# Elastic Scattering from Deuterium*†

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Scattering of elementary particles from deuterons at BeV energies is considered. Corrections to the Glauber theory, resulting from violations of its high-energy and small-momentum-transfer assumptions, are found significant for experiments already performed. The most important correction results from including the principal-value part of the propagator in double scattering. Triple and higher-order multiple scattering are calculated in a model and are predicted to be important at momentum transfers of $-t>4$ BeV$^2$. Available data on the energy dependence of the "screening correction" to deuteron total cross sections are shown to disagree with the Glauber theory, and a possible phenomenological treatment of that disagreement is discussed.

## I. INTRODUCTION

THE deuteron can be considered a nonrelativistic bound state of neutron and proton, to a certain accuracy. One can therefore hope to calculate elastic scattering of elementary particles on deuterons in terms of amplitudes for scattering on neutrons and protons, making use of the fact that the deuteron is loosely bound to treat the neutron and proton as free particles. This should be a good approximation, because the average $np$ separation ($\sim$3 F) is large compared to the range of the $np$ interaction ($\sim\hbar/m_\pi c=1.4$ F).

Elastic scattering from deuterium has been discussed many times using the "high-energy approximation" of Glauber.[1] I wish to reexamine the validity of that approximation, while staying within the framework of treating the deuteron nonrelativistically. Various attempts to extend this framework, e.g., by including certain relativistic effects in the deuteron-neutron-proton vertices,[2] or by "Reggeizing" the scattering amplitudes,[3] will not be discussed.

The organization of this paper is as follows. Section II contains a derivation of the *multiple-scattering series*, which relates the amplitude for scattering on a deuteron to the slightly-off-mass-shell amplitudes for scattering on free protons and neutrons. The derivation makes use of formal scattering theory, and is similar to one given by Everett, following ideas of Chew and Goldberger.[4] The relation of this series to the Glauber approximation, and to a certain class of Feynman diagrams, is then discussed. Section III contains model calculations appropriate to $pd$, $\pi d$, $Kd$, and $\bar{p}d$ scattering at BeV energies. The model consists in assuming simple Gaussian expressions for the deuteron wave function and the nucleon amplitudes, in order to facilitate calculation. More accurate expressions will not be difficult to handle by computer when new experimental data make their use appropriate. Despite its simplicity, the model calculation is shown to agree reasonably well with $pd$ data of Palevsky *et al.*[5] Predictions of the theory for total cross sections are discussed in Sec. IV. They are shown to be *inconsistent* with available $\pi d$ data, and probably also $pd$ data. Experimental confirmation of this discrepancy, and theoretical understanding of it, should be sought. For the present, the standard procedures for extracting cross sections of $p$, $K$, $\bar{p}$, on neutrons from experiments on deuterium must be considered unreliable. Some possible flaws in the theory are discussed in Sec. V, and recommendations are made for future experiments.

## II. MULTIPLE-SCATTERING FORMALISM

In formal scattering theory, the scattering of a particle $x$ on a deuteron is described by the Hamiltonian

$$H=H_0+V\,,$$

where

$$\begin{aligned}H_0&=K_x+K_n+K_p+V_{np}\,,\\ V&=V_{xn}+V_{xp}\,.\end{aligned}\tag{1}$$

Here $K_p$, $K_n$, and $K_x$ are the kinetic energies of the proton, the neutron, and the particle $x$; $V_{np}$ is the deuteron binding potential; and $V_{xn}$, $V_{xp}$ are the two-body interaction potentials which cause the scattering. Three-body forces are assumed negligible. Relativistic kinematics are used, e.g., $K_x=(q_x^2+m_x^2)^{1/2}$, so the incident particle need not be assumed nonrelativistic in the laboratory frame. The scattering amplitude $T$ satisfies the Lipp-

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[4] A. Everett, Phys. Rev. **126**, 831 (1962); G. Chew and M. Goldberger, *ibid.* **87**, 778 (1952). A very careful discussion of multiple scattering and the impulse approximation, applied to low-energy $Nd$ and $Kd$ scattering, is given by N. M. Queen, Nucl. Phys. **55**, 177 (1964); **66**, 673 (1965); see also his references.
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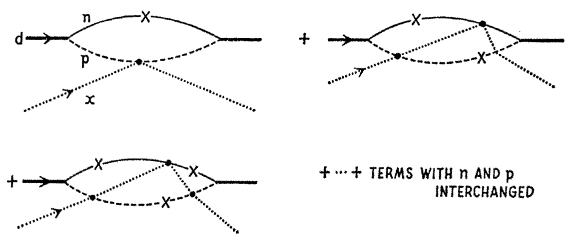


FIG. 1. Multiple-scattering contributions to $xd$ elastic scattering. The lines marked by crosses are put on-mass-shell.

mann-Schwinger equation

$$T=V+VGT,\quad G=1/(E-H_0+i\epsilon)\,, \tag{2}$$

where $E=E_x+E_d$ is the energy of the initial (or final) state.[6] The two-body amplitudes which describe $xn$, $xp$ scattering similarly satisfy

$$\begin{aligned} t_{xn}&=V_{xn}+V_{xn}gt_{xn}\,,\\ t_{xp}&=V_{xp}+V_{xp}gt_{xp}\,,\\ g&=1/(E'-K_x-K_n-K_p+i\epsilon)\,. \end{aligned} \tag{3}$$

The inclusion of both $K_n$ and $K_p$ in defining the propagator $g$ is permissible because for $xn$ ($xp$) scattering, $K_p(K_n)$ is a constant. The energy $E'$ is a sum of free-particle energies: $E'=E_x+E_n+E_p$.

The *impulse approximation* consists in setting $G$ equal to $g$, i.e., neglecting the deuteron binding potential for the duration of the collision. This approximation is expected to be a good one, provided the duration of the scattering process is short compared to the characteristic interaction time of the deuteron, which could be defined as the average separation of the neutron and proton divided by their average relative velocity. In general, that will be the case if the particle $x$ is relativistic; it may be false, however, if the $xN$ interactions are dominated by resonances which provide long time delays, as in $\pi d$ scattering below 2 BeV. Making the impulse approximation, Eqs. (3) and (4) can be iterated

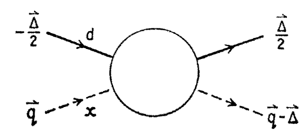


FIG. 2. Kinematics of $xd$ scattering in the Breit frame.

and then combined to yield

$$T\cong(t_{xn}+t_{xp})+(t_{xn}gt_{xp}+t_{xp}gt_{xn})+(t_{xn}gt_{xp}gt_{xn}+t_{xp}gt_{xn}gt_{xp})+\cdots. \tag{4}$$

The successive terms of this series have obvious interpretations as single, double, triple, etc., scatterings separated by free propagation as illustrated in Fig. 1. The multiple-scattering terms can be shown to equal what one obtains by interpreting the pictures in Fig. 1 as Feynman diagrams, if one (a) includes form factors at the $d$-$n$-$p$ vertices, making the appropriate identification of the form factor with the deuteron wave function, (b) includes form factors at the $x$-$x$-$n$-$n$, $x$-$x$-$p$-$p$ vertices appropriate to off-shell $xn$ and $xp$ scattering, and (c) retains only the mass-shell parts of the propagators for the lines marked by crosses.[3]

