suitable pion conspirator, having $I^G(J^P)C=1^-(0^+)+$ at $t\cong 0\cong (m_\pi)^2$. We have used the Gell-Mann[13] ghost-killing mechanism, which implies that the sense-sense residue function vanishes at $\alpha_2=0$ (i.e., the trajectory chooses nonsense), preventing a physical manifestation of a 0^+ particle at the pion mass.[7,8]

A similar treatment may provide other conjectured conspirators, e.g., the K conspirator[7] may be the $(5,27)$ [10] counterpart of the $K^*(1400)$. That these new mesons have small widths was assumed[1] in order to reproduce the symmetric two-peak structure in the case of the A_2,[1,2] and also explains why these mesons may be hard to detect experimentally. Application of this formalism to other relevant reactions involving V_2 exchange is now underway and will be reported in a later paper.

A general dipole[14] (i.e., the amplitude is obtained by differentiating the single Regge pole form for all t) fit to reaction (1) was also tried, as well as some more complicated forms of the mixed-meson amplitudes. The best fit we were able to obtain for the general dipole had $\chi^2=67$, with 7 parameters and 32 data points. This rather poor quality fit supports the common view expressed in Ref. 1 that multiple poles in the S matrix are produced by the (probably accidental) coalescence of single poles and that therefore in the Regge picture trajectories crossing at $J=2$, $M=1.3$ GeV would be responsible for the double pole character of the experimental[2] mass distribution.

The authors are especially grateful to Vesa Ruuskanen for useful discussion and suggestions.

[13] M. Gell-Mann, in *Proceedings of the International Conference on High-Energy Physics, Geneva, 1962*, edited by J. Prentki (CERN Scientific Information Service, Geneva, 1962), p. 539; M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[14] T. Sawada, Nuovo Cimento **48**, 534 (1967); R. Gatto, Instituto di Fisica dell "Universita," Firenze, report, 1967 (unpublished).

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Asymptotic Behavior of Form Factors for Some Composite Models

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The asymptotic behavior of electromagnetic form factors is examined for bound states treated by means of the Bethe-Salpeter equation in the ladder approximation. Results are found which depend on the behavior of the interaction at small distances, and the models examined are accordingly divided into regular and singular cases. For spin-0 and spin-$\frac{1}{2}$ bound states with regular interaction, the form factors go to zero as $(1/q^2)^2$ (apart from logarithmic factors). For singular cases (e.g., a spinless N-$\bar{N}$ bound state) it is shown that the asymptotic behavior is worse and depends on the strength of the interaction. In all cases a behavior more convergent than $1/q^2$ seems to occur, and to be related to the compositeness of the system rather than to the structure of the interaction.

I. INTRODUCTION

There has recently been some interest in the asymptotic behavior of electromagnetic form factors.[1] This is in part due to the experimental result[2] that the nucleon form factors decrease as $(1/q^2)^2$ for large four-momentum transfer, q.

The fact that the form factor goes to zero indicates that the bare electromagnetic charge is zero, while the fact that there is no $1/q^2$ term suggests that the bare strong-interaction coupling constant also vanishes. As this shows the nonelementary nature of the particle in question, it seems worthwhile to examine simplified composite models which can be treated rigorously.

The nonrelativistic potential model for s-wave bound states gives results which depend on the behavior of the potential at the origin. If the potential diverges as $1/r$, it follows in a straightforward way that the behavior is $(1/q^2)^2$. If the potential goes like $(-\lambda/r^2)$, then the behavior depends on the strength of the singular part and is given by $(1/q^2)^{1+\sqrt{(\frac{1}{4}-\lambda)}}$ $(0<\lambda<\frac{1}{4})$.[3]

A more realistic model is given by the ladder Bethe-

[1] D. Amati, R. Jengo, and E. Remiddi, Phys. Letters **22**, 674 (1966); I. G. Halliday and P. V. Landshoff, Nuovo Cimento **51A**, 980 (1967); J. Harte, Phys. Rev. **165**, 1557 (1968).

[2] See, e.g., W. K. H. Panofsky, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1968), p. 371; D. H. Coward *et al.*, Phys. Rev. Letters **20**, 292 (1968).

[3] For potentials with hard core, also exponentially falling behaviors can be obtained. Other conditions for exponential falloff in nonrelativistic models have been examined by S. D. Drell, A. C. Finn, and M. H. Goldhaber [Phys. Rev. **157**, 1402 (1967)].

Salpeter (BS) equation (Sec. II) in which the relativistic structure of the composite particle is, at least qualitatively, taken into account. In this paper we shall rigorously treat a regular interaction model, i.e., $g\Phi^3$ theory for spinless particles, and we shall also deal with particles with spin and with singular interactions, though on less rigorous grounds. In doing this we shall exploit systematically the Deser-Gilbert-Sudarshan-Ida (DGSI)[4] spectral representation of the wave function. This representation embodies in a simple way both analytic properties and asymptotic behavior of the wave function in the relative momentum.

It is found[5] in this way (Secs. II and III) that, in the $g\Phi^3$ theory, the behavior of the charge form factor is $(1/q^2)^2\ln(q^2/M^2)$ for a spinless compound particle, while it is $(1/q^2)^3\ln(q^2/M^2)$ for a spin-1 particle. [In this case the magnetic moment form factor goes simply as $(1/q^2)^3$.] For a spin-$\frac{1}{2}$ particle composed of spin-0 and spin-$\frac{1}{2}$ particles bound together via the exchange of a scalar boson, we find analogous results (Sec. IV). No claim of rigor is made for this case.

It must be remarked that, in both cases, the assumption of an interaction mediated by a scalar particle is essential to get the above-mentioned results. In fact this assumption makes the interaction "regular" in the sense that the asymptotic behavior of the wave function does not depend on the strength of the interaction. Things behave differently, at least in the ladder approximation, in more realistic cases, as, for example, the pion as a N-$\bar{N}$ (or quark-antiquark) bound state.

These singular cases are discussed in Sec. IV, and it is pointed out that, while a behavior better than $1/q^2$ seems always to occur, the $(1/q^2)^2$ behavior is peculiar to the regular case.

