}^0)\sum_r c_r \sum_{k_r} U_{i_r,k_r}\delta(m_A-m_B)_{k_r}U_{j_r,k_r}=0 \quad (3.11)$$

and

$$\sum_r c_r \sum_{k_r} U_{i_r,k_r}\delta(m_A+m_B)_{k_r}U_{j_r,k_r}=0\,, \quad (3.12)$$

where the second sum runs over all channels with magnetic quantum numbers characterized by r, and the symbols $(m_A-m_B)_{k_r}$ and $(m_A+m_B)_{k_r}$ stand, respectively, for the difference and sum of the masses of the particles in the channel k_r. The same two unperturbed masses $m_A{}^0$, $m_B{}^0$ have to occur for all the channels appearing in (3.1) because of the condition (2.14) and unitarity. The factor $m_A{}^0-m_B{}^0$ has been written on the left-hand side of Eq. (3.11) to emphasize that a nontrivial relation corresponding to (3.11) is obtained only when the channels consist of two unequal-mass particles in the unperturbed limit; writing the corresponding factor $m_A{}^0+m_B{}^0$ on the left-hand side of (3.12) is redundant because we do not consider zero-mass particles. Equations (3.11) and (3.12) constitute the mass formulas in the S-matrix approach. The crucial role played by the conditions (2.14) in simplifying the consequence of unitarity applied to the stationary principle (2.13) into these simple mass formulas should be noted.

B. M-Matrix Stationary Principle and S-Matrix Stationary Principle

Strictly speaking, our primary postulate (2.13) is stated in terms of the invariant-amplitude matrix M rather than the S-matrix elements, and it is not obvious that a similar relation between the S-matrix elements holds, since the phase-space matrix in the perturbed situation is different from a multiple of the unit matrix even for a given set of magnetic quantum numbers r. We shall show here that our assumptions (2.13) subject to (2.14) indeed imply identical relations between the S-matrix elements, provided that the mass formulas dictated by unitarity are satisfied. From Eq. (2.2) we obtain

$$\sum_r c_r(U\delta SU^T)_{i_r,j_r}=\sum_r c_r\delta_{i_r,j_r}+2i\sum_r c_r[\rho_r{}^0\delta\Lambda_{i_r,j_r} + \tfrac{1}{2}(\Lambda^0{}_{i_r,i_r}+\Lambda^0{}_{j_r,j_r})(U\delta\rho U^T)_{i_r,j_r}]. \quad (3.13)$$

The first sum on the right-hand side of (3.13) vanishes on account of (2.13) and (2.14), and $\rho_r{}^0$ and $\Lambda^0{}_{i_r,i_r}+\Lambda^0{}_{j_r,j_r}$ can be taken outside the summation on account of (2.14). Using (2.13), we then deduce

$$\sum_r c_r(U\delta SU^T)_{i_r,j_r}=0\,. \quad (3.14)$$

As in Sec. III A, we have assumed that for a certain energy region the channels included above are the only open channels. As before, the specific form of the stationary principle plays a crucial role.

C. Choice of Fixed Kinematic Variables in Exact and Broken Internal-Symmetry Amplitude Relations

In comparing scattering amplitude relations with experiment, one of the important questions to be answered is: Which of the energy variables c.m. energy, Q value, etc., and which of the angle variables, c.m. scattering angle, momentum transfer, etc., are to be considered fixed in the arguments of the scattering amplitudes?