Before proceeding with Eq. (4), it is necessary to discuss kinematics. The best coordinate frame for considering the deuteron to be nonrelativistic is the Breit frame, in which the deuteron's kinetic energy is kept as small as possible for given momentum transfer. In that frame (see Fig. 2),

$$\begin{aligned} t&=\mathbf{\Delta}^2=-2\mathbf{q}\cdot\mathbf{\Delta}\,,\\ s&=m_d^2+m_x^2+\tfrac{1}{2}\mathbf{\Delta}^2+2(\mathbf{q}^2+m_x^2)^{1/2}(\tfrac{1}{4}\mathbf{\Delta}^2+m_d^2)^{1/2}\,,\\ \frac{d\sigma}{dt}&=16\pi^5\frac{(s-m_d^2-m_x^2+\frac{1}{2}t)^2}{[s-(m_d+m_x)^2][s-(m_d-m_x)^2]}|T|^2. \end{aligned} \tag{5}$$

At small momentum transfer, the Breit frame is similar to the laboratory frame.

Denoting the momentum-space deuteron wave function at relative momentum $\mathbf{k}$ by $\varphi(\mathbf{k})$, Eq. (4) can be written explicitly as

$$\begin{aligned} \langle-\tfrac{1}{2}\mathbf{\Delta},\mathbf{q}|T|\tfrac{1}{2}\mathbf{\Delta},\mathbf{q}-\mathbf{\Delta}\rangle&=\int d\mathbf{k}\;\varphi(\mathbf{k}+\tfrac{1}{4}\mathbf{\Delta})\varphi(\mathbf{k}-\tfrac{1}{4}\mathbf{\Delta})t_{xp}(-\tfrac{1}{2}\mathbf{\Delta}-\mathbf{k},\,\mathbf{q}\rightarrow\tfrac{1}{2}\mathbf{\Delta}-\mathbf{k},\,\mathbf{q}-\Delta)\\ &+\int d\mathbf{k}d\mathbf{s}\;\varphi(\mathbf{s}+\tfrac{1}{2}\mathbf{k})\varphi(\mathbf{s}-\tfrac{1}{2}\mathbf{k})\cdot t_{xp}(-\mathbf{s}-\tfrac{1}{2}\mathbf{k}-\tfrac{1}{4}\mathbf{\Delta},\,\mathbf{q}\rightarrow-\mathbf{s}+\tfrac{1}{2}\mathbf{k}+\tfrac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{k}-\tfrac{1}{2}\mathbf{\Delta})\\ &\times\frac{t_{xn}(\mathbf{s}+\frac{1}{2}\mathbf{k}-\frac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{k}-\frac{1}{2}\mathbf{\Delta}\rightarrow\mathbf{s}-\frac{1}{2}\mathbf{k}+\frac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{\Delta})}{(\mathbf{q}^2+m_x^2)^{1/2}-[(\mathbf{q}-\mathbf{k}-\frac{1}{2}\mathbf{\Delta})^2+m_x^2]^{1/2}+i\epsilon}+\int d\mathbf{k}d\mathbf{r}d\mathbf{s}\;\varphi(\mathbf{s}+\tfrac{1}{2}\mathbf{k})\varphi(\mathbf{s}-\tfrac{1}{2}\mathbf{k})\\ &\times t_{xp}(-\mathbf{s}-\tfrac{1}{2}\mathbf{k}-\tfrac{1}{4}\mathbf{\Delta},\,\mathbf{q}\rightarrow-\mathbf{s}+\mathbf{r},\,\mathbf{q}-\mathbf{r}-\tfrac{1}{2}\mathbf{k}-\tfrac{1}{4}\mathbf{\Delta})\frac{t_{xn}(\mathbf{s}+\frac{1}{2}\mathbf{k}-\frac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{r}-\frac{1}{2}\mathbf{k}-\frac{1}{4}\mathbf{\Delta}\rightarrow\mathbf{s}-\frac{1}{2}\mathbf{k}+\frac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{r}+\frac{1}{2}\mathbf{k}-\frac{3}{4}\mathbf{\Delta})}{(\mathbf{q}^2+m_x^2)^{1/2}-[(\mathbf{q}-\mathbf{r}-\frac{1}{2}\mathbf{k}-\frac{1}{4}\mathbf{\Delta})^2+m_x^2]^{1/2}+i\epsilon}\\ &\times\frac{t_{xn}(-\mathbf{s}+\mathbf{r},\,\mathbf{q}-\mathbf{r}+\frac{1}{2}\mathbf{k}-\frac{3}{4}\mathbf{\Delta}\rightarrow-\mathbf{s}+\frac{1}{2}\mathbf{k}+\frac{1}{4}\mathbf{\Delta},\,\mathbf{q}-\mathbf{\Delta})}{(\mathbf{q}^2+m_x^2)^{1/2}-[(\mathbf{q}-\mathbf{r}+\frac{1}{2}\mathbf{k}-\frac{3}{4}\mathbf{\Delta})^2+m_x^2]^{1/2}+i\epsilon}+\cdots+\text{terms with } n \text{ and } p \text{ interchanged}. \end{aligned} \tag{6}$$

[6] M. Goldberger and K. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

In writing Eq. (6), I have neglected the contributions to the energy denominators which result from recoil of the nucleon. In others, I have approximated both of the propagators $G$ and $g$ by $g_0=(E_x-K_x+i\epsilon)^{-1}$. The approximations $G\cong g\cong g_0$ are expected to be very good, as can be shown using the model described in Sec. III. The lowest-order correction resulting from replacement of $G$ by $g$ (impulse approximation) can be written as

$$\begin{aligned}(t_{xn}+t_{xp})(G-g)(t_{xn}+t_{xp}) &= (t_{xn}+t_{xp})G[V_{np},g(t_{xn}+t_{xp})] \\ &\cong (t_{xn}+t_{xp})g_0^2[V_{np},\, t_{xn}+t_{xp}], \quad (7)\end{aligned}$$

making use of the fact that $T$ is to operate on a state for which $K_n+K_p+V_{np}-E_d=0$ by the Schrödinger equation for the deuteron. The further correction resulting from replacement of $g$ by $g_0$ (neglect of recoil) can be written in lowest order as

$$t_{xn}(g-g_0)t_{xp}+t_{xp}(g-g_0)t_{xn} \cong t_{xn}g_0^2[K_p,t_{xp}]+t_{xp}g_0^2[K_n,t_{xn}]. \quad (8)$$

The $xN$ amplitudes in Eq. (6) are to be obtained from experimental $xN$ differential cross sections via

$$|t_{xN}(\mathbf{p},\mathbf{q}\to\mathbf{p}',\mathbf{q}')|^2=\frac{1}{64\pi^5}\times\frac{[s-(m_x+m_N)^2][s-(m_x-m_N)^2]}{(\mathbf{q}^2+m_x^2)^{1/2}(\mathbf{q}'^2+m_x^2)^{1/2}(\mathbf{p}^2+m_N^2)^{1/2}(\mathbf{p}'^2+m_N^2)^{1/2}}\times\frac{d\sigma}{dt}(s,t). \quad (9)$$