II. FORM FACTORS IN THE LADDER APPROXIMATION

In the following we shall consider a bound state of two spinless bosons with equal mass M. The matrix elements of the current in the ladder approximation are given by[6]

$$\langle 2|J_\mu|1\rangle = 2i\int d^4p\bar{\Phi}_2(p+\tfrac{1}{2}P_2)(p^2+M^2) \times[\tfrac{1}{2}(P_1+P_2)_\mu+p_\mu]\Phi_1(p+\tfrac{1}{2}P_1)\,, \quad (1)$$

where[7] P_1,P_2 are the initial and final momenta of the bound state, and

$$\Phi_i(p)=(2\pi)^{-2}\int d^4x e^{ipx}\varphi_i(x)\,,$$
$$\varphi_i(x)\equiv\langle 0|T[\Psi_a(\tfrac{1}{2}x)\Psi_b(-\tfrac{1}{2}x)]|i,P_i\rangle\,, \quad (2)$$
$$\bar{\varphi}_i(x)\equiv\langle i,P_i|T[\Psi_a^\dagger(\tfrac{1}{2}x)\Psi_b^\dagger(-\tfrac{1}{2}x)]|0\rangle.$$

The relation between Φ and $\bar{\Phi}$ is given, through the analytic continuation in p_0,[8] by[9]

$$\bar{\Phi}_i(p)=\int\frac{d^4p}{(2\pi)^2}e^{-ipx}\bar{\varphi}_i(x)=-[\Phi(\mathbf{p},p_0^*)]^*. \quad (3)$$

In Eq. (1), the BS wave functions, $\Phi_1(p)$ and $\Phi_2(p)$, depend on the momenta P_1 and P_2 of the bound state. They can be expressed in terms of the rest-frame wave function $\Phi(p)$ by means of a Lorentz transformation:

$$\Phi'(p)=\Phi(L^{-1}p)\,. \quad (4)$$

As we are interested in the spacelike region $q^2\to\infty$, we shall evaluate the form factors using the Breit system:

$$P_1=(0,0,-q;iK_0),\quad P_2=(0,0,q;iK_0)\,, \quad (5)$$

where $K_0=(q^2+m^2)^{1/2}$ and m is the mass of the bound state. Then, for a spinless bound state, we have

$$F(q^2)=2i\int d^4p\bar{\Phi}\left(p_1,p_2,\frac{K_0}{m}p_3-\frac{q}{m}p_0,\frac{K_0}{m}p_0-\frac{q}{m}p_3+\tfrac{1}{2}m\right) \times(p^2+M^2)\left(1+\frac{p_0}{K_0}\right) \times\Phi\left(p_1,p_2,\frac{K_0}{m}p_3+\frac{q}{m}p_0,\frac{K_0}{m}p_0+\frac{q}{m}p_3+\tfrac{1}{2}m\right). \quad (6)$$

In contrast with the nonrelativistic case, it is clear from Eq. (6) that the asymptotic behavior in q^2 depends in a rather involved way on the structure of the wave function near the light cone. The use of the DGSI representation of the wave function allows all the p integrations to be explicitly performed; in this way unambiguous results are obtained.

For a spinless bound state of mass m this representation is given by[4,10]

$$\Phi(p)=i\int_{-1}^{+1}dz\int_0^\infty dt\frac{g(z,t)}{(p^2-mp_0z+\rho^2+t)^3}\,, \quad (7)$$
$$(\rho^2\equiv M^2-\tfrac{1}{4}m^2)$$

[4] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959); M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960).

[5] Similar results for the regular case have been recently obtained by D. Amati, R. Jengo, H. Rubinstein, G. Veneziano, and M. Virasoro [Phys. Letters **27B**, 38 (1968)] by summing the asymptotic behaviors of ladder diagrams, and by J. S. Ball and F. Zachariasen [Phys. Rev. **170**, 1541 (1968)] by inducing the asymptotic properties of the vertex functions by looking at the BS equations.

[6] S. Mandelstam, Proc. Roy. Soc. **A233**, 248 (1955); K. Nishijima, Progr. Theoret. Phys. (Kyoto) **13**, 305 (1955); M. Ciafaloni and P. Menotti, Nuovo Cimento, **46A**, 162 (1966).

[7] We shall use 4-vectors with imaginary fourth component and Hermitian γ matrices. A covariant normalization of initial and final states is understood.

[8] G. C. Wick, Phys. Rev. **96**, 1125 (1954).

[9] M. Ciafaloni and P. Menotti, Phys. Rev. **140**, B929 (1965); Y. Ohnuki and K. Watanabe, Nuovo Cimento **39**, 772 (1965).

[10] M. Ida and K. Maki, Progr. Theoret. Phys. (Kyoto) **26**, 470 (1961); N. Nakanishi, Phys. Rev. **130**, 1230 (1963).

[$g(zt)$ can be chosen to be real; in this way $\bar{\Phi}(p)=\Phi(p)$]. Clearly some properties of $g(z,t)$ are needed. In the $g\Phi^3$ model, if the mass μ of the exchanged particle vanishes, it has been shown by Wick[8] that

$$g(z,t)=g(z)\delta(t),\quad g(z) \underset{z\to\pm1}{\sim} (1\mp z). \tag{8}$$

If $\mu\neq 0$ the analogous properties are

$$\int |g(z,t)|dt<\infty,\quad g(z,t) \underset{z\to\pm1}{\sim} (1\mp z). \tag{9}$$

More precisely, we shall prove in Sec. III that

$$\int dt|g(z,t)| \leq \text{const}\times(1-z^2). \tag{10}$$

The properties (8), (9), and (10) are sufficient to prove our results.

By substituting the representation (7) into Eq. (6), and performing the p integrations by Feynmann methods, we obtain

$$F(q^2)=\int_{-1}^{+1} dzdz'd\alpha \int_0^\infty dtdt'g(z,t)g(z',t')(1-\alpha^2)\times\frac{\pi^2}{24} \times\frac{\partial^2}{\partial t\partial t'}\left[\left(\frac{M^2+K'^2}{D^2}+\frac{2}{D}\right)\left(1-\frac{K_0'}{K_0}\right)+\frac{K_0'}{K_0}\frac{1}{D}\right], \tag{11}$$

where

$$K'=\tfrac{1}{4}(1+\alpha)(1-z)P_1+\tfrac{1}{4}(1-\alpha)(1-z')P_2,$$
$$D=\rho^2+\tfrac{1}{2}(1+\alpha)t+\tfrac{1}{2}(1-\alpha)t'+\tfrac{1}{4}m^2[\tfrac{1}{2}(1+\alpha)(1-z) +\tfrac{1}{2}(1-\alpha)(1-z')-1]^2+\tfrac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2\geq\rho^2 +\tfrac{1}{2}(1+\alpha)t+\tfrac{1}{2}(1-\alpha)t'+\tfrac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2. \tag{12}$$

