Coherent Soft-Photon States and Infrared Divergences. II. Mass-Shell Singularities of Green's Functions*

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This is the second in a series of four papers in which a new, field-theoretic approach to the problem of the infrared divergences of quantum electrodynamics is presented. The primary aim of the present paper is the study of the mass-shell singularities of the Green's functions, which are branch points rather than simple poles. This is an essential preliminary to the discussion of asymptotic states and scattering matrix elements in subsequent papers. A conventional separation is introduced between hard- and soft-photon regions of momentum space. It is shown that the soft-photon contribution to an arbitrary Green's function may be isolated in a single function of the external momenta, independent of the spins of the particles involved. An explicit expression for this function is obtained and its mass-shell singularities are studied in detail. In particular, it is shown that, in contrast to the case where there are no massless particles, the mass-shell singularities in different momenta are not independent, but depend on the order in which the various momenta are allowed to approach their mass shells.

1. INTRODUCTION

In a preceding paper[1] (here referred to as I) we discussed the problem of scattering by a prescribed external classical current distribution. A set of generalized coherent states of the radiation field were defined, which can contain infinitely many soft photons and which span a nonseparable Hilbert space $\mathcal{H}_{\mathrm{IR}}$. We showed that it is possible to define a unitary scattering operator on this space, all of whose matrix elements are finite.

It has been known since the classic paper of Bloch and Nordsieck[2] that the infrared divergences of quantum electrodynamics appear because an accelerated charged particle can emit an infinite number of soft photons with finite total energy. Thus the assumption which is implicit in the conventional perturbation calculation, that the asymptotic states belong to the Foch space, is invalid. It is the principal aim of this series of papers to show that quantum electrodynamics may be treated like any other renormalizable field theory, provided that one drops this assumption and instead allows the theory itself to determine the nature of the asymptotic states between which scattering matrix elements are to be evaluated.

Similar ideas have been presented by Chung[3] and more recently by Storrow[4] in the context of S-matrix theory. These authors were concerned to show that if the asymptotic states of the radiation field are chosen to be coherent states, then all the S-matrix elements can be made finite. Their calculations were performed by making a formal expansion of the coherent states in terms of states with definite numbers of photons, and summing the relevant Feynman diagrams to an exponential form which can be given a meaning even when the expansion in question is invalid, that is, when the expectation value of the total photon number is infinite.

Our approach will be rather different, and more strictly field-theoretical. That is to say, the basic quantities in our work are not the S-matrix elements but the field operators. We do not regard the structure of the space of asymptotic states as given *a priori*. Rather, their properties are to be determined from an examination of the singularity structure of the Wightman functions or Green's functions, which contain complete information about the theory. The present paper is devoted to a study of this singularity structure. In subsequent papers, we shall use the information so obtained to define asymptotic states and extract the scattering matrix elements of the theory.

We begin in Sec. 2 by introducing a conventional separation of the four-dimensional momentum space into hard- and soft-photon regions. In this section, we also investigate the asymptotic behavior of a function which will play an important role later. Then in Sec. 3 we show that the soft-photon contribution to an arbitrary Green's function for values of the momenta close to their mass shells may be isolated in a single function of these momenta, independent of the spins of the particles involved. Section 4 is devoted to an examination of the mass-shell singularities of these functions. These singularities are now branch points rather than simple poles. Moreover, in contrast to the situation in a theory without massless particles, the singularities in different momenta are no longer independent, but depend on the order in which the various momentum variables are allowed to approach their mass shells. The conclusions are briefly summarized in Sec. 5.

2. SEPARATION OF HARD AND SOFT PHOTONS

Let us begin by recalling some definitions and notation.

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[1] T. W. B. Kibble, J. Math. Phys. 9, 315 (1968). (This paper will be referred to as I.) T. W. B. Kibble, in *Applications of Mathematics to Problems in Theoretical Physics*, edited by M. Lévy (Gordon and Breach Science Publishers, Inc., New York, 1968).
[2] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).
[3] V. Chung, Phys. Rev. 140, B1110 (1965).
[4] J. K. Storrow, Nuovo Cimento 54A, 15 (1968).