Because these amplitudes are off the mass shell, the Lorentz-invariant kinematic variables $s$ and $t$ are not uniquely defined; possible choices for them are, for example,

$$\begin{aligned}s_1&=m_x^2+m_N^2-2\mathbf{q}\cdot\mathbf{p}+2(\mathbf{p}^2+m_N^2)^{1/2}(\mathbf{q}^2+m_x^2)^{1/2},\\ s_2&=m_x^2+m_N^2-2\mathbf{q}'\cdot\mathbf{p}'+2(\mathbf{p}'^2+m_N^2)^{1/2}(\mathbf{q}'^2+m_x^2)^{1/2}, \quad (10)\\ t_1&=2m_N^2+2\mathbf{p}\cdot\mathbf{p}'-2(\mathbf{p}^2+m_N^2)^{1/2}(\mathbf{p}'^2+m_N^2)^{1/2},\\ t_2&=2m_x^2+2\mathbf{p}\cdot\mathbf{p}'-2(\mathbf{p}^2+m_x^2)^{1/2}(\mathbf{p}^2+m_N^2)^{1/2}.\end{aligned}$$

At high energy, elastic cross sections change only slowly with energy. They are also sharply peaked in the forward direction, which implies that only small momentum transfer, and a fairly small region of energy, is important in the multiple-scattering integrals of Eq. (6), because the deuteron wave function allows only a restricted range of Fermi momenta. It is therefore possible to make the approximation

$$|t_{xN}(\mathbf{p},\mathbf{q}\to\mathbf{p}',\mathbf{q}')|^2\cong\frac{\mathbf{q}^2}{\mathbf{q}^2+m_x^2}\frac{1}{16\pi^5}\frac{d\sigma}{dt}(s,t), \quad (11)$$

where $s=2m_N(\mathbf{q}^2+m_x^2)^{1/2}$ and $t=-(\mathbf{p}'-\mathbf{p})^2$. [At low energies, $(d\sigma/dt)(s,t)$ may vary rapidly with $s$ because of direct-channel resonances. In such cases, e.g., for $\pi d$ scattering below 2 BeV, it is essential to include this variation when carrying out the double-scattering integral. Either of the forms $s_1$ or $s_2$ given in Eq. (10) can be used for $s$.]

In obtaining the two-body amplitudes from Eq. (9), it is necessary to make some assumption about the phase of the amplitude as well as to choose a way of going off the mass shell. Unfortunately, phases of elastic scattering amplitudes are generally known only in the forward direction where they can be obtained from measurements of Coulomb interference or by the optical theorem. It would also be necessary to postulate the spin dependence of the amplitude, unless one is willing to ignore spin entirely, as I shall do here, in the hope that spin-flip elastic amplitudes are relatively small.

When the $xN$ amplitudes can be considered to vary only with the magnitude of three-momentum transfer, Eq. (6) can be written in terms of the deuteron spatial form factor:

$$S(\mathbf{p})=\int\varphi(\mathbf{k}+\tfrac{1}{2}\mathbf{p})\varphi(\mathbf{k}-\tfrac{1}{2}\mathbf{p})d\mathbf{k}=\int|\psi(\mathbf{x})|^2e^{i\mathbf{p}\cdot\mathbf{x}}d\mathbf{x}, \quad (12)$$

where $\psi(\mathbf{x})$ is the coordinate-space wave function for separation distance $\mathbf{x}$. Also making the high-energy, small-momentum-transfer approximation

$$(\mathbf{q}^2+m_x^2)^{1/2}-[(\mathbf{q}-\mathbf{p})^2+m_x^2]^{1/2}=\frac{\mathbf{q}\cdot\mathbf{p}}{(\mathbf{q}^2+m_x^2)^{1/2}}\times[1+O(|\mathbf{p}|/|\mathbf{q}|)] \quad (13)$$

in the energy denominators, and assuming $\mathbf{q}\cdot\boldsymbol{\Delta}=|\boldsymbol{\Delta}|/2|\mathbf{q}|\cong 0$, the result is

$$\begin{aligned}\langle-\tfrac{1}{2}\boldsymbol{\Delta},\mathbf{q}|T|\tfrac{1}{2}\boldsymbol{\Delta},\,\mathbf{q}-\boldsymbol{\Delta}\rangle&\cong t_{xp}(|\boldsymbol{\Delta}|)S(\tfrac{1}{2}\boldsymbol{\Delta})+\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{1/2}\int d\mathbf{k}\frac{S(\mathbf{k})t_{xp}(|\mathbf{k}+\frac{1}{2}\boldsymbol{\Delta}|)t_{xn}(|\mathbf{k}-\frac{1}{2}\boldsymbol{\Delta}|)}{\hat{q}\cdot\mathbf{k}+i\epsilon}\\ &+\left(\frac{\mathbf{q}^2+m_x^2}{q^2}\right)\int d\mathbf{k}d\mathbf{r}\frac{S(\mathbf{k})t_{xp}(|\mathbf{r}+\frac{1}{2}\mathbf{k}+\frac{1}{4}\boldsymbol{\Delta}|)t_{xn}(|\mathbf{k}-\frac{1}{2}\boldsymbol{\Delta}|)t_{xp}(|\mathbf{r}-\frac{1}{2}\mathbf{k}-\frac{1}{4}\boldsymbol{\Delta}|)}{[\hat{q}\cdot(\mathbf{r}+\frac{1}{2}\mathbf{k})+i\epsilon][\hat{q}\cdot(\mathbf{r}-\frac{1}{2}\mathbf{k})+i\epsilon]}\\ &+\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{3/2}\int d\mathbf{k}d\mathbf{r}d\mathbf{s}\;S(\mathbf{k})t_{xp}(|\mathbf{r}+\tfrac{1}{2}\mathbf{k}+\tfrac{1}{2}\boldsymbol{\Delta}|)t_{xn}(|\mathbf{s}-\mathbf{k}-\mathbf{r}+\tfrac{1}{4}\boldsymbol{\Delta}|)\\ &\times\frac{t_{xp}(|\frac{1}{2}\mathbf{k}-\mathbf{r}+\frac{1}{4}\boldsymbol{\Delta}|)t_{xn}(|\mathbf{r}-\mathbf{s}+\frac{1}{4}\boldsymbol{\Delta}|)}{[\hat{q}\cdot(\mathbf{r}+\frac{1}{2}\mathbf{k})+i\epsilon][\hat{q}\cdot(\mathbf{s}-\frac{1}{2}\mathbf{k})+i\epsilon][\hat{q}\cdot(\mathbf{s}-\mathbf{r})+i\epsilon]}+\cdots+\text{terms with } n,p \text{ interchanged}. \quad (14)\end{aligned}$$