It is then clear that the leading terms in Eq. (11) are of the form

$$\int\frac{dzdz'dtdt'd\alpha\, g(z,t)g(z',t')(1-\alpha^2)^2}{[M^2+\frac{1}{2}(1+\alpha)t+\frac{1}{2}(1-\alpha)t'+\frac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^3}. \tag{13}$$

By using either the property (8) for $\mu=0$ or (10) for $\mu\neq0$, the leading term of Eq. (13) is written as

$$\int_0^1 \lambda d\lambda \int_0^1 \mu d\mu \int_0^1 \nu^2 d\nu (M^2+\lambda\mu\nu q^2)^{-3}=I(q^2). \tag{14}$$

This gives the behavior[11]

$$F(q^2)\simeq\text{const}\times\frac{1}{(q^2)^2}\ln\left(\frac{q^2}{M^2}\right), \tag{15}$$

which is valid for the $g\Phi^3$ model.

The extension of this result to higher partial waves is not difficult. For simplicity, we shall limit ourselves to the case $\mu=0$, where the following spectral representation holds for a $J=1$ bound state[12]:

$$\Phi^{(i)}(p)=p_i\int_{-1}^{+1}\frac{dz\, g(z)}{(p^2-mp_0z+\rho^2)^4},\quad (i=1,2,3) \tag{16a}$$

$$g(z)\simeq(1\mp z)^2 \quad\text{as}\quad z\to\pm1. \tag{16b}$$

The definition of form factors is given by

$$1/e\langle 2|J_\mu|1\rangle=F_s(q^2)K_\mu\bar{\epsilon}_2\cdot\epsilon_1+F_M(q^2)(\bar{\epsilon}_2\cdot q\epsilon_{1\mu}-\bar{\epsilon}_{2\mu}\epsilon_1\cdot q) +F_Q(q^2)K_\mu(4\bar{\epsilon}_2\cdot q\;\epsilon_1\cdot q-\bar{\epsilon}_2\cdot\epsilon_1 q^2), \tag{17}$$
$$K_\mu=P_{1\mu}+P_{2\mu},\quad q_\mu=\tfrac{1}{2}(P_{2\mu}-P_{1\mu}),\quad \bar{\epsilon}=(\boldsymbol{\epsilon}^*,i\epsilon_0^*).$$

By using the expression of the current already given, one obtains[13] ($a=1,2$, means transverse polarization in the Breit system)

$$F_s-q^2F_Q=2i\int d^4p\bar{\Phi}_2{}^a(p+\tfrac{1}{2}P_2)(p^2+M^2) \times(1+p_0/K_0)\Phi_1{}^a(p+\tfrac{1}{2}P_1), \tag{18a}$$

$$F_M=-2i\int d^4p\bar{\Phi}_2{}^a(p+\tfrac{1}{2}P_2)(p^2+M^2) \times(p_0/K_0)\Phi_1{}^a(p+\tfrac{1}{2}P_1), \tag{18b}$$

and a more involved expression for longitudinal-longitudinal transitions which gives the combination $F_s+q^2F_Q-F_M$.

Due to the representation (16), the leading term in Eq. (18a) is given by the integral

$$\int_0^1\frac{d\lambda\,\lambda^2 d\mu\,\mu^2 d\nu\,\nu^3}{(M^2+\lambda\mu\nu q^2)^4}=-\frac{1}{3}\frac{d}{dq^2}I(q^2)\simeq\left(\frac{1}{q^2}\right)^3\ln\left(\frac{q^2}{M^2}\right). \tag{19}$$

On the other hand, an extra convergence factor $\lambda=(1-z)$ is obtained from Eq. (18b) because of the presence of p_0/K_0 only, in the p integration. It follows[11] that $\ln(q^2)$ disappears in the magnetic-moment form factor. The final result is then

$$F_s(q^2)\simeq(1/q^2)^3\ln(q^2/M^2),\quad F_M(q^2)\simeq(1/q^2)^3, \quad F_Q(q^2)\simeq(1/q^2)^4\ln(q^2/M^2). \tag{20}$$

This completes the treatment of the $g\Phi^3$ model, except for some mathematical details which will be given in Sec. III. For further generalizations (Sec. IV), it is important to investigate what the asymptotic behavior of the form factors can be when the properties of $g(z,t)$ are different from those used above [Eqs. (8)–(10)]. We shall see how the behavior of $g(z,t)$ for $z\to\pm1$ and $t\to\infty$ controls both the asymptotic properties of the wave function in momentum space and of the form factor in q^2.

[11] The evaluation of $I(q^2)$ and of other more general integrals is given in the Appendix.

[12] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

[13] M. Ciafaloni and P. Menotti (Ref. 6).

For the wave functions, two asymptotic limits are of particular interest: $p^2\to\infty$ and $p_3=p_0+\frac{1}{2}m\to\infty$. We notice that the latter appears naturally in the expression (6) for the form factor. From (7), for large p^2, we have

$$\Phi(p)\simeq\int_0^\infty dt\, g_1(t)[p^2+\rho^2+t]^{-3}, \qquad (21)$$

where we defined

$$g_1(t)=\int_{-1}^{+1}dz\, g(z,t).$$

If $\int_0^\infty g_1(t)dt<\infty$, we clearly have $\Phi(p)\simeq(1/p^2)^3$. This is the regular case and the corresponding form factor has been treated above. If $g_1(t)\sim t^{-\delta}$ for $t\to\infty$ and $0<\delta<1$, we have

$$\int^\infty dt\, t^{-\delta}(p^2+\rho^2+t)^{-3} \simeq\int_0^\infty d\tau\, \tau^{-\delta}(1+\tau)^{-3}(p^2)^{-2-\delta}. \qquad (22a)$$

On the other hand, the behavior of $g(z,t)$ for $z\to 1$[14] controls the behavior for $p_3=\frac{1}{2}m+p_0\to\infty$. In fact if we put, for $z\to 1$, $g(z,t)\simeq(1-z)^\gamma g(t)$, then we have, for $|\gamma|<1$,

$$\Phi(p_1,p_2,p_0+\tfrac{1}{2}m,p_0) =\int\frac{dzdt\, g(zt)}{[p_1^2+p_2^2+mp_0(1-z)+M^2+t]^3} \simeq\left(\frac{1}{mp_0}\right)^{1+\gamma}\int_0^\infty\frac{dtdx\, g(t)}{(p_1^2+p_2^2+x+M^2+t)^3}. \qquad (22b)$$

From these remarks it is clear that the properties of the spectral function $g(z,t)$ for $z\to 1$ and $t\to\infty$ are related to the nature of the interaction at small distances. In particular we notice that, when $\delta>0$, the vertex function vanishes in the limit $p^2\to\infty$, so that the bare (strong) coupling constant between the bound state and its constituents is zero.

