Elastic Unitarity and Phase Equality of Scattering Amplitudes

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Recently Törnqvist suggested that πN scattering below the inelastic threshold is compatible with the assumption that the phases of the $\frac{1}{2}$ and $\frac{3}{2}$ isospin amplitudes differ by $n\pi$. We show in the scalar case that this phase relation cannot hold over the whole angular interval; for elastic unitarity would then, in general, force the two amplitudes to have equal moduli also.

1. INTRODUCTION

IN a recent paper,[1] Törnqvist has analyzed the experimental data on πN scattering below inelastic threshold in the light of charge independence. He found that the phases of the $\frac{1}{2}$ and $\frac{3}{2}$ isospin amplitudes differ by $n\pi$ for the spin-nonflip and spin-flip amplitudes, separately. This relation between the phases of the isospin amplitudes seems to hold quite well in the forward and in the backward region, whereas in the intermediate angular region the data have large errors and can only be said to be compatible with the above phase relation.

In this paper, we show that Törnqvist's relation,[2]

$$\varphi^{(1)}(k,\theta)=\varphi^{(3)}(k,\theta)+n\pi \quad (n=\text{integer}) \tag{1}$$

(k=c.m. momentum; θ=c.m. scattering angle), is inconsistent with elastic unitarity if assumed valid for all θ's and for $0\leq k<k_{\max}$, where $k_{\max}$ is below the inelastic threshold. In fact, we shall prove that if Eq. (1) holds, the moduli of the $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes must be equal, as a consequence of elastic unitarity. The resulting equality of the cross sections rules out, then, on empirical grounds, the general validity of Eq. (1).

Our result is complementary to the one due to Bessis and Martin,[3] who showed that the knowledge of the modulus of the amplitude in the physical region fixes the amplitude entirely.

In this paper, we limit ourselves to the case of scalar particles.

2. STATEMENT OF THE PROBLEM

Let us define the amplitudes for the scattering of a pion on a scalar nucleon as

$$\begin{aligned}T^{(i)}(k,\theta)&=A^{(i)}(k,\theta)+iB^{(i)}(k,\theta)\\&\equiv\tau^{(i)}(k,\theta)e^{i\varphi^{(i)}(k,\theta)},\quad(i=1,3)\end{aligned} \tag{2}$$

where the index i labels the two isospin channels $I=\frac{1}{2}$ and $I=\frac{3}{2}$. We shall assume the validity of Eq. (1).

In terms of angular-momentum decomposition, the real and imaginary parts of the amplitudes $T^{(i)}(k,\theta)$ are given by

$$\begin{aligned}A^{(i)}(k,\theta)&=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)A_l^{(i)}(k)P_l(\cos\theta)\,,\\B^{(i)}(k,\theta)&=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)B_l^{(i)}(k)P_l(\cos\theta)\,,\end{aligned} \tag{3}$$

where

$$\begin{aligned}A_l^{(i)}(k)&=\sin\delta_l^{(i)}(k)\cos\delta_l^{(i)}(k)\,,\\B_l^{(i)}(k)&=\sin^2\delta_l^{(i)}(k)\,,\end{aligned} \tag{4}$$

and, below the inelastic threshold, the phase shifts $\delta_l^{(i)}(k)$ are real functions of k [and so, therefore, are $A_l^{(i)}(k)$ and $B_l^{(i)}(k)$].

The elastic unitarity condition for the $A_l^{(i)}$'s and $B_l^{(i)}$'s reads

$$B_l^{(i)}(k)=[A_l^{(i)}(k)]^2+[B_l^{(i)}(k)]^2\,. \tag{5}$$

We now assume that the following properties of $\delta_l(k)$ are verified:

(i) The phase shifts $\delta_l(k)$ are odd functions of k for k real below the inelastic threshold:

$$\delta_l(-k)=-\delta_l(k)\,. \tag{6}$$

(ii) The phase shifts $\delta_l(k)$ satisfy, in the limit $k\to 0$, the threshold condition

$$\delta_l(k)\underset{k\to 0}{\longrightarrow}0(k^{2l+1})\,. \tag{7}$$

(iii) The partial-wave amplitudes possess the well-known analyticity properties in the complex k plane.[4] This ensures that we can expand them as power series in k converging inside a circle centered at the origin. As a consequence of the above conditions, below inelastic threshold we can now write

$$\begin{aligned}A_l^{(i)}(k)&=\sum_{n=0}^{\infty}A_{ln}^{(i)}k^{2l+1+2n}\,,\\B_{l'}^{(i)}(k)&=\sum_{m=0}^{\infty}B_{l'm}^{(i)}k^{4l'+2+2m}\,.\end{aligned} \tag{8}$$