If the double-scattering integral in Eq. (14) is written in spherical coordinates, its principal-value part is seen to vanish. The remaining ($\delta$-function) part yields a two-dimensional integral over a plane perpendicular to the beam direction. If triple and higher-order multiple scattering effects are neglected, Eq. (14) becomes

$$\langle -\tfrac{1}{2}\mathbf{\Delta},\mathbf{q}|T|\tfrac{1}{2}\mathbf{\Delta},\mathbf{q}-\mathbf{\Delta}\rangle \cong S(\tfrac{1}{2}|\mathbf{\Delta}|)[t_{xn}(|\mathbf{\Delta}|)+t_{xp}(|\mathbf{\Delta}|)]-2\pi i\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{1/2}\int_0^\infty dk\, kS(k)\int_0^{2\pi} d\theta$$
$$\times t_{xn}((k^2+\tfrac{1}{4}\mathbf{\Delta}^2-k|\mathbf{\Delta}|\cos\theta)^{1/2})t_{xp}((k^2+\tfrac{1}{4}\mathbf{\Delta}^2+k|\mathbf{\Delta}|\cos\theta)^{1/2}), \quad (15)$$

which is the standard result of the Glauber approximation. Important corrections to this formula arise from two sources: the small-momentum-transfer approximations and the neglect of higher-order multiple-scattering effects. The size of these corrections is examined in Sec. III by means of a model. The small-momentum-transfer approximations are found to be rather good, with correction terms being of order $t/4m_d^2$ and $(t/16|\mathbf{q}|)\times$(radius of deuteron). However, the correction terms can be significant at energies of a few BeV or less, because they alter the phase of the double-scattering term by making its principal-value part nonzero. This affects the shape of the differential cross section in the region around $t=-0.35$ BeV$^2$ where single and double scattering interfere strongly.

According to the model, triple and higher-order multiple scattering is negligible at small momentum transfer, but very important at large momentum transfer. In calculations made by Glauber and his followers for scattering on deuterium, triple and higher-order multiple scattering is excluded from the outset, on the grounds that in order for the incident particle to interact with a given nucleon more than once, it would have to be scattered through a large angle at least once. This argument is incorrect because of the finite range of the two-body interactions. The situation in coordinate space is illustrated in Fig. 3. The neutron and proton, as seen by the incident particle, can be represented as spheres of radius about 1 F. [Experimental elastic cross sections behave like $d\sigma/dt \propto \exp(10\text{ BeV}^{-2}t)$, which corresponds to diffraction from a Gaussian density distribution of rms radius 1.1 F; this radius is somewhat smaller than the range associated with pion exchange, $\hbar/m_\pi c=1.4$ F, because diffraction scattering is associated mainly with nonperipheral processes.] If the $np$ separation in the deuteron is $\gtrsim 2$ F (probability about 67%), only single and double scattering are possible. If the separation is between about 1 and 2 F (probability about 25%), triple and higher orders of multiple scattering are possible in addition to single and double. If the separation is less than 1 F (probability 5–10%), the neutron and proton are able to interact strongly with one another. In that case, the impulse approximation may fail; the nucleons may be relativistic; many-body states may become important—in short, the entire treatment breaks down. One must therefore be prepared to find the present theory in error by a few percent. Data on the magnetic component of electron-deuteron elastic scattering have successfully been explained[7] in terms of deuteron wave functions (calculated from models of the $np$ interaction) for momentum transfers up to 12 F$^{-2}$; this suggests that $S(p)$ is known reasonably well for $p\lesssim 0.3$ BeV, or $\psi(x)$ for $x\gtrsim 0.6$ F.

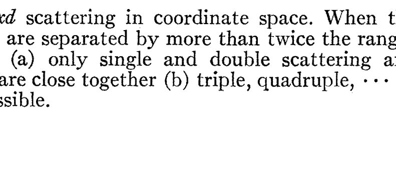

FIG. 3. $xd$ scattering in coordinate space. When the neutron and proton are separated by more than twice the range of the $xn$ interaction (a) only single and double scattering are possible; when they are close together (b) triple, quadruple, $\cdots$ scatterings are also possible.

## III. MODEL CALCULATION

As a model of the $xN$ scattering amplitudes, I use

$$t_{xn}(|\mathbf{k}|)=t_{xp}(|\mathbf{k}|)=Ce^{-\gamma\mathbf{k}^2/2}, \quad (16)$$

which corresponds to

$$\left(\frac{d\sigma}{dt}\right)_{xn}=\left(\frac{d\sigma}{dt}\right)_{xp}\cong 16\pi^5\left(\frac{\mathbf{q}^2}{\mathbf{q}^2+m_x^2}\right)^{1/2}|C|^2e^{\gamma t} \quad (17)$$

at small momentum transfer. This form is roughly compatible with $\pi^\pm p$, $K^\pm p$, $pp$, $\bar{p}p$ elastic scattering data at energies from 5 to 25 BeV and momentum transfers from 0 to 1 BeV$^2$.[8] The best value of $\gamma$ varies somewhat with energy and the type of particle, but is never very far from 10 BeV$^{-2}$. Typical values of $|C|$ in BeV$^{-2}$ are 0.26 for $\bar{p}p$, 0.21 for $pp$, 0.13 for $\pi^\pm p$, 0.12 for $K^-p$, 0.10 for $K^+p$ [9]; these values decrease slowly with energy because total cross sections do. The phase of $C$ is such that Re$C$/Im$C$ falls from $-0.33$ to $-0.16$ in $pp$, from $-0.22$ to $-0.14$ in $\pi^-p$, and from $-0.15$ to $-0.13$ in $\pi^-p$ between 8 and 25 BeV.[10]

[7] C. Buchanan and M. Yearian, Phys. Rev. Letters **15**, 303 (1965); N. Glendenning and G. Kramer, Phys. Rev. **126**, 2159 (1962).

[8] K. Foley, R. Gilmore, S. Lindenbaum, W. Love, S. Ozaki, E. Willen, R. Yamada, and L. Yuan, Phys. Rev. Letters **15**, 45 (1965).

[9] W. Galbraith, E. Jenkins, T. Kycia, B. Leontic, R. Phillips, and A. Read, Phys. Rev. **138**, B913 (1965); W. Baker, E. Jenkins, T. Kycia, R. Phillips, A. Read, K. Riley, and H. Ruderman, in *Proceedings of the Sienna International Conference on Elementary Particles and High Energy Physics*, edited by G. Bernardini and G. Puppi (Societa Italiana de Fisica, Bologna, 1963).

[10] K. Foley, R. Jones, S. Lindenbaum, W. Love, S. Ozaki, E. Platner, C. Quarles, and E. Willen, Phys. Rev. Letters **19**, 193 (1967); **19**, 857 (1967).

As a model deuteron wave function I use the Gaussian

$$\varphi(\mathbf{k})=(\alpha/\pi)^{3/4}e^{-\alpha\mathbf{k}^2/2}. \tag{18}$$

This provides an adequate, though not precise, description of the deuteron. The value $\alpha=134\ \mathrm{BeV}^{-2}$ was determined in a variational calculation[11]; it corresponds to a rms separation of neutron and proton equal to 2.8 F, which is not unreasonable. The spatial form factor corresponding to this wave function is $S(\mathbf{p})=e^{-\alpha\mathbf{p}^2/4}$.

