Current Algebra, Crossing Symmetry, and Unitarity for s-Wave $\pi\pi$ Scattering

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We calculate s-wave pion-pion scattering lengths and phase shifts, using the algebra of currents and dispersion relations. We disperse three pions, and in the nonsoft off-shell $q_\pi^2 \to 0$ limit for these pions we use the algebra of chiral $SU(2)\times SU(2)$ and the σ model to obtain an amplitude which is explicitly crossing-symmetric. This amplitude is used to provide the N function for the s-wave pion-pion amplitude written in the usual N/D form. The calculated scattering lengths and phase shifts depend on a parameter related to the $\sigma A_1\pi$ coupling. Our results agree reasonably well with the recent experimental estimates by Walker *et al.* even when the $\sigma A_1\pi$ coupling vanishes.

RECENT interest in the calculation of s-wave pion-pion scattering parameters follows the work of Weinberg.[1] In his calculation of scattering lengths, Weinberg disperses two pions, and using partially conserved axial-vector currents (PCAC), the algebra of currents, and soft-pion techniques, he obtains a scattering amplitude which is not, by itself, crossing-symmetric. He imposes the requirements of crossing symmetry by comparing the amplitude in current algebra with a crossing-symmetric expansion around $s=u=t=0$; this, together with the use of Adler's PCAC consistency condition[2] and the σ model,[3] enables him to evaluate the parameters necessary for the determination of all s-wave scattering lengths. We also remark that, as in all calculations using the algebra of currents, Weinberg's $\pi\pi$ amplitude does not exhibit unitarity.

We attempt to study the modifications in the pion-pion scattering amplitude when

(a) the calculation using the algebra of currents is explicitly crossing-symmetric at all stages and employs the off-shell limit[4] $q_\pi^2 \to 0$ in contradistinction to the soft-pion limit $q_\pi \to 0$;

(b) the requirements of unitarity are subsequently imposed on the amplitude given by current algebra. The N/D formalism is used to unitarize the partial wave amplitude.

In the present calculation of $\pi\pi$ scattering using the algebra of currents, we use PCAC for three pions and symmetrize the resulting amplitude. The structure of this amplitude may be found in Chang.[5] We allow the square of the four-momentum of each of the dispersed pions ($\equiv q^2$) to go to zero. Thus $s+u+t=1-3q^2 \to 1$. In addition to the chiral algebra of $SU(2)\otimes SU(2)$, we also use the current commutation relations given by the σ model. In simplifying our matrix elements, we make extensive use of the form for the three-point functions of vector and axial-vector currents obtained by Schnitzer and Weinberg.[6] We also use the following relation which is a consequence of the σ model:

$$\int d^4x d^4y\, e^{-iq\cdot x+ip\cdot y}\langle T\{\partial_\mu A_\alpha{}^\mu(x), A_\beta{}^\nu(y), \sigma(0)\}\rangle_0$$
$$\equiv (-i)\delta_{\alpha\beta}\frac{f_\pi}{(q^2+1)}\left[g_{A_1}\left(g^{\nu\eta}+\frac{p^\nu p^\eta}{m_{A_1}{}^2}\right)\right.$$
$$\times\frac{1}{(p^2+m_{A_1}{}^2)}\frac{f_\sigma m_\sigma{}^2}{(k^2+m_\sigma{}^2)}\Gamma[(q+k)_\eta+\delta'(q-k)_\eta]$$
$$\left.+\frac{f_\pi p^2}{(p^2+1)}\frac{f_\sigma m_\sigma{}^2}{(k^2+m_\sigma{}^2)}\Gamma'\right]. \quad (1)$$

Here $g_{A_1}=\sqrt{2}m_\rho f_\pi$, f_σ is the decay constant of the σ meson, and Γ and δ' are the two (unknown) parameters which determine the $\sigma A_1\pi$ coupling. In our off-shell limits, the second term in the expression above does not contribute to the scattering amplitude. The final form of the off-shell amplitude is as follows:

$$T_{\beta\gamma,\alpha\delta}{}^{\mathrm{C.A.}}(s,t,u)=-\frac{1}{16\pi(\sqrt{s})f_\pi{}^2}$$
$$\times\left[\delta^{\alpha\delta}\delta^{\beta\gamma}\left(\tfrac{1}{3}(s+t+u)+(\tfrac{2}{3}-s)+(t-s)\frac{m_\rho{}^2+\frac{1}{2}u(1-\delta)}{m_\rho{}^2-u}\right.\right.$$
$$\left.+(u-s)\frac{m_\rho{}^2+\frac{1}{2}t(1-\delta)}{m_\rho{}^2-t}+xm_\sigma{}^2\frac{1-s}{m_\sigma{}^2-s}\right)$$
$$\left.+\delta^{\alpha\beta}\delta^{\gamma\delta}\{s\leftrightarrow t,\,u\}+\delta^{\alpha\gamma}\delta^{\beta\delta}\{s\leftrightarrow u,\,t\}\right]. \quad (2)$$

Here[7]

$$x=(g_{A_1}/m_{A_1}{}^2)(f_\sigma/f_\pi)\Gamma\delta',\quad \delta=-1.$$

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[1] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); see also R. Arnowitt *et al.*, Phys. Rev. Letters **20**, 475 (1968); I. S. Gerstein and H. J. Schnitzer Phys. Rev. **170**, 1638 (1968).

[2] S. Adler, Phys. Rev. **137**, B1022 (1965).

[3] M. Gell-Mann and M. Lévy, Nuovo Cimento **46**, 707 (1966).

[4] S. Okubo, R. E. Marshak, and V. S. Mathur, Phys. Rev. Letters **19**, 407 (1967).

[5] L. Chang, Phys. Rev. **162**, 1497 (1967). We use the definitions and notation of Chang and Weinberg (Ref. 1).

[6] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

[7] Schnitzer and Weinberg, Ref. 6, use $\delta=-\frac{1}{2}$. K. C. Gupta and J. S. Vaishya [Phys. Rev. **170**, 1530 (1968)] find better agreement with observed decay widths using $\delta=-1$. Variations of δ in this region alter our results minimally; hence we work with $\delta=-1$.

We note here the following features of the amplitude given in Eq. (2):

(i) The amplitude is explicitly crossing symmetric.
(ii) No expansion is used to determine any parameters.
(iii) In addition to the constant and linear terms in s, t, and u, the amplitude has contributions from ρ, σ, and A_1 exchanges in the crossed channels. These may be regarded as providing terms quadratic (and of higher powers) in the kinematic variables.[8]

The amplitude (2) with three zero-mass off-shell pions is an unphysical amplitude in an unphysical kinematic domain $s+u+t=1$. To recover the physical amplitude, this amplitude must first be continued analytically in the masses of the external pions. This problem of mass extrapolation has been studied by several authors.[9] Meiere and Sugawara[10] have studied this problem in dispersion theory for $\pi\pi$ scattering. Their conclusion is that for a pion pion amplitude with two pions off their mass shells, continuation in the pion mass from 0 to 1 changes threshold parameters by about 5%; one may expect scattering at higher energies to be even less affected. We therefore make an assumption of smooth extrapolation in the masses of the external pions for an amplitude with three pions off their mass shells. Thus we assume that the amplitude (2) is the physical pion-pion amplitude even just below the unitarity cut. This assumption, however, needs further investigation.

Even though the amplitude (2) is not, by itself, unitary, it may still be used to compute threshold parameters, once the assumption of smooth extrapolation is made. We note, however, that the pion-pion phase shifts at the kaon mass are of interest, being related to the $K_1^0-K_2^0$ mass difference[11] and to the branching ratios of the various decay modes of the K_1^0.[12] An amplitude which has no unitarity cut cannot reasonably be used to compute scattering parameters far from threshold. We therefore consider the effects of imposing unitarity on this amplitude.

The problem of unitarizing the results of current algebra has been studied by several authors.[13,14] In general, there seem to be two approaches to the problem. Raman and others[13] employ the results of current algebra to evaluate the amplitude at an unphysical point in the domain of validity of current algebra. This number is then continued into the physical region by suitable use of analyticity and unitarity. Since the value of the amplitude at a point is continued analytically, the process of continuation is not unique and additional dynamical assumptions[15] have to be made to perform the continuation. We remark here that if current algebra is used only in the soft-pion limit $q_\pi \to 0$, it is difficult to justify the use of current algebra except to evaluate the amplitude at a point.