On the other hand, it is clear from Eq. (11) (integrated by parts in t and t') that the coefficient of $1/q^2$, for $q^2\to\infty$, is given by

$$\text{const}\times\left[\int\frac{dtdz}{1-z}\frac{d}{dt}g(z,t)\right]^2. \qquad (23)$$

This means that, if $g(z,\infty)=0=g(1,t)$ (vanishing of the bare coupling constant), then the form factor goes to zero more rapidly than $1/q^2$ for $q^2\to\infty$. More precisely, the asymptotic behavior of the form factor is connected with δ and γ as follows: If

$$g(z,t)\sim t^{-\delta}g_2(z) \quad\text{as}\quad t\to\infty,$$

we have, after performing in Eq. (11) the change of variables:

$$t=2(1+\alpha)^{-1}[M^2+\tfrac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]\tau$$

and the analogous change for t', that

$$F(q^2)\simeq\text{const}\times\int_0^\infty\frac{d\tau d\tau'\tau^{-\delta}\tau'^{-\delta}}{(1+\tau+\tau')^3} \times\int\frac{dzdz'g_2(z)g_2(z')d\alpha(1-\alpha^2)^{1+\delta}}{[M^2+\frac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^{1+2\delta}}. \qquad (24)$$

In order for the first integral to converge we must require $-\frac{1}{2}<\delta<1$. We can distinguish various cases:

(1) $g_2(z)\simeq(1-z)^\gamma, \gamma>0$ and $0<\delta<1$.

(*a*) $2\delta<\gamma<1$.

Then

$$F(q^2)\simeq\text{const}\times\left[\int_{-1}^{+1}\frac{dz\, g_2(z)}{(1-z)^{1+2\delta}}\right]^2 \times\int_{-1}^{+1}d\alpha(1-\alpha^2)^{-\delta}\left(\frac{1}{q^2}\right)^{1+2\delta}. \qquad (25a)$$

(*b*) $0<\gamma<2\delta$.

In this case one performs the change of variables

$$(1-z)=2xq^{-1}M(1-\alpha^2)^{-1/2},$$

and the analogous change for z', to obtain[11]

$$F(q^2)\lesssim\text{const}\times\int_0^{q/M}\frac{dxdx'x^\gamma x'^\gamma}{(1+xx')^{1+2\delta}} \times\int_{-1}^{+1}d\alpha(1-\alpha^2)^{\delta-\gamma}\left(\frac{1}{q^2}\right)^{1+\gamma} \qquad (25b)$$

$$\simeq\text{const}\times(1/q^2)^{1+\gamma}\ln(q^2/M^2).$$

(2) $g_2(z)\simeq(1-z)^{-\gamma'}$, $0<\gamma'<1$, $\delta=-\delta'$, $0<\delta'<\frac{1}{2}$.

(*a*) $0<\gamma'<2\delta'$.

With the same method as above, we get

$$F(q^2)\simeq\text{const}\times\left[\int\frac{dz\, g_2(z)}{(1-z)^{1-2\delta'}}\right]^2\left(\frac{1}{q^2}\right)^{1-2\delta'}. \qquad (26a)$$

(*b*) $0<2\delta'<\gamma'<1$.

$$F(q^2)\simeq\text{const}\times(1/q^2)^{1-\gamma'}\ln(q^2/M^2). \qquad (26b)$$

[14] We shall limit ourselves to wave functions which are even in p_0 and therefore, have positive norm (Ref. 9). This implies that $g(z,t)$ is even in z.

We see therefore that according to the various cases the asymptotic behavior of the form factor is dominated by the behavior of g as $t\to\infty$ or as $z\to 1$. In the last section we shall comment on the relation between such behavior of the spectral function g, and the interaction at small distances between the particles that form the bound state.

III. PROPERTIES OF THE INTEGRAL OPERATOR FOR THE SPECTRAL FUNCTION

The purpose of this section is to prove, for the $g\Phi^3$ model, the relation (10) which has been used in deriving the asymptotic properties of the form factor. Before doing this, one has to give a characterization of the function space in which the eigenvalue equation has to be understood. We shall prove that the integral operator $\bar{K}$ for the function

$$\bar{g}(z,t)\equiv(1-z^2)^{-1/2}g(z,t) \tag{27}$$

is bounded[15] and compact in the space of L^2 functions. This result insures that the spectrum of the bound states is discrete, and the finiteness of the L^2 norm for $\bar{g}$ combined with some properties of the integral operator will be sufficient to prove (10).

The two-dimensional integral operator K for the spectral function g has been given by various authors.[10,16] For the s wave, and using the representation (7) with the cube in the denominator, K can be written in the form

$$K(z,t;z',t')=\frac{d}{dt}\left[\frac{1}{2}\int_0^1\frac{dx}{x^2u^2}t\theta(t)\theta(Ru-t)\right], \tag{28}$$

where

$$u(x;t',z')\equiv\mu^2(1-x)^{-1}+x^{-1}[t'+Q(z')]-Q(z'), \tag{29a}$$

$$Q(z')\equiv M^2-\tfrac{1}{4}m^2(1-z'^2), \tag{29b}$$

$$R(z,z')\equiv(1-z)/(1-z') \quad\text{for}\quad z>z'$$
$$\equiv(1+z)/(1+z') \quad\text{for}\quad z<z'. \tag{29c}$$

Clearly

$$\bar{K}(z,t;z',t')=(1-z^2)^{-1/2}K(z,t;z',t')(1-z'^2)^{1/2}, \tag{30}$$

and the eigenvalue equation we consider is

$$\bar{g}=\lambda\bar{K}\bar{g}. \tag{31}$$

The main difficulty in dealing with the kernel $\bar{K}$ is due to the presence of a square-root singularity along the line

$$t/R(z,z')=\tau_0(z',t')\equiv t'+\mu^2+2\mu[t'+Q(z')]^{-1/2}. \tag{32}$$

[15] In the Appendix it is shown that $\bar{K}$ is bounded also in the Banach space with the norm

$$\|\bar{g}\|'=\max_z(\textstyle\int|\bar{g}(z,t)|^2dt)^{1/2}.$$

This of course implies boundedness in the L^2 space.