The contribution to the scattering amplitude $T$ due to the two single-scattering terms in Eq. (6) is, in the model,

$$T_1=2Ce^{(\gamma/2+\alpha/16)t}. \tag{19}$$

The contribution due to the double-scattering terms is

$$T_2=2C^2\exp(-\tfrac{1}{4}\gamma\boldsymbol{\Delta}^2)\times\int\frac{d\mathbf{k}\ e^{-(\gamma+\alpha/4)\mathbf{k}^2}}{(\mathbf{q}^2+m_x^2)^{1/2}-[(\mathbf{q}-\mathbf{k}-\frac{1}{2}\boldsymbol{\Delta})^2+m_x^2]^{1/2}+i\epsilon}. \tag{20}$$

Using the kinematic relations $2\mathbf{q}\cdot\boldsymbol{\Delta}=\boldsymbol{\Delta}^2=-t$, defining $Q=(\mathbf{q}^2-\frac{1}{4}\boldsymbol{\Delta}^2)^{1/2}$, and performing the angular integrations, Eq. (20) becomes

$$T_2=\frac{8\pi C^2}{4\gamma+\alpha}\,e^{\gamma t/4}\int_{-\infty}^{\infty}dk\ e^{-(\gamma+\alpha/4)k^2}\times\frac{1+k/Q}{(\mathbf{q}^2+m_x^2)^{1/2}-(Q^2+m_x^2+k^2+2Qk)^{1/2}+i\epsilon}. \tag{21}$$

Let us now assume that $\boldsymbol{\Delta}^2/4\mathbf{q}^2\ll1$. This corresponds to a small-angle approximation in the Breit frame, for $\boldsymbol{\Delta}^2/4\mathbf{q}^2=[\sin(\frac{1}{2}\theta_{\mathrm{Breit}})]^2$; however, it is a very good approximation in most situations of interest—even those involving relatively large scattering angles in the center-of-mass frame—for $\mathbf{q}$ is approximately the laboratory momentum of the incident particle. Simplifying the energy denominator in Eq. (21) using $\boldsymbol{\Delta}^2/4\mathbf{q}^2\ll1$ and the fact that large values of $k$ are made unimportant by the factor $e^{-(\gamma+\alpha/4)k^2}$, one obtains

$$T_2\cong C^2\left(\frac{-8i\pi^2}{4\gamma+\alpha}\right)\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{1/2}\times e^{\gamma t/4}w\left(\frac{-t}{16|\mathbf{q}|}(4\gamma+\alpha)^{1/2}\right), \tag{22}$$

where

$$w(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{dp\ e^{-p^2}}{z-p+i\epsilon}=e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{p^2}dp\right) \tag{23}$$

is a type of error function.[12]

The result one obtains for double scattering using the Glauber formula [Eq. (15)] is equal to that given in Eq. (22) except for replacement of the function $w$ by unity. For small momentum transfer,

$$w\left(\frac{-t}{16|\mathbf{q}|}(4\gamma+\alpha)^{1/2}\right)\cong1-i\left(\frac{4\gamma+\alpha}{64\pi}\right)^{1/2}\frac{t}{|\mathbf{q}|}; \tag{24}$$

the numerical value of $[(4\gamma+\alpha)/64\pi]^{1/2}$ is about 1 BeV, so the correction to the Glauber formula can be quite large at low energies. Particularly important is the alteration in the *phase* of double scattering which results from retaining the principal-value part of the propagator. For example, Palevsky *et al.*[5] have concluded, from the shape of their $pd$ differential cross section in the region of strong interference between single and double scattering, that the phases of the $pn$ and $pp$ amplitudes at $t=-0.35\ \mathrm{BeV}^2$ differ radically from the phases at $t=0$. That situation improves considerably when one includes the principal-value contribution to double scattering. Figure 4 shows Palevsky's data, together with results of a model calculation in which $\mathrm{Re}t_{pp}/\mathrm{Im}t_{pp}=\mathrm{Re}t_{pn}/\mathrm{Im}t_{pn}=-0.4$ was assumed. The solid curve, which corresponds to the model, gives a much better fit near $t=-0.35\ \mathrm{BeV}^2$ than the dashed curve, which corresponds to the Glauber approximation, i.e., replacing the function $w$ by 1 in Eq. (22). It is difficult to draw a quantitative conclusion because the values of $\mathrm{Re}t_{pp}/\mathrm{Im}t_{pp}$ and $\mathrm{Re}t_{pn}/\mathrm{Im}t_{pn}$ in the forward direction are uncertain. However, they are probably not larger in magnitude than 0.4, so some variation in the phases with $t$ is probably still necessary to explain the interference being less destructive than the model otherwise predicts. (The parameters used in the model were $\gamma=5.23\ \mathrm{BeV}^{-2}$ and $C$ corresponding to $\sigma_{pN}=48.2$ mb,

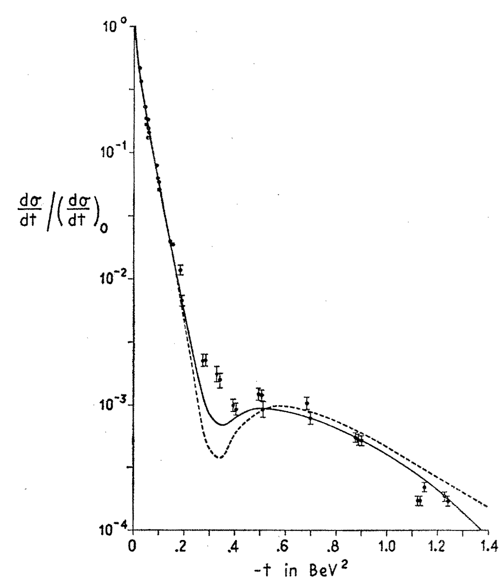


FIG. 4. Proton-deuteron elastic scattering at 1.7 BeV/$c$ (see Ref. 5). The solid curve is the prediction of single plus double scattering; the dashed curve results from making the high-energy and small-momentum-transfer approximations of Glauber theory.

[11] M. Verde, Helv. Phys. Acta **22**, 339 (1949).

[12] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. **55**.

as used by Bassel and Wilkin.[13] The deuteron form factor was taken from Glendenning and Kramer[7]; their form factor is nearly equal to the one corresponding to Moravcsik's wave function,[14] and is a distinct improvement over the Gaussian.)

Now assume the energy sufficiently high for the Glauber type of approximation [Eq. (14)] to be correct. In the model, the $n$th-order multiple-scattering term then has the form

$$T_n=C^n\left[\frac{-i}{\gamma}\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{1/2}\right]^{n-1} f_n \times\begin{cases}\exp\left(\dfrac{\gamma}{2n}t\right), & \text{if } n \text{ is even}\\[2ex] \exp\left(\dfrac{8\gamma+n\alpha}{8\gamma+(n-1/n)\alpha}\dfrac{\gamma}{2n}t\right), & \text{if } n \text{ is odd}\end{cases} \tag{25}$$

where

$$\begin{aligned} f_1&=2\,,\\ f_2&=\frac{8\pi^2}{4+\alpha/\gamma}\,,\\ f_3&=\frac{32\pi^3}{3+\alpha/\gamma}\operatorname{arccot}[(3+\alpha/\gamma)^{1/2}]\,,\\ f_4&=\frac{32\pi^5}{2+\alpha/\gamma}\operatorname{arccot}\left[(3+\alpha/\gamma)^{1/2}+\frac{2+\alpha/2\gamma}{(1+\alpha/2\gamma)^{1/2}}\right]. \end{aligned} \tag{26}$$

Each term falls exponentially with momentum transfer. The slopes of the exponentials decrease monotonically with $n$, being given by $\gamma/2n$ for even $n$ and very nearly by $\gamma/2n$ for odd $n>1$. The approximate independence of the slopes on $\alpha$ implies that the momentum-transfer dependence of multiple scattering is insensitive to the choice of a model wave function, whenever $t_{xn}(|\mathbf{\Delta}|) \propto t_{xp}(|\mathbf{\Delta}|) \propto e^{-\frac{1}{2}\gamma\Delta^2}$. The momentum-transfer dependence of single scattering, on the other hand, depends strongly on the wave function: By coherence, it is mainly determined by the "size" of the deuteron.