However, the results of current algebra in the weaker off-shell limit $q_\pi^2 \to 0$ may be regarded as giving a valid analytic form for the amplitude in a certain unphysical kinematic stu domain. The equal-time commutator, taken in conjunction with the "weak" amplitude, now includes t- and u-channel exchanges and gives an analytic form for the amplitude even for $t\neq 0$. We therefore adopt the viewpoint that our crossing symmetric amplitude from current algebra in the off-shell $q_\pi^2 \to 0$ limit may be regarded as giving the input forces in a dynamical theory. This amplitude has the characteristics of the input forces which would arise in a dynamical theory from crossed channel exchanges; in addition, there are terms of the form $a+bs$ representing the background effect arising from distant singularities. The strength of all the singularities is given by current algebra. Thus this current algebra amplitude, modified by the exclusion of the direct channel σ resonance,[16] may be used as the input force giving the Born terms for the N function in a unitarized N/D formalism for the s-wave $\pi\pi$ amplitude. Thus, if

$$f_0^I(s) \equiv e^{i\delta_0^I(s)} \sin\delta_0^I(s)/|\mathbf{q}|\,, \tag{3}$$

we define

$$g_0^I(s) \equiv (\sqrt{s}) f_0^I(s)$$

and perform the N/D decomposition for $g_0^I(s)$. We assume that $D_0^I(s)$ satisfies a once-subtracted dispersion relation in s:

$$D_0^I(s) = 1 + \frac{(s-s_0)}{\pi}\int_4^\infty ds' \frac{\operatorname{Im} D_0^I(s')}{(s'-s_0)(s'-s)}. \tag{4}$$

[8] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).
[9] See, e.g., S. Adler, Phys. Rev. **140**, B736 (1965); K. Raman and E. C. G. Sudarshan, *ibid.* **154**, 1499 (1967); S. Fubini and G. Furlan, M.I.T. report, 1968 (unpublished), and references listed therein.
[10] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1709 (1967).
[11] S. Patil, Phys. Rev. Letters **13**, 454 (1964); T. Truong, *ibid.* **17**, 1102 (1966). See also papers by R. Rockmore and T. Yao, Phys. Rev. Letters **18**, 501 (1967); K. Kang and D. J. Lane, *ibid.* **18**, 503 (1967); S. L. Glashow, *ibid.* **18**, 524 (1967).
[12] S. Okubo *et al.*, Phys. Rev. **112**, 665 (1958).
[13] K. Raman, Phys. Rev. Letters **17**, 983 (1966); K. Kang and T. Akiba, Phys. Rev. **164**, 1836 (1967); S. C. Bhargava *et al.*, Phys. Rev. Letters **20**, 558 (1968); J. Sucher and C. Woo, *ibid.* **18**, 723 (1967).
[14] L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968).
[15] Thus, Bhargava *et al.*, Ref. 13, assume that the N function in an N/D formalism is given by a single pole on the negative real s axis and that the strength of the pole is determined by current algebra in the soft-pion limit. Akiba and Kang, Ref. 13, assume that the absorptive parts in a fixed-t dispersion relation are given by the Chew-Mandelstam or by the nonrelativistic effective-range approximation.
[16] A direct-channel resonance would, in an N/D formalism, appear as a zero of D. Our D function shows no such zero at the energy which characterizes the mass of the exchanged particle. This shows that current algebra is not "strong" enough to generate an s-wave $\pi\pi$ resonance; it also shows that the amplitude has lost crossing symmetry as a result of analytic continuation in s. This is a reflection of the well-known difficulties in satisfying unitarity and crossing symmetry in any approximate calculation.

TABLE I. s-wave $\pi\pi$ scattering lengths (a^0,a^2) and effective ranges (r^0,r^2) in units of m_π^{-1}. $s_0=1.0$; $\Lambda=50\ m_\pi^2$.

x	a^0	a^2	r^0	r^2
+2	0.11	−0.095	−16.6	+14.0
0	0.15	−0.075	−14.2	+14.6
−2	0.20	−0.05	−11.8	+12.8

We choose $s_0=1$,[17] since it is sufficiently far from the unitarity cut and is within the domain of validity of current algebra. As a consequence of our preceding discussion, we write[18]

$$N_0{}^I(s)=(\sqrt{s})f_0{}^I(s)\,|_{\text{C.A.}}.$$

We use elastic unitarity to obtain $\text{Im}D_0{}^I(s)$, $s\geqslant 4$. In view of the linear asymptotic behavior of the N function, the integral (4) diverges; we use a straight cutoff Λ which we vary between 50 and 100 (in units of $m_\pi{}^2$)[19] We also vary the parameter x over a reasonable range of values.[20] The scattering lengths and phase shifts in the channels $I=0$, 2 are shown in Table I and Fig. 1.