To be specific, let us consider the baryon-meson amplitude relations given by exact and broken $SU(3)$. We first state the somewhat obvious answer in the exact-symmetry case. The amplitude relations here imply via unitarity (as in BCR) that all the baryon-meson channels appearing in any amplitude relation are kinematically identical. Hence, fixed s and fixed $\cos\theta$ imply fixed Q value and fixed momentum transfer, and similarly fixed values for other pairs of kinematic variables usually considered. Thus amplitude relations with a certain pair of kinematic variables fixed imply the corresponding relations with other pairs of variables fixed. However, it is found, for example, by Meshkov and Yodh,[5] that the exact $SU(3)$ amplitude relations with certain pairs of kinematic variables fixed fit the experimental data for the ranges of energy considered much better than the corresponding relations with other pairs of fixed variables which are equally good according to exact $SU(3)$. We have therefore to consider the approximate validity of the exact $SU(3)$ relations with only a particular pair of fixed variables, not as a verification of the exact $SU(3)$ relations, but as an interesting empirical observation requiring theoretical understanding. It is, of course, well known that the deviations of the experimental masses from the exact $SU(3)$ predictions are quite large; the point we are making is that the validity of the $SU(3)$ amplitude relations and the $SU(3)$ mass formulas are connected by unitarity, and cannot be considered separately. The broken-$SU(3)$ mass formulas, on the other hand, are reasonably well satisfied for certain multiplets, and for them it is relevant to ask what variables to fix in comparing the broken-$SU(3)$ amplitude relations with experiment. Each amplitude in an amplitude relation will suffer first-order changes in the symmetry breaking when we switch from one pair of fixed variables to the other, and first-order changes are not neglected in the broken-symmetry relations. The question here is therefore nontrivial and is answered in detail below.

We shall prove that amplitude relations of the form (2.13) subject to (2.14) evaluated at fixed s and $\cos\theta$, plus the mass formulas required by unitarity, imply that the corresponding amplitude relations evaluated at fixed s and t are also valid provided that only first-order terms in the perturbations are retained.

The fixed $(s, \cos\theta)$ relation can be written in the form

$$\sum_r c_r U_{i_r,k_r} M_{k_r,l_r}[s,t(k_r,l_r)] U_{j_r,l_r} = 0\,, \qquad (3.15)$$

where the $t(k_r,l_r)$ are the momentum-transfer values for the various amplitudes at fixed s and $\cos\theta$. The amplitudes can be expanded around the unperturbed momentum-transfer value t^0 common to all the amplitudes in (3.15) to give

$$M_{k_r,l_r}[s,t(k_r,l_r)] \cong M_{k_r,l_r}(s,t^0) + \delta M_{k_r,l_r}(s,t^0)\,, \qquad (3.16)$$

where

$$\delta M_{k_r,l_r}(s,t^0) = [t(k_r,l_r) - t^0]\partial M^0_{k_r,l_r}(s,t)/\partial t|_{t=t^0}\,, \qquad (3.17)$$

$$t(k_r,l_r) - t^0 = \delta\hat{t}_{k_r,k_r} + \delta\hat{t}_{l_r,l_r}\,, \qquad (3.18)$$

and

$$\begin{aligned}\delta\hat{t}_{k_r,k_r} =& \tfrac{1}{2}[1-(m_A{}^0-m_B{}^0)^2/s](m_A{}^0+m_B{}^0)\delta(m_A+m_B)_{k_r} \\ &+\tfrac{1}{2}[1-(m_A{}^0+m_B{}^0)^2/s](m_A{}^0-m_B{}^0)\delta(m_A-m_B)_{k_r} \\ &-\{t^0+\tfrac{1}{2}s-[(m_A{}^0)^2+(m_B{}^0)^2]+[(m_A{}^0)^2-(m_B{}^0)^2]^2/2s\} \\ &\times\{[s-(m_A{}^0+m_B{}^0)^2]^{-1}(m_A{}^0+m_B{}^0)\delta(m_A+m_B)_{k_r} \\ &+[S-(m_A{}^0-m_B{}^0)^2]^{-1}(m_A{}^0-m_B{}^0)\delta(m_A-m_B)_{k_r}\}\,.\end{aligned} \qquad (3.19)$$

In (3.18), $\delta\hat{t}$ is defined to be a diagonal matrix whose diagonal elements are given by (3.19). We then deduce

$$\sum_r c_r U_{i_r,k_r}\delta M_{k_r,l_r}(s,t^0) U_{j_r,l_r} = \sum_r c_r (U\delta\hat{t}U^T)_{i_r,j_r} \times\left[\frac{\partial\Lambda^0{}_{i_r,i_r}(s,t)}{\partial t}\bigg|_{t=t_0} + \frac{\partial\Lambda^0{}_{j_r,j_r}(s,t)}{\partial t}\bigg|_{t=t_0}\right]. \qquad (3.20)$$