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[1] N. A. Törnqvist, Phys. Rev. **161**, 1591 (1967).
[2] In a recent paper [Nucl. Phys. **B6**, 187 (1968)], Törnqvist claims that the analysis of forward data suggests $n=0$, but $n=1$ in the backward region. His conclusions are in agreement with the results of the present paper.
[3] D. Bessis and A. Martin, Nuovo Cimento **52**, 719 (1967).
[4] See, for instance, M. Froissart and R. Omnès, in *High-Energy Physics*, edited by C. De Witt and M. Jacob (Gordon and Breach Science Publishers, Inc., New York, 1965), p. 91.

Inserting Eq. (8) into the unitarity relation Eq. (5), we get

$$\sum_{n=0}^{\infty} k^{4l+2n}B_{ln}{}^{(i)}=\sum_{m,r=0}^{\infty} k^{4l+2m+2r}A_{lm}{}^{(i)}A_{lr}{}^{(i)} +\sum_{m,r=0}^{\infty} k^{8l+2m+2r+2}B_{lm}{}^{(i)}B_{lr}{}^{(i)}. \quad (9)$$

For the terms of order $2N$ with

$$N=2l+n, \quad (10)$$

equating equal powers of k on both sides of Eq. (9), we obtain the elastic unitarity condition for the coefficients $A_{ln}{}^{(i)}$ and $B_{ln}{}^{(i)}$:

$$B_{l,N-2l}{}^{(i)}=\sum_{n=0}^{N-2l} A_{l,N-2l-n}{}^{(i)}A_{ln}{}^{(i)} +\sum_{n=0}^{N-4l-1} B_{l,N-4l-n-1}{}^{(i)}B_{ln}{}^{(i)}\theta(N-4l-1), \quad (11)$$

where $\theta(x)$ is the usual step function.

Turning now to Törnqvist's condition, Eq. (1), we note that it implies

$$A^{(1)}(k,\theta)B^{(3)}(k,\theta)-A^{(3)}(k,\theta)B^{(1)}(k,\theta)=0. \quad (12)$$

Using the partial-wave decomposition (3), Eq. (12) becomes

$$\sum_{l,l'=0}^{\infty} (2l+1)(2l'+1)P_l(\cos\theta)P_{l'}(\cos\theta) \times[A_l{}^{(1)}(k)B_{l'}{}^{(3)}(k)-A_l{}^{(3)}(k)B_{l'}{}^{(1)}(k)]=0. \quad (13)$$

We can now use the completeness of the set of Legendre polynomials to express the product $P_l(\cos\theta)P_{l'}(\cos\theta)$ in terms of $P_{l''}(\cos\theta)$, and we finally obtain

$$\sum_{l=0}^{\infty}\sum_{l'=|l-l''|}^{l+l''} (2l+1)(2l'+1)C_{ll'l''} \times[A_l{}^{(1)}(k)B_{l'}{}^{(3)}(k)-A_l{}^{(3)}(k)B_{l'}{}^{(1)}(k)]=0, \quad (14)$$

where $C_{ll'l''}$ is the square of the Clebsch-Gordan coefficient given by

$$C_{ll'l''}=\frac{1}{2\pi}\frac{\Gamma(\frac{1}{2}(l+l'+l'')+1)\Gamma(\frac{1}{2}(l+l'-l''+1))\Gamma(\frac{1}{2}(l-l'+l''+1))\Gamma(\frac{1}{2}(-l+l'+l''+1))}{\Gamma(\frac{1}{2}(l+l'+l''+3))\Gamma(\frac{1}{2}(l+l'-l'')+1)\Gamma(\frac{1}{2}(l-l'+l'')+1)\Gamma(\frac{1}{2}(-l+l'+l'')+1)}$$

if $l+l'+l''=$ even integer and $|l-l'|\leq l''\leq l+l'$,

$=0$ otherwise. (15)

Equation (14) is a set of infinitely many coupled equations for each possible value of l'' [with l and l' constrained by the conditions specified in Eq. (15)].