The relative magnitudes of the various multiple-scattering amplitudes are shown in Fig. 5 for the case of high-energy $pd$ scattering ($\alpha=134$ BeV$^{-2}$, $\gamma=10$ BeV$^{-2}$, $|C|=0.20$ BeV$^{-2}$). Single scattering dominates the cross section at small momentum transfer, and double scattering at somewhat larger momentum transfer, as was first concluded by Coleman and Franco.[1] At momentum transfers of $-t\gtrsim 3$ BeV$^2$, higher-order multiple scattering becomes important.

If the amplitudes $t_{xn}$ and $t_{xp}$ are pure-imaginary, as has been conjectured for elastic amplitudes at small momentum transfer and very high energy, then double

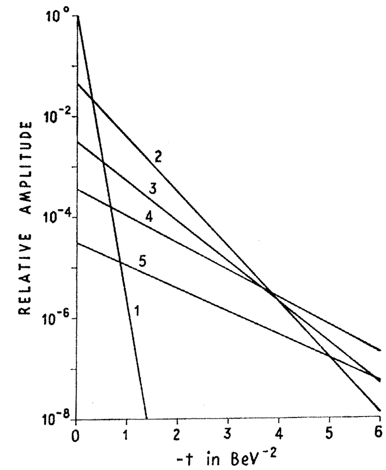


FIG. 5. Relative sizes of the multiple-scattering amplitudes in the model calculation of proton-deuteron scattering. Curve 1 is single scattering, etc.

scattering is 180° out of phase with single scattering, as required by its correspondence with classical "shadowing": Double scattering must act to reduce the total cross section. As a result, the differential cross section should have a zero at the point where single and double scattering are equal in magnitude. (Higher-order multiple-scattering contributions would in principle shift the position of the zero very slightly.) At available accelerator energies, elastic amplitudes are known to have sizable real parts,[10] so the "dips" in deuteron cross sections are expected to be filled in to some extent, as in Fig. 4.

The shape of the differential cross section beyond the double-scattering region cannot be predicted reliably without using accurate expressions for the deuteron wave function and $xN$ amplitudes, because it results from the delicate interference of several terms. With that qualification, the cross section given by the model for $\mathrm{Re}\,t_{xN}/\mathrm{Im}\,t_{xN}=-0.2$ is shown in Fig. 6. Fifth- and higher-order multiple-scattering terms are negligible at the momentum transfers shown. Third and fourth order tend to cancel in such a way that substantial deviations from pure single+double scattering do not occur until $-t>4$ BeV$^2$.

The multiple-scattering amplitudes for $\pi d$, $\bar{p}d$, and $Kd$ are similar to those shown in Fig. 5 for $pd$. However, $\pi N$ and $KN$ cross sections are smaller than $pN$, causing the dip to move from about $t=-0.33$ BeV$^2$ to about $t=-0.38$ BeV$^2$; $\bar{p}N$ cross sections are larger, so the dip moves in to about $-0.29$ BeV$^2$.

It may be possible to predict the scattering amplitude successfully at relatively large momentum transfer, while treating the deuteron nonrelativistically, because the amplitude is dominated there by high orders of multiple scattering. For example, to calculate the single-scattering amplitude for $-t\gtrsim 0.5$ BeV$^2$, one would have to understand the coordinate-space wave function in the

[13] R. Bassel and C. Wilkin, Phys. Rev. Letters 18, 871 (1967).
[14] M. Moravcsik, Nucl. Phys. 7, 113 (1958).

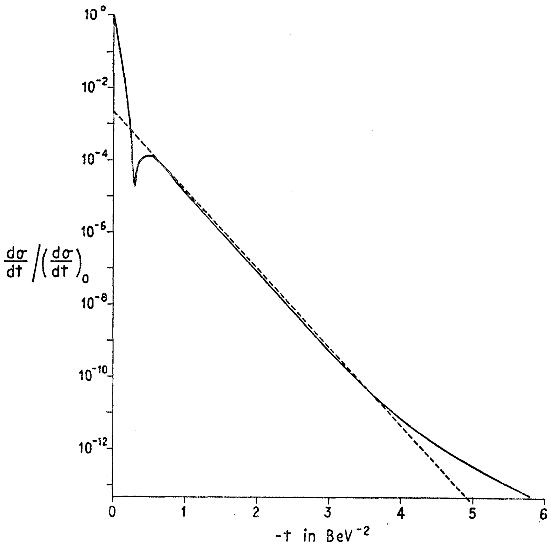


FIG. 6. Model calculation of proton-deuteron elastic scattering (solid curve), together with the result of pure double scattering (dashed curve).

nether regions of radii $\lesssim 0.5$ F. Fortunately, however, single scattering is negligible at the momentum transfers for which it cannot be calculated. In $n$th-order multiple scattering, a three-momentum transfer $\mathbf{\Delta}$ is achieved by means of $n$ interactions of three-momentum transfer $\approx \mathbf{\Delta}/n$; thus $xN$ scattering with $-t \lesssim 1$ BeV$^2$ accounts for double scattering with $-t \lesssim 4$ BeV$^2$, triple scattering with $-t \lesssim 9$ BeV$^2$, etc.

The coordinate-space description of scattering from a deuteron, discussed in Sec. II, can be verified by expressing the multiple-scattering amplitude in terms of the wave function:

$$T_1 = 8\pi C e^{-\gamma\Delta^2/2} \int_0^\infty dx\, x^2 |\psi(x)|^2 \frac{\sin(\frac{1}{2}|\mathbf{\Delta}|x)}{(\frac{1}{2}|\mathbf{\Delta}|x)},$$

$$T_2 = \frac{-8\pi^3 i C^2}{\gamma}\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{1/2} \times e^{-\gamma\Delta^2/4} \int_0^\infty dx\, x^2 |\psi(x)|^2 \frac{D(x/2\sqrt{\gamma})}{(x/2\sqrt{\gamma})},$$

$$T_3 \cong \frac{-128\pi^4 C^3}{3\sqrt{3}\gamma^2}\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right) e^{-\gamma\Delta^2/6} \int_0^\infty dx\, x^2 |\psi(x)|^2 \times \frac{\sin(|\mathbf{\Delta}|\frac{1}{6}x)}{(|\mathbf{\Delta}|\frac{1}{6}x)} e^{-x^2/3\gamma}, \quad (27)$$

$$T_4 \cong \frac{64\pi^6(\sqrt{2}-1) i C^4}{\gamma^3}\left(\frac{\mathbf{q}^2+m_x^2}{\mathbf{q}^2}\right)^{3/2} \times e^{-\gamma\Delta^2/8} \int_0^\infty dx\, x^2 |\psi(x)|^2 e^{-x^2/2\gamma},$$

where

$$D(z) = e^{-z^2} \int_0^z e^{t^2} dt \quad (28)$$

is Dawson's integral.[12] From Eq. (25) it is apparent that single scattering "probes" the wave function to distances of order $\hbar/|\mathbf{\Delta}|$. Triple scattering "probes" only to $3\hbar/|\mathbf{\Delta}|$. Both triple and quadruple scattering are cut off exponentially when neutron and proton are too far apart for the incident particle to interact with them simultaneously. The momentum-transfer dependence of the even orders of multiple scattering is independent of the wave function; as was stated above, this result depends heavily on the Gaussian model of the $xN$ interaction. Equation (27) could be used for employing a more accurate deuteron wave function, while retaining the high-energy, small-momentum-transfer approximations involved in Eq. (14), and keeping Eq. (15) as the form for the two-body amplitudes. Additional minor approximations were made in $T_3$ and $T_4$, however.