The last two terms within the brackets in (3.20) can be brought outside the summation, and using (3.19) and the mass formulas (3.11) and (3.12) we can prove that

$$\sum_r c_r (U\delta\hat{t}U^T)_{i_r,j_r} = 0\,, \qquad (3.21)$$

and hence from (3.20)

$$\sum_r c_r U_{i_r,k_r}\delta M_{k_r,l_r}(s,t^0) U_{j_r,l_r} = 0\,. \qquad (3.22)$$

Using (3.15), (3.16), and (3.22), we obtain

$$\sum_r c_r \Lambda_{i_r,j_r}(s,t^0) = 0\,, \qquad (3.23)$$

which is the desired relation at fixed s and fixed momentum transfer. It can similarly be proved that amplitude relations of the form (2.13) subject to (2.14) at fixed $(s, \cos\theta)$, plus the mass formulas required by unitarity, imply the corresponding amplitude relations with the fixed energy variable changed to Q value, laboratory energy, etc., provided that only first-order terms in the perturbations are retained.

D. Use of Crossing Relations

We shall show that the use of the crossing relations remains as simple in first-order perturbation theory as in the unperturbed limit discussed in BCR. The extra complications when higher-order perturbations are retained will also be seen. The crossing relations have the general form

$$A_i(stu) = \sum_j c_j A_j(tsu) = \sum_k c_k A_k(uts)\,. \qquad (3.24)$$

Each amplitude will be separated into parts odd and even under $\cos\theta \leftrightarrow -\cos\theta$, as in BCR, Thus,

$$A_i(stu) = A_i{}^{(o)}(stu) + A_i{}^{(e)}(stu)\,, \qquad (3.25)$$

$$A_j(tsu) = A_j{}^{(o)}(tsu) + A_j{}^{(e)}(tsu)\,, \qquad (3.26)$$

and

$$A_k(uts) = A_k{}^{(o)}(uts) + A_k{}^{(e)}(uts)\,, \qquad (3.27)$$

where the superscripts in (3.25)–(3.27) refer to the reflection property under $\cos\theta_s \leftrightarrow -\cos\theta_s$, $\cos\theta_t \leftrightarrow -\cos\theta_t$, and $\cos\theta_u \leftrightarrow -\cos\theta_u$, respectively. If the amplitude on the left-hand side of (3.24) describes a reaction $m_1 + m_2 \to m_3 + m_4$, where the masses are used to label the particles, we can verify that

$$\cos\theta_s \leftrightarrow -\cos\theta_s \quad \text{if} \quad t \leftrightarrow u - (m_1{}^2 - m_2{}^2)(m_3{}^2 - m_4{}^2)/s\,, \qquad (3.28)$$

and

$$\cos\theta_u \leftrightarrow -\cos\theta_u \quad \text{if} \quad t \leftrightarrow s - (m_1{}^2 - m_4{}^2)(m_3{}^2 - m_2{}^2)/u\,. \qquad (3.29)$$

Using the crossing relations in (3.24) and the reflection properties mentioned above, we obtain

$$2\begin{pmatrix}A_i{}^{(o)}(stu)\\ A_i{}^{(e)}(stu)\end{pmatrix} = \sum_j c_j A_j(tsu) \mp \sum_k c_k(-A_k{}^{(o)} + A_k{}^{(e)})\left(t + \frac{(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)}{s},\ s - \frac{(m_1{}^2-m_4{}^2)(m_3{}^2-m_2{}^2)}{t+(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)/s},\right.$$
$$\left. u - \frac{(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)}{s} + \frac{(m_1{}^2-m_4{}^2)(m_3{}^2-m_2{}^2)}{t+(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)/s}\right). \qquad (3.30)$$