If we now use in Eq. (14) the power-series expansions (8), Törnqvist's condition, Eq. (1), is finally equivalent to the infinite set of coupled equations

$$\sum_{n,m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{l'=|l-l''|}^{l+l''} (2l+1)(2l'+1)C_{ll'l''}k^{2l+4l'+2n+2m} \times[A_{ln}{}^{(1)}B_{l'm}{}^{(3)}-A_{ln}{}^{(3)}B_{l'm}{}^{(1)}]=0. \quad (16)$$

Equations (16) and (9) are the basic set of equations to be solved. One trivial solution of these equations consists of

$$A_{ln}{}^{(3)}=A_{ln}{}^{(1)},\quad B_{l'm}{}^{(3)}=B_{l'm}{}^{(1)} \quad (\text{for all } ln,\ l'm). \quad (17)$$

In the next section, we shall solve, at each order N, Eqs. (9) and (16), where, in the latter,

$$N=l+2l'+m+n, \quad (18)$$

and we will show that the trivial solution (17) is, in general, the only one, since it holds at every order.

Once Eqs. (17) are proved, from Eqs. (3) and (8), we obtain

$$A^{(3)}(k,\theta)=A^{(1)}(k,\theta),\quad B^{(3)}(k,\theta)=B^{(1)}(k,\theta), \quad (19)$$

and the theorem will be proved.

3. PROOF OF THE THEOREM

We shall prove the validity of Eqs. (17) by using a recurrence procedure, namely, we shall prove that they hold at the zeroth order; then we shall assume that they are valid at the order $N-1$ and show that they are still valid at the order N.

For $N=0$ [N as defined by Eq. (10) and Eq. (18) for the sets of equations (9) and (16), respectively], we simply get

$$A_{00}{}^{(1)}B_{00}{}^{(3)}-A_{00}{}^{(3)}B_{00}{}^{(1)}=0, \quad (9')$$

and

$$B_{00}{}^{(1)}=[A_{00}{}^{(1)}]^2,\quad B_{00}{}^{(3)}=[A_{00}{}^{(3)}]^2. \quad (16')$$

Inserting (16′) into (9′), since $A_{00}{}^{(1)}\neq0$ and $A_{00}{}^{(3)}\neq0$ [because otherwise the S waves would not satisfy the threshold condition (ii)], we obtain

$$A_{00}^{(3)}=A_{00}^{(1)}\,,\qquad B_{00}^{(3)}=B_{00}^{(1)}=[A_{00}^{(1)}]^2\,,\tag{20}$$

i.e., Eqs. (17) hold in the lowest order.

We now assume that Eqs. (17) hold up to the (arbitrary) order $N-1$, i.e.,

$$A_{ln}^{(3)}=A_{ln}^{(1)}\tag{21}$$

for all l,n such that

$$l+n\leqslant N-1\,,\tag{22}$$

and

$$B_{l'm}^{(3)}=B_{l'm}^{(1)}\tag{23}$$

for all l',m such that

$$2l'+m\leqslant N-1\,,\tag{24}$$

and we set to prove that Eqs. (17) are still valid at the order N (i.e., for all l,n and l',m such that $l+n\leqslant N$ and $2l'+m\leqslant N$).

First of all, because of the structure of Eqs. (16) and because of (21), (22), (23), (24), it is evident that the only contributions to Eqs. (16) come, for any given $l''\leqslant N$, from

(a) $l=l''$, $l+n=N$, $l'=m=0$,

(b) $l'=l''$, $2l'+m=N$, $l=n=0$.

Moreover, the term (b) is absent if $N<2l''$.

In other words, the only contributions which survive arise when the power k^{2N} originates from either $A(k,\theta)$ or $B(k,\theta)$. All other contributions must vanish because of the equalities (21) and (23) already assumed at the order $N-1$.