[16] G. Wanders, Helv. Phys. Acta **30**, 417 (1957).

However, it is proved in the Appendix that

$$|K|\leqslant C_1[t/R(t/R-\mu^2-t')]^{-1}$$
$$\times[(t/R-\mu^2-t')^2-4\mu^2(t'+Q(z'))]^{1/2},\ (t/R>\tau_0)$$
$$\leqslant C_2[t'+Q(z')]^{-1}\{t'+\mu^2+2\mu[t'+Q(z')]^{1/2}\}^{-1},$$
$$(t/R<\tau_0). \tag{33}$$

The boundedness of K is then shown by using Tiktopoulos's method,[17] choosing as test functions $\sigma_1(z,t)=1/\sqrt{t}$, $\sigma_2(z',t')=1$ and using the bounds (33).

For proving the compactness of $\bar{K}$ we approximate it uniformly in the L^2 norm with operators $\bar{K}_\epsilon$ which do not have the square-root singularity; K_ϵ is defined to be equal to K except for the supplementary restriction

$$|K_\epsilon|\leqslant C_1[t/R(t/R-\mu^2-t')]^{-1}$$
$$\times\{[(t/R-\mu^2-t')^2-4\mu^2[t'+Q(z')]]^2+\epsilon^2\}^{-1/4} \tag{34}$$

for $t/R>\tau_0(z',t')$. From (33) and (34) it follows directly that $\bar{K}_\epsilon$ is square-integrable; i.e., it has a finite Schmidt norm:

$$|\bar{K}_\epsilon|^2\equiv\int dzdtdz'dt'|\bar{K}_\epsilon(z,t;z't')|^2<\infty. \tag{35}$$

In the Appendix it is proved furthermore that

$$\|\bar{K}_\epsilon-\bar{K}\|\to 0 \quad\text{as}\quad \epsilon\to 0. \tag{36}$$

Then from a well-known theorem[18] the compactness of $\bar{K}$ follows.

We come now to prove the relation

$$\int_0^\infty dt|g(z,t)|\leqslant\text{const}\times(1-z^2), \tag{37}$$

where g is a solution of the eigenvalue equation (31).

By using again the bound (33), it is shown in the Appendix that

$$\int_0^\infty dt|g(z,t)|$$
$$\leqslant\text{const}\times\int dz'R(z,z')(1-z'^2)^{1/2}\|\bar{g}\|_t(z'), \tag{38a}$$

$$\leqslant\text{const}\times\int dz'R(z,z')\int_0^\infty dt'|g(z',t')|, \tag{38b}$$

where

$$\|\bar{g}\|_t(z)\equiv\left[\int_0^\infty dt|\bar{g}(z,t)|^2\right]^{1/2}. \tag{39}$$

From the definition of R, and using Schwartz's inequality one gets, from (38a),

$$\int_0^\infty dt|g(z,t)|\leqslant(1-z^2)\{C'[\ln(1-z^2)]^{1/2}+C''\}. \tag{40}$$

[17] G. Tiktopoulos, J. Math Phys. **6**, 573 (1965).

[18] See, e.g., F. Riesz and B. Sz. Nagy, *Leçons d'analyse fonctionelle* (Academie des Sciences de Hongrie, 1952), Chap. IV.

Then, by substituting (40) into Eq. (38b), one obtains the final result, Eq. (37).

IV. GENERALIZATIONS TO SPIN-$\frac{1}{2}$ PARTICLES AND SINGULAR INTERACTIONS

In this section we shall first examine a spin-$\frac{1}{2}$ bound state with "regular potential" and then some singular cases. First we consider a spin-$\frac{1}{2}$ particle interacting with a spinless particle through the exchange of a scalar meson to give a spin-$\frac{1}{2}$, even-parity bound state.[19] In Sec. II we have seen how the large q^2 behavior of the form factors is related to the behavior of the spectral function for $t\to\infty$ and $z\to\pm1$. We shall now induce the behavior of $g(z,t)$ for $t\to\infty$ from the asymptotic properties of the c.m. wave function for $p^2\to\infty$; these can be derived from the zero-energy equation ($m=0$) which, as a rule, takes a rather simple form.[20] In our case the general form of a spin-$\frac{1}{2}$ wave function is

$$\phi=\begin{pmatrix}\bar{A}\chi\\ \bar{B}\sigma\cdot\mathbf{p}\chi\end{pmatrix}=[A(\mathbf{p}^2,p_0)+ip\cdot\gamma B(\mathbf{p}^2,p_0)]\begin{pmatrix}\chi\\ 0\end{pmatrix},\quad(41)$$

where χ is a two-component (constant) spinor and

$$\bar{A}=A-p_0B,\quad \bar{B}=-B.\quad(42)$$

The BS equation has the form

$$\phi=-G(p)[-i(\tfrac{1}{2}P-p)\cdot\gamma+M](\lambda i/\pi^2)\times\int d^4k[(p-k)^2+\mu^2]^{-1}\phi(k),\quad(43)$$

$$G(p)\equiv[(\tfrac{1}{2}P+p)^2+M^2]^{-1}[(\tfrac{1}{2}P-p)^2+M^2]^{-1},\quad(44)$$

and, for $P=0$, it can be easily reduced to the system of differential equ tions ($\mu=0$):

$$\frac{d^2}{ds^2}[s(s+M^2)(MA+sB)]=-\lambda A,\quad(s\equiv p^2)\quad(45)$$

$$\frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}[s^2(s+M^2)(A-MB)]\right\}=\lambda B.$$

There are two possible asymptotic behaviors of A and B in Eq. (45): The irregular[21] behavior is given by

$$A\simeq s^{-2},\quad B\simeq\rho_1(\lambda)s^{-2},\quad(46a)$$

while the regular one is

$$A\simeq s^{-3},\quad B\simeq\rho_2(\lambda)s^{-4},\quad(46b)$$

and ρ_1 and ρ_2 are simple functions of λ which can be easily deduced from (45). If the asymptotic behavior in p^2 can be correctly derived from the zero-energy case (as is usually assumed), then the following spectral representations hold:

$$A=\int\frac{dzdt\,a(z,t)}{(p^2-mp_0z+\rho^2+t)^3},\quad B=\int\frac{dzdt\,b(z,t)}{(p^2-mp_0z+\rho^2+t)^4},\quad(47)$$

where a and b are integrable functions.