As in BCR, (3.30) substituted into an s-channel amplitude relation between the A_i's yields a new relation between the A_j's and the A_k's with the same arguments in all amplitudes, provided that all the masses are equal. Even in the unperturbed limit we can have $m_1^0 \neq m_2^0$ ($m_1^0 = m_3^0$, $m_2^0 = m_4^0$), and special considerations are then necessary to deduce relations between amplitudes with a common set of arguments, as in the derivation of isotopic-spin-invariance predictions in the scattering of the π's and the K's considered in BCR. We shall consider here the perturbed situation corresponding to the unperturbed limit $m_1^0 = m_2^0 = m_3^0 = m_4^0$, and leave the other cases for individual consideration. Our main observation from (3.30), then, is that all the arguments on the right-hand side of (3.30) become identical if terms of second and higher orders in the perturbations in the masses from the symmetry limit are neglected; hence no extra complication arises in the use of the crossing relations in first-order perturbation theory. In higher-order perturbation theory one has to make perturbation expansions of the exact crossing relation (3.30).

IV. VIOLATIONS OF ISOTOPIC SPIN

We shall consider first the derivation of the patterns of breaking of isotopic-spin conservation in the scattering of the π^+, π^0, and π^- mesons from each other. The assumptions will be that a stationary principle of the form (2.13) subject to (2.14) holds, and that all further amplitude relations obtained as consequences separate into relations each satisfying the conditions (2.14). The notation for the various $\pi\pi \to \pi\pi$ amplitudes will be taken to be the same as in BCR, and is listed in Table I together with the crossing relations; the spatially odd and even parts of amplitudes are given odd and even subscripts, respectively.

The matrices Λ and Λ^0 corresponding to the subspace of charge-zero channels and even spatial parity are given by

$$\Lambda = U\begin{pmatrix} A_6 & A_8 \\ A_8 & A_{10} \end{pmatrix} U^T \equiv \begin{pmatrix} \Lambda_6 & P_8 \\ P_8 & \Lambda_{10} \end{pmatrix} \tag{4.1}$$

and

$$\Lambda^0 = U\begin{pmatrix} A^0_6 & A^0_8 \\ A^0_8 & A^0_{10} \end{pmatrix} U^T \equiv \begin{pmatrix} \Lambda^0_6 & 0 \\ 0 & \Lambda^0_{10} \end{pmatrix}, \tag{4.2}$$

where the matrix U has been determined in BCR to within certain irrelevant phases. The arguments (s, $\cos\theta$) etc., of the amplitudes will be suppressed except when crucial. Explicitly, Eqs. (4.1) read

$$\Lambda_6 = \tfrac{2}{3}A_6 + \tfrac{2}{3}\sqrt{2}A_8 + \tfrac{1}{3}A_{10}, \tag{4.3}$$

$$P_8 = -\tfrac{1}{3}\sqrt{2}A_6 + \tfrac{1}{3}A_8 + \tfrac{1}{3}\sqrt{2}A_{10}, \tag{4.4}$$

and

$$\Lambda_{10} = \tfrac{1}{3}A_6 - \tfrac{2}{3}\sqrt{2}A_8 + \tfrac{2}{3}A_{10}. \tag{4.5}$$

Λ^0_6 and Λ^0_{10} are given by equations analogous to (4.3) and (4.5). The unperturbed eigenamplitudes divide themselves into the following sets of equal amplitudes:

TABLE I. Amplitudes and crossing relations for the elastic scattering of pions. Factors of 2 and $\sqrt{2}$ are included so that all the A_i will satisfy the usual two-particle unitarity relations.