At the order N, Eqs. (16) become, then, for any $l\leqslant N$,

$$A_{l,N-l}^{(1)}B_{00}^{(3)}-A_{l,N-l}^{(3)}B_{00}^{(1)}+[A_{00}^{(1)}B_{l,N-2l}^{(3)}-A_{00}^{(3)}B_{l,N-2l}^{(1)}]\theta(N-2l)=0\,.\tag{25}$$

Let us now turn to the unitarity conditions (11). First we restrict ourselves to

$$2l\leqslant N<4l+1\Rightarrow l>\tfrac{1}{4}(N+1)>0\,,\tag{26}$$

in which case

$$B_{l,N-2l}^{(i)}=\sum_{n=0}^{N-2l}A_{l,N-2l-n}^{(i)}A_{ln}^{(i)}\,.\tag{27}$$

If we consider the maximum index on the right-hand side of Eq. (27), we have

$$(l+n)_{\max}=(l+N-2l-n)_{\max}=N-l\leqslant N-1\,,\tag{28}$$

where the last inequality follows from (26). Therefore, because of (21) and (22), we get

$$B_{l,N-2l}^{(3)}=B_{l,N-2l}^{(1)}\quad\text{for}\quad 4l+1>N\geqslant 2l\,.\tag{29}$$

Using (29) in (25) and (20), we then obtain

$$A_{l,N-l}^{(3)}=A_{l,N-l}^{(1)}\quad\text{for}\quad 4l+1>N\geqslant l\,.\tag{30}$$

We can then limit our consideration of Eq. (25) to the case

$$N\geqslant 4l+1\,,\tag{31}$$

when Eqs. (20) become

$$A_{00}^{(1)}(A_{l,N-l}^{(1)}-A_{l,N-l}^{(3)})+(B_{l,N-2l}^{(3)}-B_{l,N-2l}^{(1)})=0\,.\tag{32}$$

Taking the difference $B_{l,N-2l}^{(3)}-B_{l,N-2l}^{(1)}$ from the unitarity equations (11), we have

$$B_{l,N-2l}^{(3)}-B_{l,N-2l}^{(1)}=\sum_{n=0}^{N-2l}[A_{l,N-2l-n}^{(3)}A_{ln}^{(1)}-A_{l,N-2l-n}^{(1)}A_{ln}^{(3)}]+\sum_{n=0}^{N-4l-1}[B_{l,N-4l-n-1}^{(3)}B_{ln}^{(1)}-B_{l,N-4l-n-1}^{(1)}B_{ln}^{(3)}]\,.\tag{33}$$

In the second sum on the right-hand side of Eq. (33), the consideration of the maximum index gives

$$(2l+n)_{\max}=(2l+N-4l-n-1)_{\max}=N-2l-1\leqslant N-1\tag{34}$$

and therefore this sum is identically zero because of Eqs. (23) and (24). The first sum on the right-hand side of Eq. (33) has, on the other hand, a sum of highest powers such that

$$(l+n)_{\max}=(l+n-2l-n)_{\max}=N-l\,.\tag{35}$$

Therefore, it gives a nonzero contribution only for $l=0$. In other words

$$B_{l,N-2l}^{(3)}-B_{l,N-2l}^{(1)}=0\,,\quad (N\geqslant 2l>0)\tag{36a}$$

$$B_{0N}^{(3)}-B_{0N}^{(1)}=2(A_{0N}^{(3)}A_{00}^{(3)}-A_{0N}^{(1)}A_{00}^{(1)})\,.\tag{36b}$$

From Eqs. (32) and (36a) we then get

$$A_{l,N-l}^{(3)}=A_{l,N-l}^{(1)}\,,\quad (N\geqslant 2l>0)\tag{37}$$

whereas from (32) and (36b)

$$A_{00}^{(1)}(A_{0N}^{(1)}-A_{0N}^{(3)})+(B_{0N}^{(3)}-B_{0N}^{(1)})=0\,,\qquad 2A_{00}^{(1)}(A_{0N}^{(1)}-A_{0N}^{(3)})+(B_{0N}^{(3)}-B_{0N}^{(1)})=0\,,\tag{38}$$

and therefore, finally,

$$B_{0N}^{(3)}=B_{0N}^{(1)}\,,\qquad A_{0N}^{(3)}=A_{0N}^{(1)}\,,$$

which completes the proof of the validity of Eqs. (17) and thus of Eqs. (19).