Now, the form factors F_1, F_2 and G_E, G_M are given by the relations

$$(1/e)\langle2|J_0|1\rangle=G_E(q^2)\chi_2^\dagger\chi_1,\quad G_E\equiv F_1-(q^2/m^2)F_2,$$
$$(1/e)\langle2|J_1|1\rangle=G_M(q^2)(q/m)(i\chi_2^\dagger\sigma_2\chi_1),\quad(48)$$
$$G_M\equiv F_1+F_2,$$

where, if the charge belongs to the spin-$\frac{1}{2}$ constituent we have[6]

$$(1/e)\langle2|J_\mu|1\rangle=-\int d^4p\bar{\phi}_2(p+\tfrac{1}{2}P_2)\times\gamma_\mu\phi_1(p+\tfrac{1}{2}P_1)(p^2+M^2).\quad(49)$$

By substituting the wave function (41) into Eq. (49) after performing the appropriate Lorentz transformations, we get

$$G_E(q^2)=i\int d^4p(p^2+M^2)[\bar{A}_qA_{-q}+\bar{B}_qB_{-q}(\mathbf{p}^2+p_0^2+\tfrac{1}{4}m^2+K_0p_0)-\bar{B}_qA_{-q}(K_0p_0/m-qp_3/m+\tfrac{1}{2}m)-\bar{A}_qB_{-q}(K_0p_0/m+qp_3/m+\tfrac{1}{2}m)],\quad(50)$$

$$G_M(q^2)=i\int d^4p(p^2+M^2)[\bar{A}_qA_{-q}+\bar{B}_qB_{-q}(p_3^2-p_0^2+\tfrac{1}{4}m^2)-\bar{B}_qA_{-q}(mp_3/q+\tfrac{1}{2}m)-\bar{A}_qB_{-q}(-mp_3/q+\tfrac{1}{2}m)],\quad(51)$$

$$A_q(p)\equiv A(p_1,p_2,(K_0/m)p_3-(q/m)p_0,(K_0/m)p_0-(q/m)p_3+\tfrac{1}{2}m),\quad(52)$$
$$\bar{A}_0(p)\equiv-[A_0(\mathbf{p},p_0^*)]^*.$$

The leading terms of Eq. (51) are of the same form of the spinless case, while those of Eq. (50) are of the form

$$\int\frac{dz\,dtdz'\,dt'a(z,t)b(z',t')(1-\alpha^2)^3(1-z)q^2}{[M^2+\tfrac{1}{2}(1+\alpha)t+\tfrac{1}{2}(1-\alpha)t'+\tfrac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^4}.\quad(53)$$

[19] This model is clearly unrealistic. Moreover, we shall limit ourselves to the equal-mass case; this simplifies the formalism somewhat.

[20] The behavior for $z\to1$ depends on the asymptotic behavior of ϕ for $p_3=p_0+\frac{1}{2}m\to\infty$, which cannot be deduced, of course, from the zero-energy case.

[21] The irregular behavior gives a divergent normalization [cf. Eq. (50)].

The asymptotic behavior of (53) can be derived if one knows the behavior of $a(z,t)$, $b(z,t)$ for $z\to\pm1$, i.e., the behavior of A and B for $p_3=p_0+\frac{1}{2}m\to\infty$ (Sec. II). As $G(p)\simeq(mp_0)^{-1}$, the form of Eq. (44) suggests[22] that

$$A\simeq p_0^{-2},\ B\simeq p_0^{-2} \quad \text{for} \quad p_3=p_0+\tfrac{1}{2}m\to\infty\,. \tag{54}$$

This means that a and b go to zero as $(1\mp z)$ for $z\to\pm1$. Then the leading part of (53) is reduced to the integral[11]

$$q^2\int_0^1\frac{d\lambda\ \lambda d\mu\ \mu^2 d\nu\ \nu^3}{(M^2+\lambda\mu\nu q^2)^4}\simeq\frac{1}{q^2}. \tag{55}$$

This means that

$$G_M(q^2)\simeq(1/q^2)^2\ln(q^2/M^2),\quad G_E(q^2)\simeq1/q^2, \tag{56}$$

$$F_1(q^2)\simeq(1/q^2)^2\ln(q^2/M^2),\quad F_2(q^2)\simeq(1/q^2)^2. \tag{57}$$

This is in qualitative agreement with the experimental results on nucleon form factors. It must be remarked that the assumed interaction is unrealistic; it is also more regular than that due to the exchange of an elementary ρ meson or that due to an N^* intermediate state in the π-N interaction. These singular interactions (see the following) should give different asymptotic behaviors. However, the presence of form factors (e.g., in the $\rho\pi\pi$ vertex) should bring back this problem to a regular case.

We shall now examine two examples in which the interaction turns out to be singular, in the sense that the asymptotic behavior of the wave function in momentum space depends on the strength of the interaction. The simple $(1/q^2)^2$ behavior of the form factor is no longer true, at least in the ladder approximation. The results we shall obtain are, however, not rigorous because the p^2 asymptotic behavior will be deduced from the zero-energy case.

Let us first consider a spinless, odd-parity, N-$\bar{N}$ bound state (e.g., the pion in the Fermi-Yang model). The ladder BS equation takes the form[23]

$$[i(\tfrac{1}{2}P+p)\gamma+M]\phi[-i(\tfrac{1}{2}P-p)\gamma+M] = -i\frac{\lambda}{\pi^2}\int\frac{d^4k}{(p-k)^2+\mu^2}\gamma_5\phi\gamma_5, \tag{58}$$

while the form factor is given by

$$F(q^2)=-K_0^{-1}\int d^4p \times\mathrm{Tr}[\bar{\phi}_2(p+\tfrac{1}{2}P_2)\gamma_4\phi_1(p+\tfrac{1}{2}P_1)(ip\cdot\gamma+M)]. \tag{59}$$

In the zero-energy case a plausible 0^- wave function is of the form $\phi=\gamma_5\varphi(p^2)$, where φ satisfies the Goldstein equation (taken for $\mu=0$, as we are interested in the large-p behavior)

$$\frac{d^2}{ds^2}[s(s+M^2)\varphi]=-\lambda\varphi,\quad (s\equiv p^2). \tag{60}$$

The indicial equation near the $s=\infty$ singularity gives, for the regular behavior $(0<\lambda<\frac{1}{4})$,

$$\varphi\simeq s^{-1-\delta},\quad \delta=\tfrac{1}{2}+(\tfrac{1}{4}-\lambda)^{1/2}<1. \tag{61}$$