	(stu)	(tsu)	(uts)
$\pi^+\pi^+ \to \pi^+\pi^+$	$2A_2$	$=(-A_5+A_6)=$	(A_5+A_6)
$\pi^+\pi^0 \to \pi^+\pi^0$	$(A_3+A_4)=$	$\sqrt{2}A_8 =$	(A_3+A_4)
$\pi^+\pi^- \to \pi^+\pi^-$	$(A_5+A_6)=$	$(A_5+A_6) =$	$2A_2$
$\pi^+\pi^- \to \pi^0\pi^0$	$\sqrt{2}A_8$	$= (A_3+A_4)$	$=(-A_3+A_4)$
$\pi^0\pi^0 \to \pi^0\pi^0$	$2A_{10}$	$= 2A_{10}$	$= 2A_{10}$

$$A^0_3 = A^0_5, \quad A^0_2 = A^0_4 = \Lambda^0_{10}, \quad \Lambda^0_6, \quad P^0_8 = 0. \tag{4.6}$$

Hence the primary stationary principle satisfying (2.13) and (2.14) can be taken to be any of the following:

(a) $$c_3A^0_3 + c_5A^0_5 = c_3A_3 + c_5A_5 = 0, \tag{4.7}$$

(b) $$c_2A^0_2 + c_4A^0_4 + c_{10}\Lambda^0_{10} = c_2A_2 + c_4A_4 + c_{10}\Lambda_{10} = 0, \tag{4.8}$$

or

(c) $$P_8 = 0. \tag{4.9}$$

The procedure will be illustrated for case (b) and the results for the other cases stated. Substituting for Λ_{10} from (4.5) into the perturbed amplitude relation in (4.8) and using crossing, we deduce

$$\tfrac{1}{2}c_2(-A_5+A_6) + \tfrac{1}{2}c_4(\sqrt{2}A_8 - A_3 + A_4) + \tfrac{1}{3}c_{10} \times[\tfrac{1}{2}(A_5+A_6+2A_2) - 2(A_3+A_4) + 2A_{10}] = 0. \tag{4.10}$$

Requiring $\Lambda^0_{10} \neq 0$, we have $c_2 + c_4 + c_{10} = 0$. Substituting for c_{10}, the odd-parity part of (4.10) becomes

$$(4c_2 + c_4)(A_3 - A_5) = 0, \tag{4.11}$$

which yields either

$$A_3 = A_5 \tag{4.12a}$$

or

$$c_2 = -\tfrac{1}{4}c_4. \tag{4.12b}$$

The even-parity part of (4.10) on separation into relations subject to (2.14) yields

$$2(c_2+c_4)A_2 - (4c_2+7c_4)A_4 + (2c_2+5c_4)\Lambda_{10} = 0 \tag{4.13a}$$

and

$$(4c_2 + c_4)P_8 = 0, \tag{4.14}$$

the relation involving Λ_6 being an identity. If (4.12b) is true, (4.13) gives

$$A_2 - 4A_4 + 3\Lambda_{10} = 0, \tag{4.13b}$$

which is also obtained by substituting (4.12b) in (4.8); (4.14) is automatically satisfied in this case, and no further amplitude relations can be obtained by using crossing. Unitarity applied to Eq. (4.13b) gives the mass formula

$$m_{\pi^+} = m_{\pi^-}, \tag{4.15}$$

but allows m_{π^0} to be different.

TABLE II. List of the 25 odd- and even-parity pseudoscalar amplitudes. Subscripts denote isotopic spins. Note that there are four two-by-two and one three-by-three amplitude matrices indicated by brackets. The numbers correspond to subscripts for amplitudes used in the text.