We have shown how the calculation of the $\pi\pi$ amplitude in current algebra may be performed crossing-symmetrically and in the off-shell, nonsoft limit $q_\pi{}^2\rightarrow 0$. The resulting amplitude has been unitarized. The scattering lengths obtained by Weinberg are essentially stable to both modifications. This conclusion is in agreement with the results of Akiba and Kang.[13] The calculation exhibits how large phase shifts may be consistent with small scattering lengths if current algebra and dispersion theory are suitably combined. In particular, our results show a positive scattering length in the channel $I=0$ corresponding to an attractive interaction, a weak repulsion in the channel $I=2$, and phase shifts at the kaon mass such that

$$+37^\circ(x=2)\lesssim\delta^0-\delta^2\lesssim+50^\circ(x=-2).$$

The sign of $\delta^0-\delta^2$ agrees with determinations from strong interaction data[21] and disagrees with that demanded by CP-violating weak decays.[22]

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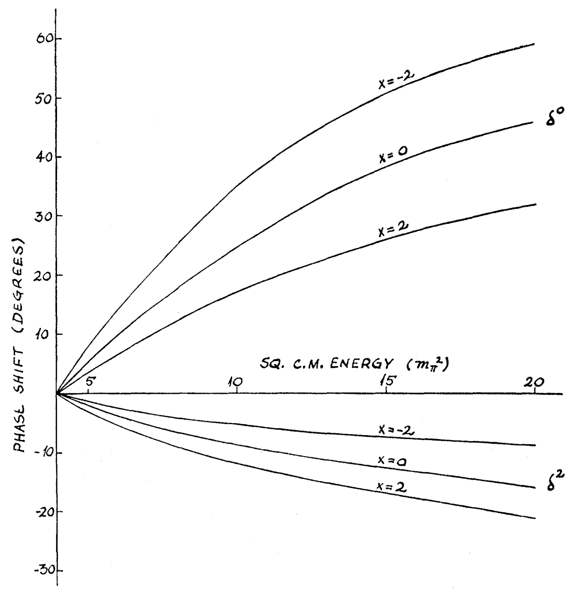

FIG. 1. s-wave $\pi\pi$ phase shifts in degrees for $I=0$, 2 plotted against the square of the c.m. energy (in units of the pion mass squared) for $x=2$, 0, -2.

[17] We have varied s_0 in the range $0\leqslant s_0\leqslant\frac{4}{3}$, which includes both the off-shell and on-shell symmetry points. The numerical results are completely insensitive to this variation.

[18] $f_0{}^I(s)$ is the partial-wave projection of the amplitude given in Eq. (2) using on-mass-shell kinematics. This results in $f_0{}^I(s)$ having the correct threshold behavior at $s=4$. An alternate procedure would be to project partial waves from (2) using off-mass-shell kinematics and assume smooth extrapolation in the masses of the external pions for the resulting partial-wave amplitude. Use of this latter partial-wave amplitude as the N function is harder to justify since its threshold behavior at $s=4$ is unphysical. We also recall that in perturbation theory, off-mass-shell continuation is more easily performed for the invariant rather than for the partial-wave amplitude.

[19] In this region of variation, the results vary between 3 and 10%. Use of a very much larger cutoff (~1000) does, indeed, affect the results. However, we use elastic unitarity and, experimentally, large inelastic effects start around $s=40$. We would, of course, get a convergent integral in (4) if we use the parametrization of the N function due to B. R. Desai [Phys. Rev. Letters **6**, 497 (1961)]. That would, however, merely be a reparametrization of the N function with the pole position replacing the cutoff; further, it uses current algebra at two neighboring points in a domain only.

[20] We vary x in the region $-2\leqslant x\leqslant+2$. The numerical results for $x=1$, -1 lie between those exhibited for $x=2$, 0, -2.

[21] W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967).
[22] S. Bennett *et al.*, Phys. Rev. Letters **19**, 997 (1967).