From the discussion of Sec. II it follows that, if for $m\neq0$ we also take $\phi=\gamma_5\varphi(p)$,[24] with

$$\varphi(p)=\int\frac{dzdt\ f(z,t)}{(p^2-mp_0z+\rho^2+t)^2}, \tag{62}$$

then

$$f(z,t)\simeq t^{-\delta}f_1(z)\quad\text{for}\quad t\to\infty\,. \tag{63}$$

By substituting (62) into (59), we easily obtain

$$F(q^2)\simeq\mathrm{const}\times\int\frac{dz\,dz'dtdt'd\alpha\ f(z,t)f(z',t')(1-\alpha^2)(1+\alpha)(1-z)}{[M^2+\frac{1}{2}(1+\alpha)t+\frac{1}{2}(1-\alpha)t'+\frac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^2}. \tag{64}$$

By changing variables as in Sec. II, one gets

$$F(q^2)\simeq\mathrm{const}\times\int\frac{dzdz'\ f_1(z)f_1(z')(1-\alpha^2)^\delta(1+\alpha)(1-z)}{[M^2+\frac{1}{4}(1-\alpha^2)(1-z)(1-z')q^2]^{2\delta}}. \tag{65}$$

The asymptotic behavior of (65) can depend on the behavior of $f_1(z)$ near $z=\pm1$. If $f_1(z)\sim(1-z)^\gamma$, $\gamma>0$, one has simply

$$F(q^2)\simeq\mathrm{const}\times(q^2)^{-\gamma-1}\ln(q^2/M^2)\quad\text{if}\quad(1+\gamma)<2\delta, \tag{66a}$$

$$\simeq\mathrm{const}\times(q^2)^{-2\delta}\quad\text{if}\quad(1+\gamma)>2\delta. \tag{66b}$$

From Eq. (66) it is clear that the λ-dependent behavior $(1/q^2)^{2\delta}$ is the best one can expect, and that the

[22] Equation (55) is based on the assumption that, in Eq. (44), the integral over the interaction term goes as $1/p_0$ for $p_3=p_0+\frac{1}{2}m\to\infty$. This can be justified, although not rigorously, by noting that the k integral, which gives the coefficient of the $1/p_0$ behavior, is convergent if the behavior (46b) is assumed.

[23] Equation (58) is written for symmetric pseudoscalar interaction in the isospin-1 channel; therefore it is not quite the Fermi-Yang model which assumes a vector interaction, but can be treated in the same way. One should notice that no proof is available, to our knowledge, for the existence of solutions of this equation. Therefore, until this proof is produced, no rigorous treatment of the form factor can be made.

[24] This form of the wave function is certainly inconsistent with Eq. (58) for $m\neq0$. However, terms of this form clearly occur in the form factor and our argument assumes that these are dominant.

behavior $(1/q^2)^2$ is possibly reached only for $\lambda\to 0$. However, if $\gamma>0$, the form factor goes to zero faster than $1/q^2$.

Analogous remarks hold for the case of a (neutral) vector interaction between two scalar particles bound together to give an s-wave bound state. After subtraction of the divergent part by means of a $g\phi^4$ counterterm, the ladder BS equation is written as

$$\phi(p)=G(p)(-i\lambda/\pi^2)\int d^4k[p\cdot k+\tfrac{1}{4}(m^2-\mu^2)]\times[(p-k)^2+\mu^2]^{-1}\phi(k). \quad (67)$$

The zero-energy equation reduces to the differential equation $(\mu=0)$

$$\frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}[s(s+M^2)^2\varphi(s)]\right\}=-\lambda\varphi(s), \quad (68)$$

which admits the regular behavior

$$\varphi(s)\simeq s^{-2-\delta},\quad \bar{\delta}\equiv(1-\lambda)^{1/2},\quad (0<\lambda<1). \quad (69)$$

If the same behavior is assumed for $m\neq 0$, from the discussion of Sec. II it follows that[25]

$$F(q^2)\simeq(q^2)^{-1-2\delta},\quad (2\bar{\delta}<1), \quad (70)$$

is the best behavior one can expect for $1>\lambda>\frac{3}{4}$.

In conclusion, while a behavior of the form factors more convergent that $1/q^2$ seems to be a common feature of these composite models, the actual behavior is model-dependent. For the lowest partial waves (spin-0 and spin-$\frac{1}{2}$ cases) the $(1/q^2)^2$ behavior is reached [Eqs. (15) and (56)] only if the relativistic interaction is regular at the origin; otherwise a behavior occurs which is dependent on the strength of the singular part (Eqs. (66) and (70)] as in the nonrelativistic case. For higher partial waves, one has similar results, with a scaling of one power of q^2 (Eq. 20) in the regular case; in the singular case the l dependence is more complicated because the critical value of λ is by itself l dependent.

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[25] Equation (70) refers to the form factor of Eq. (6). For vector interaction there is another contribution to the form factor in the ladder approximation, which is not examined here.

APPENDIX

1. Evaluation of an Integral

In the text we have used several times the asymptotic behavior of the integral

$$\int_0^1 d\lambda\,\lambda^a\int_0^1 d\mu\,\mu^b\int_0^1 d\nu\,\nu^c(1+\lambda\mu\nu q^2)^{-1-d}, \quad (A1)$$

$$(0\leqslant a\leqslant b\leqslant c\leqslant d).$$

By using the change of variables

$$\nu=\nu'/\mu',\quad \mu=\mu'/\lambda',\quad \lambda=\lambda'/q^2, \quad (A2)$$

one obtains

$$(q^2)^{-1-a}\int_0^{q^2} d\lambda'\lambda'^{a-b-1}\int_0^{\lambda'} d\mu'\mu'^{b-c-1}\times\int_0^{\mu'} d\nu'\nu'^c(1+\nu')^{-1-d}. \quad (A3)$$

It is now clear that the behavior of this integral is

$$\begin{aligned}\text{const}\times(1/q^2)^{1+a}\times\;&1 &&\text{if } a<b,\\ &\times\ln q^2 &&\text{if } a=b<c,\\ &\times(\ln q^2)^2 &&\text{if } a=b=c<d,\\ &\times(\ln q^2)^3 &&\text{if } a=b=c=d.\end{aligned} \quad (A4)$$

In the same way one can obtain the asymptotic behavior of (A1) in the case $c>d$.