Odd-parity amplitudes	Even-parity amplitudes
1. $\langle K\bar{K}\|K\bar{K}\rangle_0$	11. $\langle K\bar{K}\|K\bar{K}\rangle_0$ ⎤
2. $\langle K\bar{K}\|K\bar{K}\rangle_1$ ⎤	12. $\langle K\bar{K}\|\pi\pi\rangle_0$ ⎥
3. $\langle K\bar{K}\|\pi\pi\rangle_1$ ⎥	13. $\langle \pi\pi\|\pi\pi\rangle_0$ ⎥
4. $\langle \pi\pi\|\pi\pi\rangle_1$ ⎦	14. $\langle \eta\eta\|\eta\eta\rangle_0$ ⎥
5. $\langle \pi K\|\pi K\rangle_{1/2}$ ⎤	15. $\langle \eta\eta\|\pi\pi\rangle_0$ ⎥
6. $\langle \pi K\|\eta K\rangle_{1/2}$ ⎥	16. $\langle \eta\eta\|K\bar{K}\rangle_0$ ⎦
7. $\langle \eta K\|\eta K\rangle_{1/2}$ ⎦	17. $\langle K\bar{K}\|K\bar{K}\rangle_1$ ⎤
8. $\langle \pi K\|\pi K\rangle_{3/2}$	18. $\langle K\bar{K}\|\eta\pi\rangle_1$ ⎥
9. $\langle KK\|KK\rangle_0$	19. $\langle \eta\pi\|\eta\pi\rangle_1$ ⎦
10. $\langle \eta\pi\|\eta\pi\rangle_1$	20. $\langle KK\|KK\rangle_1$
	21. $\langle \pi K\|\pi K\rangle_{1/2}$ ⎤
	22. $\langle \pi K\|\eta K\rangle_{1/2}$ ⎥
	23. $\langle \eta K\|\eta K\rangle_{1/2}$ ⎦
	24. $\langle \pi K\|\pi K\rangle_{3/2}$
	25. $\langle \pi\pi\|\pi\pi\rangle_2$

If (4.12a) is satisfied we are really assuming case (a), Eq. (4.7). It can be readily seen that cases (a) and (c) are equivalent and each leads to the following amplitude relations and mass formulas:

$$A_3=A_5\,,\quad P_8=0\,,\quad 2A_2-A_4=\Lambda_{10}\,, \tag{4.16}$$

and

$$m_{\pi^+}=m_{\pi^-}=m_{\pi^0}\,. \tag{4.17}$$

Note that the relation (4.16) cannot be further decomposed via the three-eigenamplitude theorem in BCR, because Eqs. (4.16) have been proved to be true only up to first order in the perturbation, and unitarity up to this order will be guaranteed.

We finally have two distinct solutions to our stationary problem. One is contained in the relation (4.13b) and the mass formula (4.15). In the conventional group-theoretic language, this solution can be shown to correspond to the scattering operator $\hat{M}$ in Sec. II, having no $I=4$ part. The other solution is contained in the Eqs. (4.16) and (4.17), and can be shown to correspond to the operator $\hat{M}$ having no $I=2$ part. If $\hat{M}$ has neither any $I=2$ part nor any $I=4$ part, we recover all the relations of the exact-symmetry case for these amplitudes. It may be noted that the electromagnetic nature of the isotopic-spin violations has nowhere been used in our approach; we have found the only stationary principles satisfying (2.13) and (2.14) consistent with unitarity and crossing.

V. PATTERNS IN THE BREAKING OF UNITARY SYMMETRY

We now describe briefly the rather tedious derivation of broken-$SU(3)$ predictions in our approach which constituted the original motivation for the present work. We consider the scattering of the eight pseudoscarar mesons (π, K, η) from each other assuming isotopic-spin invariance and adopt the labels for the various amplitudes given in BCR; these labels are restated in Table II for reference. The crossing relations are listed in BCR. The real orthogonal matrices U that diagonalize the scattering matrices in the internal-symmetry limit have also been found there. In the perturbed situation we define

$$U\begin{pmatrix}A_2 & A_3\\ A_3 & A_4\end{pmatrix}U^T\equiv\begin{pmatrix}\Lambda_2 & P_3\\ P_3 & \Lambda_4\end{pmatrix}, \tag{5.1}$$

and similarly define the amplitudes

$$\Lambda_5,\ P_6,\ \Lambda_7,\ \Lambda_{11},\ P_{12},\ \Lambda_{13},\ \Lambda_{14},$$
$$P_{15},\ P_{16},\ \Lambda_{17},\ P_{18},\ \Lambda_{19},\ \Lambda_{21},\ P_{22},\text{ and }\Lambda_{23}.$$