## IV. SCREENING CORRECTION TO TOTAL CROSS SECTION

The total cross section for scattering a particle $x$ on deuterons differs from the sum of total cross sections on neutrons and protons by an amount $\sigma_{xn}+\sigma_{xp}-\sigma_{xd}$ which is called the screening correction. It may be thought of as the reduction of the deuteron cross section which results from one nucleon lying in the "shadow" of the other. A procedure for calculating the screening correction is necessary for extracting neutron cross sections from experiments on deuterium.

Assuming that the amplitudes $t_{xn}$ and $t_{xp}$ are functions of the magnitude of three-momentum transfer only, neglecting higher-order multiple scattering and recoil effects, and applying the optical theorem, Eq. (6) yields

$$\sigma_{xd} = \sigma_{xn}+\sigma_{xp} - \frac{\sigma_{xn}\sigma_{xp}}{4\pi} \frac{1}{\operatorname{Im}t_{xn}(0)\operatorname{Im}t_{xp}(0)} \times \int_0^\infty dk\, kS(k)\operatorname{Re}[-t_{xn}(k)t_{xp}(k)]. \quad (29)$$

Inclusion of recoil and triple-scattering effects would modify the screening correction by at most a few percent. Equation (29) can be rewritten in the form

$$\text{``}\left\langle\frac{1}{r^2}\right\rangle\text{''} \equiv \frac{4\pi}{\sigma_{xn}\sigma_{xp}}(\sigma_{xn}+\sigma_{xp}-\sigma_{xd}) = \frac{1}{\operatorname{Im}t_{xn}(0)\operatorname{Im}t_{xp}(0)} \int_0^\infty dk\, kS(k) \times \operatorname{Re}[-t_{xn}(k)t_{xp}(k)]. \quad (30)$$

The experimental quantity $4\pi(\sigma_{xn}+\sigma_{xp}-\sigma_{xd})/\sigma_{xn}\sigma_{xp}$

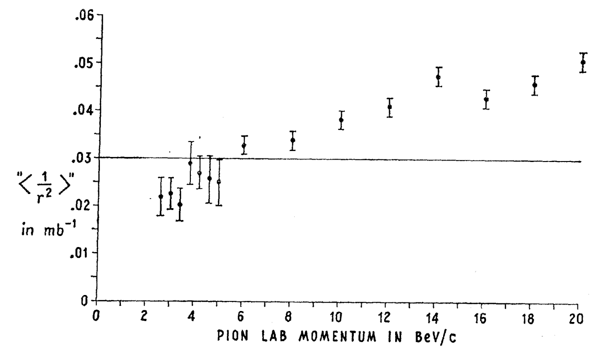


FIG. 7. Experimental $\pi d$ screening correction: "$\langle 1/r^2\rangle$"$=(4\pi/\sigma_{\pi^+p}\sigma_{\pi^-p})[\sigma_{\pi^+p}+\sigma_{\pi^-p}-\frac{1}{2}(\sigma_{\pi^+d}+\sigma_{\pi^-d})]$ as a function of energy (see Ref. 9). A constant value of about 0.03 mb$^{-1}$ is predicted on the basis of double scattering.

is denoted by "$\langle 1/r^2\rangle$" in the literature, based on the fact that the right-hand side of Eq. (30) would equal the expectation value of $1/\mathbf{r}^2$ in the deuteron wave function, if the $xN$ amplitudes were purely imaginary and purely $s$ wave. The name is misleading, for unlike the calculated value of "$\langle 1/r^2\rangle$," $\langle 1/\mathbf{r}^2\rangle$ is independent of the $xN$ amplitudes and extremely sensitive to the wave function at small distances. Numerically, the two quantities differ by $\sim$30% in typical applications.

The ratio $\mathrm{Re}[-t_{xn}(k)t_{xp}(k)]/[\mathrm{Im}t_{xn}(0)\ \mathrm{Im}t_{xp}(0)]$ is found experimentally to be nearly independent of energy for small $k$—i.e., the shape of the diffraction peak is about constant. Since only small values of $k$ are important for the integral in Eq. (30), the Glauber theory predicts "$\langle 1/r^2\rangle$" to be independent of energy. This can be tested using $\pi^\pm p$ and $\pi^\pm d$ measurements, assuming $\sigma_{\pi^\pm n}=\sigma_{\pi^\mp p}$ by charge symmetry. (It is desirable to measure both $\sigma_{\pi^+d}$ and $\sigma_{\pi^-d}$, even though these quantities are expected to be equal by charge symmetry, in order to cancel certain systematic errors.) Experimental results are shown in Fig. 7.[9]

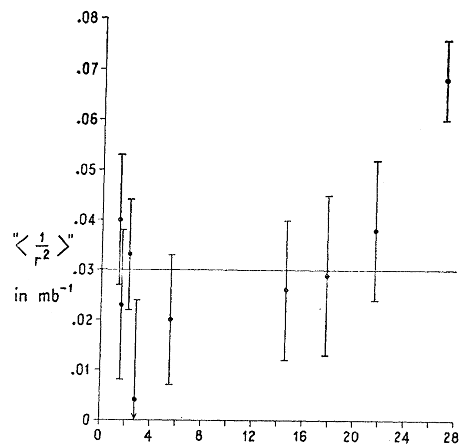


FIG. 8. Experimental $pd$ or $nd$ screening correction (see Ref. 15). A constant value of about 0.03 mb$^{-1}$ is predicted on the basis of double scattering.

The screening correction appears to increase with energy, in disagreement with the theory. The predicted value of the constant is about 0.030 mb$^{-1}$ (this is the result of the model, namely $2/(4\gamma+\alpha)$; if wave functions such as those given by Glendenning and Kramer[7] are used instead of the Guassian, the predicted value stays about the same: 0.028 mb$^{-1}$). The data seem to be in agreement with this value at low energies. However, energy-independent systematic errors in the experiment would allow the entire set of values for "$\langle 1/r^2\rangle$" to be shifted by $\pm$0.005 mb$^{-1}$, so that agreement is uncertain. The relative normalization of the data above and below 5.5 BeV is also uncertain, because two separate experiments are involved.