2. Bounds on the Kernel K

We shall prove here the inequalities (33) for the kernel K. For $t/R>\tau_0$, K can be rewritten as follows:

$$K=\frac{1}{2}\int_0^1\frac{dx}{x^2u^2}\theta(Ru-t)-\tfrac{1}{2}R\int_0^1\frac{dx}{x^2u}\delta(Ru-t). \quad (A5)$$

Defining

$$H(t/R,z',t')\theta(t/R-\tau_0)=\tfrac{1}{2}R\int_0^1\frac{dx}{x^2}\delta(Ru-t)=\frac{1}{2(t'+Q(z'))}\times\frac{(t/R-\mu^2-t')\theta(t/R-\tau_0)}{\{(t/R-\mu^2-t')^2-4\mu^2[t'+Q(z')]\}^{1/2}}, \quad (A6)$$

we have

$$\begin{aligned}K&=\int_0^{R(z,z')}dR'\,t^{-1}H(t/R',z',t')-(R/t)H(t/R,z',t')\\ &=\int_{t/R}^\infty\frac{d\tau}{\tau^2}\frac{d}{d\tau}H(\tau,z',t')+\int_{t/R}^\infty d\tau\frac{d}{d\tau}\left[\frac{1}{\tau}H(\tau,z',t')\right]\\ &=\int_{t/R}^\infty\frac{d\tau}{\tau}\frac{d}{d\tau}H(\tau,z',t').\end{aligned} \quad (A7)$$

For $t/R<\tau_0$, on the other hand, we have

$$K=\frac{1}{2}\int_0^1 \frac{dx}{x^2u^2}=\int_{\tau_0(t',z')}^{\infty} d\tau\, \tau^{-2}H(\tau,z',t'). \quad \text{(A8)}$$

One may note how in this region K does not depend on t and z. Noting that

$$\frac{\partial H}{\partial\tau}(\tau,z',t')<0 \quad \text{for} \quad t/R>\tau_0,$$

we have

$$|K|\leqslant\frac{R}{t}\left[-\int_{t/R}^{\infty} d\tau\, \frac{\partial}{\partial\tau}H(\tau,z',t')\right]$$

$$=\frac{R}{t}\left[H\left(\frac{t}{R},z',t'\right)-\frac{1}{2(t'+Q')}\right]\leqslant C_1\left[\frac{t}{R}\left(\frac{t}{R}-\mu^2-t'\right)\right]^{-1}$$

$$\times\left\{\left(\frac{t}{R}-\mu^2-t'\right)^2-4\mu^2[t'+Q(z')]\right\}^{-1/2}. \quad \text{(A9)}$$

For $t/R\leqslant\tau_0$ we have, from (A8),

$$|K|\leqslant\frac{1}{2(t'+Q(z'))}\int_0^{\infty}\frac{d\tau'}{(\tau')^{1/2}(\tau'+\tau_0)^{3/2}}$$

$$\leqslant C_2[t'+Q(z')]^{-1}[\tau_0(z',t')]^{-1}. \quad \text{(A10)}$$

By making use of this bound we prove now Eq. (38) of the text:

$$\int_0^{\infty} dt|g(z,t)|\leqslant C_1\int_0^{\infty} dt'\int_{-1}^{+1} dz'$$

$$\times\int_{\tau_0}^{\infty}\frac{d\tau R(z,z')|g(z',t')|}{\tau(\tau-\mu^2-t')\{(\tau-\mu^2-t')^2-4\mu^2[t'+Q(z')]\}^{1/2}}$$

$$+C_2\int dt'dz' R(z,z')[t'+Q(z')]^{-1}|g(z',t')|. \quad \text{(A11)}$$

By changing variables, the first term on the right-hand side can be put in the form

$$\int_{-1}^{+1} dz'\int_0^{\infty} dt'\int_0^{\infty} dx\frac{R(z,z')|g(z',t')|}{(x^2+\tau_0)\{x^2+2\mu[t'+Q(z')]^{1/2}\}}. \quad \text{(A12)}$$

Equation (38b) is then obvious and Eq. (38a) follows from the Schwartz inequality.

3. Compactness and Boundedness of $\bar{K}$

In order to prove the compactness of $\bar{K}$, one considers the kernels $\bar{K}_\epsilon$, defined through Eq. (34) of the text. From the inequalities (33) and (34) it follows that the kernels $\bar{K}_\epsilon$ are square-integrable, i.e., have a finite Schmidt norm [Eq. (35) of the text]. We now have to prove that

$$\|\bar{K}_\epsilon-\bar{K}\|\to 0 \quad \text{as} \quad \epsilon\to 0.$$

Of course one needs to examine only the region $t/R>\tau_0$. Here one applies the relation

$$A^{-1/2}-(A^2+\epsilon^2)^{-1/4}=\frac{1}{4}\int_0^1\frac{dy\,\epsilon^2}{(A^2+\epsilon^2y)^{5/4}}$$

$$<C(\eta)\epsilon^{2\eta}A^{-1/2-\eta} \quad \text{(A13)}$$

with $0<\eta<\frac{1}{2}$, and

$$A\equiv(t/R-\mu^2-t')^2-4\mu^2[t'+Q(z')].$$

All the reasoning on the boundedness of $\bar{K}$ can be repeated with $A^{1/2+\eta}$ replacing $A^{1/2}$ and this proves that

$$\|\bar{K}_\epsilon-\bar{K}\|<\text{const}\times\epsilon^{2\eta}\to 0, \quad (0<\eta<\tfrac{1}{2}). \quad \text{(A14)}$$

Finally we note how $\bar{K}$ is bounded also according to the more stringent norm

$$\|\bar{g}\|'=\max_z\|\bar{g}\|_t(z)\equiv\max_z\left(\int dt|\bar{g}|^2\right)^{1/2}. \quad \text{(A15)}$$

This means that $\|g\|_t(z)$ goes to zero for $z\to\pm1$ at least as $(1\mp z)^{1/2}$. To obtain this result, we note that

$$\|\bar{K}\bar{g}\|'\leqslant\max_z\int dz'\|\bar{K}\|_t(z,z')\|\bar{g}\|_t(z') \quad \text{(A16)}$$

But using Tiktopoulos's method one obtains

$$\|K\|_t(z,z')\leqslant\text{const}\times[R(z,z')]^{1/2}, \quad \text{(A17a)}$$

so that

$$\|\bar{K}\|_t(z,z')\leqslant\text{const}. \quad \text{(A17b)}$$

From (A17b) and (A16) we get the stated result.