In these definitions, P_i always stands for an off-diagonal element, Λ_i always stands for a diagonal element of UMU^T, and the subscripts correspond to those of the M-matrix element similarly situated. The unperturbed $\Lambda^0{}_i$ and $P^0{}_i$ are similarly defined, and all the $P^0{}_i$ are zero. The unperturbed amplitudes can be divided into the following sets of equal amplitudes;

$$\begin{gathered} A^0{}_1=\Lambda^0{}_4=\Lambda^0{}_7\,,\quad P^0{}_3=P^0{}_6=0\,,\\ \Lambda^0{}_2=\Lambda^0{}_5=A^0{}_8=A^0{}_9=A^0{}_{10}\,,\\ \Lambda^0{}_{11}\,,\quad P^0{}_{12}=0\,,\quad P^0{}_{16}=0\,,\quad \Lambda^0{}_{14}=\Lambda^0{}_{17}=\Lambda^0{}_{21}\,,\\ P^0{}_{15}=P^0{}_{18}=P^0{}_{22}=0\,,\\ \Lambda^0{}_{13}=\Lambda^0{}_{19}=A^0{}_{20}=\Lambda^0{}_{23}=A^0{}_{24}=A^0{}_{25}\,. \end{gathered} \tag{5.2}$$

The amplitudes $P_i{}^0$, which are all zero, have further been divided into subsets in each of which the conditions (2.14) are valid. There are now many canditates for a primary stationary principle many of which are in the end equivalent. It will be enough for our present purpose to derive all the consequences of a stationary principle stated in terms of A_1, Λ_4, and Λ_7:

$$d_1A^0{}_1+d_4\Lambda^0{}_4+d_7\Lambda^0{}_7=d_1A_1+d_4\Lambda_4+d_7\Lambda_7=0\,, \tag{5.3}$$

where d_1, d_4, and d_7 are constants, not all zero. As before, to have $A^0{}_1\neq0$ we must have $d_1+d_4+d_7=0$. Using the crossing relations in (5.3), we obtain for the odd-parity amplitudes, after some simplification,

$$\begin{aligned}(3d_1+d_4)(3A_1+\Lambda_4-4\Lambda_7)+\sqrt{2}(15d_1-13d_4)P_3\\ -(12d_1+28d_4)P_6+(24d_1+8d_4)\Lambda_2+(6d_1+14d_4)\Lambda_5\\ -(12d_1+28d_4)A_8+(6d_4-18d_1)A_9=0\,.\end{aligned} \tag{5.4}$$

We now separate this relation into three relations each satisfying the criterion (2.14). We then obtain from the part containing A_1, Λ_4, and Λ_7, either

$$\text{(a)}\quad d_4=-3d_1\,, \tag{5.5}$$

or

$$\text{(b)}\quad 3A_1+\Lambda_4-4\Lambda_7=0\,. \tag{5.6}$$

For case (a), we find the amplitude relation corresponding to (5.6) as

$$A_1-3\Lambda_4+2\Lambda_7=0. \tag{5.7}$$

Applying the unitarity relation to Eqs. (5.7) and (5.6), we find, respectively, the mass formulas

$$\text{(a)}\quad 3\delta m_\pi-2\delta m_K-\delta m_\eta=0 \tag{5.8}$$

and

$$\text{(b)}\quad \delta m_\pi-4\delta m_K+3\delta m_\eta=0. \tag{5.9}$$

Of these, (5.8) is badly violated experimentally, and (5.9) is recognized to be the famous Gell-Mann–Okubo mass formula for the pseudoscalar octet, in quite reasonable agreement with experiment. From the point of view of the present theory, however, we have no reason to expect one of these relations to be favored over the other. The further amplitude relations in the two cases (a) and (b) can be derived by repeated use of crossing and unitarity. We shall omit this painful detail; in the end, we obtain two sets of amplitude relations, each having its own mass formula and each satisfying crossing and unitarity to first-order terms in the perturbations. These two sets of amplitude relations are listed in Table III. The relations in set (a) arise in the conventional language by assuming the absence of octet-type operators in $\hat{M}$, and the relations in set (b) arise by assuming the absence of **27**-type operators in $\hat{M}$.