4. CONCLUDING REMARKS

In the present paper, we have shown that if two amplitudes that are not identically zero satisfy the Törnqvist relation (1), then their real and imaginary parts must be related through Eqs. (19), if the very general conditions (i), (ii), and (iii) (Sec. 2) hold, as well as unitarity. Since Eqs. (19) imply that the two amplitudes have equal moduli, the corresponding angular distributions should be equal.

This conclusion contradicts Törnqvist's relation, and therefore we conclude that the only possible way out is that Eq. (1) is valid only in a part of the angular interval. We are comforted by the fact that the phases of the $\frac{1}{2}$ and $\frac{3}{2}$ amplitudes constructed with the phenomenological πN phase shifts do not satisfy the Törnqvist relation.[5] Of course, our conclusion was reached only for the spinless case, but we have little doubt that the proof could be carried out also in the physically interesting case.

[5] V. Grecchi and G. Turchetti (private communication).

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Double Bootstrap of the ρ and f^0

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We obtain a "double bootstrap" of the ρ and f^0 mesons, in which the input force consists of both ρ and f^0 exchange, and both particles are obtained as ouptut resonances, with corresponding input and output values of masses and widths being equal to within about 1%. The values obtained for the resonance parameters are $m_\rho = 750$ MeV, $\Gamma_\rho = 162$ MeV, $m_f = 1240$ MeV, and $\Gamma_f = 100$ MeV, in reasonable agreement with experiment. The ρ width is somewhat larger than most measured values, but much smaller than generally obtained in bootstrap calculations in which only ρ exchange is included as an input force. Our calculations are carried out using the equivalent-potential method, and are free of arbitrary parameters. The model yields a second $I=0$, $j=2$ resonance, presumably to be associated with the $f^{0\prime}$, and also a broad second P-wave resonance. The latter may correspond to the ρ' Regge trajectory hypothesized in several Regge-pole analyses of high-energy data, especially the polarization in πN charge exchange. The parameters of the predicted resonance do not agree with those of any known resonance, but it might be difficult to observe because of its width. The output Regge trajectories predicted by the model are roughly linear. The ρ trajectory has a slope about half the generally accepted experimental value of 1 BeV^{-2}. We comment in passing that general considerations, based only on the crossing matrix, make it somewhat difficult to reconcile the latter value with the absence of an $I=0$ D-wave resonance at an energy less than 1250 MeV.

A VIGOROUS attack has been made in recent years on the problem of determining the parameters of the ρ resonance in $\pi\pi$ scattering by a "bootstrap" calculation, in which one first assumes that the dominant force producing the ρ is ρ exchange, and then seeks self-consistent values for the mass and width. A variety of such calculations have been published.[1-3] One finds in general that one can obtain the ρ from these calculations at more or less the experimental value of the mass, though in many of these calculations this is due to the freedom one has to adjust a cutoff parameter which enters because of the exchange of a vector particle. However, one invariably obtains a theoretical value for the width which is much too large. Most searches for solutions to this problem have tended in the direction of including inelastic effects, either through the inclusion of one or more inelastic channels explicitly in a multichannel calculation, or through the inclusion of an inelasticity parameter in a one-channel calculation. Examples are given in Ref. 2. While the inclusion of inelastic effects has tended to bring the theoretical and experimental widths into better agreement, the theoretical values remain too large.[2] It is possible that if one could include further inelastic channels in a correct way, the theoretical width would be further improved, though in practice this will be very difficult to do.

A second possible effect which might contribute to narrowing the theoretical resonance widths has been suggested by Chew.[4] Chew works in the context of the "new form of the strip approximation," in which it is assumed that the generalized potential (that part of the amplitude not containing direct-channel poles) can

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[1] G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961); B. H. Bransden and J. W. Moffat, *ibid.* **21**, 505 (1961); L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962); M. Bander and G. L. Shaw, *ibid.* **135**, B267 (1964); P. D. B. Collins, *ibid.* **142**, 1163 (1966).

[2] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962); L. A. P. Balázs, *ibid.* **132**, 867 (1963); J. R. Fulco, G. L. Shaw, and D. Y. Wong, *ibid.* **137**, B1242 (1965); R. Atkinson, III, and A. E. Everett, *ibid.* **154**, 1430 (1967).

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[4] G. F. Chew, Phys. Rev. **140**, B1427 (1965).