The screening correction can also be measured in $Nd$ scattering, making use of neutron beams to obtain the $np$ cross section. Data are shown in Fig. 8.[15] Again the screening correction appears to increase with energy, in contradiction to the theory. The possibility of unknown systematic errors makes this conclusion uncertain, however, especially since there is no measurement of $\sigma_{pd}$ at the crucial 27-BeV point, so systematic errors in measuring incident fluxes cannot be canceled.

In view of the results above, determinations of the energy dependence of $pn$, $\bar{p}n$, and $Kn$ cross sections from experiments on deuterium must, for the present, be considered unreliable.

## V. CONCLUSION

I have discussed two theoretical refinements of the Glauber approximation, applied to scattering elementary particles from deuterons at BeV energies. The first is a correction to the double-scattering amplitude. The correction vanishes in the limit of high energy and small momentum transfer, but may be important for experiments to be performed soon. One aspect of it is a change in the phase of double scattering, which results from the particle $x$ going off the mass shell in the intermediate state. At energies of a few BeV or less, this phase change will be significant for attempts to extract real parts of $xN$ amplitudes at nonzero momentum transfer, from interference between single and double scattering on deuterons. The shape of the differential cross section in the interference region is determined, to a first approximation, by the average relative phase between single and double scattering there, $\cong$phase of $\{t_{xn}(\frac{1}{2}\mathbf{\Delta}_{\rm dip})t_{xp}(\frac{1}{2}\mathbf{\Delta}_{\rm dip})/[t_{xn}(\mathbf{\Delta}_{\rm dip})+t_{xp}(\mathbf{\Delta}_{\rm dip})]\}\times$phase due to principal-value term. If the $xN$ phases prove to

[15] Galbraith *et al.* (see Ref. 9); Foley *et al.* (see Ref. 10); Phys. Rev. Letters **19**, 857 (1967); M. Kreisler, L. Jones, M. Longo, and J. O'Fallon, *ibid.* **20**, 468 (1968); H. Palevsky, J. Friedes, R. Sutter, R. Chrien, and R. Muether, in *Proceedings of the International Congress on Nuclear Physics, 1964, Comptes Rendus* (Centre National de la Recherche Scientifique, Paris, 1964), p. 162; D. Bugg, Salter, G. Stafford, R. George, K. Riley, and R. Tapper, Phys. Rev. **146**, 980 (1966); M. Khachaturyan and V. Pantuev, Zh. Eksperim. i Teor. Fiz. **45**, 1808 (1963) [English transl.: Soviet Phys.—JETP **18**, 1239 (1964)]; T. Coor, D. Hill, W. Hornyak, L. Smith, and G. Snow, Phys. Rev. **98**, 1369 (1955).

be slowly varying, it will be possible to obtain their average at $\mathbf{\Delta}_{\text{dip}}$ crudely by assuming their values at $\frac{1}{2}\mathbf{\Delta}_{\text{dip}}$ are equal to their values at $\mathbf{\Delta}=0$ (which are known from Coulomb interference measurements, the optical theorem, or forward dispersion relations). That procedure could be improved somewhat by requiring the average real part at $\frac{1}{2}\mathbf{\Delta}_{\text{dip}}$ to be, say, halfway between the value at 0 and at $\mathbf{\Delta}_{\text{dip}}$. Very accurate data should enable one to obtain the average phase variation over the entire region where single and double scattering are of the same order of magnitude—from $t=0$ to $t=-0.5$ BeV$^2$. This technique of extracting phases could be tested by performing a $\pi d$ experiment at 2 BeV, where phase-shift analysis of $\pi N$ scattering is possible. One could at the same time check whether the spin-flip amplitude contributes significantly in filling up the dip.

The second refinement of the Glauber theory is the inclusion of triple and higher-order multiple-scattering effects. These effects are predicted to become important at momentum transfers of $-t\gtrsim 4$ BeV$^2$. So far, no elastic scattering experiments have been done on deuterium at such large momentum transfer. These experiments should be possible, although the cross section becomes very small. If experiments are performed at relatively low energy, the calculation of triple and quadruple scattering, as in the case of double scattering, must be modified.

The energy dependence of the screening correction to deuteron total cross sections is in disagreement with the prediction based on double scattering, as pointed out in Sec. IV. Details in the calculation such as the choice of deuteron wave function or the model of $xN$ amplitudes (including real-part and spin dependence) seem incapable of resolving that disagreement. Side effects like triple scattering or recoil also do not help. Steps must therefore be taken to alter the calculation in some way.

One possible extension of the Glauber theory would be to include inelastic intermediate states in double scattering. For example in high-energy $\pi d$ elastic scattering, the intermediate state in double scattering could contain, in addition to the two nucleons, not simply a pion, but rather any particle or group of particles which can be produced diffractively in $\pi N$ scattering: e.g., the $A_1$ resonance(s), or any of the incoherent $3\pi$ states usually thought of as background to the $A_1$. In order to calculate the contribution due to an intermediate particle of mass $m$, one can simply replace $t_{xn}$ and $t_{xp}$ in the double-scattering term of Eq. (6) by the appropriate inelastic amplitudes obtained from inelastic $xN$ cross sections, and replace $[(\mathbf{q}-\mathbf{k}-\frac{1}{2}\mathbf{\Delta})^2+m_x^2]^{1/2}$ by $[(\mathbf{q}-\mathbf{k}-\frac{1}{2}\mathbf{\Delta})^2+m^2]^{1/2}$. When one then integrates over the mass spectrum, only states such that $[(m^2-m_x^2)/2|\mathbf{q}|]\times$(radius of deuteron)$\lesssim 1$ will contribute significantly, because to produce states of higher mass, more momentum transfer would be required than the deuteron wave function allows. For example, in $\pi d$ scattering at 20 BeV, states of mass up to about 2.5 BeV can be important. This calculation may overestimate the contribution of states containing several uncorrelated particles, whose relative wave function will spread somewhat before being absorbed.

The contribution of inelastic intermediate states to double scattering is in the right direction to improve agreement with the total-cross-section data: It lowers the deuteron cross section because some of the flux scattered out of the incident beam at the first nucleon gets scattered back into it at the second. The magnitude of the effect is, however, probably not large enough to explain the data as they stand. In $pd$ scattering at 20 BeV, for example, it would increase the double scattering by about 20%.[16]

A possible way to understand the remaining discrepancy with Glauber theory would be to treat scattering from the core region of the deuteron phenomenologically. At least 5% of the deuteron state vector is associated with the core, where multiple-scattering theory is unreliable, so a reduction on the order of 50% in single scattering from that region could account for the total-cross-section data. Such a phenomenological theory could be tested by examining the momentum-transfer dependence of the discrepancy between Glauber theory (with the contribution of inelastic double scattering added) and experiment. For if the correction is associated with the core, it must have a $t$ dependence appropriate to scattering from a region with that size, and therefore fall about like, or slightly faster than, double scattering. Hopefully, the energy dependence of the $\pi d$ or $pd$ differential cross sections will be measured soon; measurements in the region where single scattering is negligible will be very sensitive to the discrepancy with Glauber theory. Finally, it may be possible to relate the phenomenological treatment of the core region to effects in scattering from nuclei.[17]

## ACKNOWLEDGMENTS

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[16] M. Ross and J. Pumplin (to be published).
[17] K. Gottfried, Ann. Phys. (N. Y.) **21**, 29 (1963).