Note that the relations in set (b) are a subset of the relations[6] obtained from the often used assumption that $\hat{M}$ has singlet- and octet-type operators only. Since the relations in (b) already satisfy unitarity and crossing completely, and in addition have the mass formula which is one of the important successes of broken-$SU(3)$ predictions, it is worthwhile to investigate whether this set of amplitude relations is better satisfied experimentally than the full set of amplitude relations obtained from the usual assumptions.

TABLE III. The two sets of amplitude relations for P-P scattering and the corresponding mass formulas that follow from the assumption that a linear combination of A_1, Λ_4, and Λ_7 remains stationary under a perturbation from the $SU(3)$ symmetry limit.

Case (a)	Case (b)
$A_1-3\Lambda_4+2\Lambda_7=0$	$3A_1\Lambda_4-4\Lambda_7=0$
$P_3=-\frac{2}{3}\sqrt{2}P_6$	$P_3=\sqrt{2}P_6$
$\Lambda_5-2A_8+A_9=0$	$5\Lambda_2+2\Lambda_5-4A_8-3A_9=0$
$\Lambda_{14}-3\Lambda_{17}+2\Lambda_{21}=0$	$\Lambda_2=A_{10}$
	$3\Lambda_{14}+\Lambda_{17}-4\Lambda_{21}=0$
$4\Lambda_{13}+3\Lambda_{19}+15A_{20}$	$26\Lambda_{13}+27\Lambda_{19}-90A_{20}$
$+13\Lambda_{23}-10A_{24}-25A_{25}=0$	$+52\Lambda_{23}-40A_{24}+25A_{25}=0$
	$\Lambda_{13}+2\Lambda_{19}-5A_{20}+2\Lambda_{23}=0$
$P_{16}=0$	$\Lambda_{13}+2\Lambda_{19}-5A_{20}+2\Lambda_{23}=0$
	$P_{12}=0$
$P_{15}-2P_{18}-(\frac{4}{3}\sqrt{6})P_{22}=0$	$2\sqrt{3}P_{15}+6\sqrt{3}P_{18}-3\sqrt{2}P_{22}=0$
Mass formula	Mass formula
$3\delta m_\pi-2\delta m_K-\delta m_\eta=0$	$\delta m_\pi-4\delta m_K+3\delta m_\eta=0$

VI. CONCLUSIONS

We have shown that an S-matrix stationary principle together with unitarity and crossing can yield results conventionally deduced by assigning specific group-theoretic transformation properties to the Hamiltonian and to the mass matrix. One special virtue of this approach is the elucidation of the intimate connection via unitarity between the kinematic relations and the scattering-amplitude relations obtained in exact and broken symmetries. Further, since mass formulas are deduced from scattering-amplitude relations using unitarity and assuming the absence of open channels other than those contributing to the mass formula for a certain range of energies (large compared with the perturbations in the masses), it is readily understandable that the mass formula for the vector-meson octet can be violated because of neglecting the contribution to unitarity of another vector meson close in mass. A detailed study of this effect remains to be done. One limitation of the present results, although shared by most theories so far, is the neglect of second- and higher-order perturbations; it is hoped that the reason why some multiplets obey mass formulas better, and others obey mass-squared formulas better, would be understood in a higher-order perturbation theory. One question raised by the results of Sec. V is: Can a broken symmetry containing the correct mass formula, but less restrictive in its other predictions than the conventional broken symmetry, actually be experimentally obeyed better? Such a broken symmetry has arisen in Sec. V quite naturally. It will be interesting to confront the analogous relations, for example, for baryon-meson scattering with experiment to test this question. Another important question to ask is: What is the connection of previous S-matrix derivations of broken-symmetry results under some approximate dynamical schemes[8] with the present exact derivation? We hope to answer some of these questions later.

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