In an arbitrary gauge, the free photon propagator takes the form

$$D_{\mu\nu}(x)=\int \frac{dk}{(2\pi)^4}\gamma_{\mu\nu}(k)\frac{e^{ik\cdot x}}{k^2-i\epsilon}, \tag{2.1}$$

where

$$\gamma_{\mu\nu}(k)=g_{\mu\nu}-k_\mu l_\nu^*(k)-l_\mu(k)k_\nu. \tag{2.2}$$

In this paper, we shall consider only gauges in which $l_\mu(k)$ is a real, odd function of k, so that $\gamma_{\mu\nu}(k)$ is real, symmetric, and even in k:

$$\gamma_{\mu\nu}^*(k)=\gamma_{\nu\mu}(k)=\gamma_{\mu\nu}(-k)=\gamma_{\mu\nu}(k). \tag{2.3}$$

We shall compute the Green's functions in an arbitrary gauge, but when we consider their physical interpretation, we shall restrict our discussion to *physical* gauges which are characterized by the fact that for $k^2=0$, $l_\mu(k)$ satisfies the relations $l^2=0$ and $k\cdot l=1$, so that $\gamma_{\mu\nu}(k)$ becomes the projector on the two-dimensional subspace orthogonal to k and l. In particular, we shall consider the radiation gauge, characterized by the matrix function

$$\gamma_{\mu\nu}(k)=g_{\mu\nu}-\frac{(k_\mu n_\nu+n_\mu k_\nu)k\cdot n+k_\mu k_\nu}{k^2+(k\cdot n)^2}, \tag{2.4}$$

where n_μ is the unit timelike vector $n^\mu=(1,\mathbf{0})$, satisfying $n^2=-1$.

We now introduce a conventional separation of the four-dimensional k space into complementary hard- and soft-photon regions Ω^h and Ω^s. The soft-photon region Ω^s is defined by the inequalities

$$|\mathbf{k}|<K,\quad |k^0|<K^0, \tag{2.5}$$

where K and K^0 are constants chosen so that

$$K\ll K^0\ll m, \tag{2.6}$$

where m is the smallest mass in the theory.

Within Ω^s, we may treat the components of k^μ/m as small quantities, and thus obtain expressions for the soft-photon contributions by neglecting all but the leading terms. If $f(p)$ is any slowly varying function of the four-vector p (that is, one which changes little in a region small compared to m), then we may often neglect the difference between $f(p)$ and $f(p+k)$ so long as k lies in Ω^s. We shall make frequent use of this freedom to make small changes in the arguments of slowly varying functions.

The purpose of the first inequality in (2.6) is to allow us to extend certain integrals over k^0 to the entire real axis, and so evaluate them by contour integration.

It should be remarked that the constants K and K^0 are not to be identified with the physical cutoff ΔE imposed by the experimental design, which may well not satisfy the inequaltiy $\Delta E\ll m$. Apart from the requirement that they satisfy (2.6), these constants are arbitrary. The condition that all physical quantities be independent of the choice of K and K^0 will provide an important consistency check on the theory.

In a perturbation expansion, we may separate the contributions to each internal photon line from Ω^h and Ω^s, and regard them as contributions to distinct Feynman diagrams, in which the internal photon lines are labeled as "hard" or "soft." This corresponds to the separation of the photon propagator

$$D_{\mu\nu}(x)=D^h{}_{\mu\nu}(x)+D^s{}_{\mu\nu}(x),$$

with

$$D^{h,s}{}_{\mu\nu}(x)=\int_{\Omega^{h,s}}\frac{dk}{(2\pi)^4}\gamma_{\mu\nu}(k)\frac{e^{ik\cdot x}}{k^2-i\epsilon}. \tag{2.7}$$

We note that the soft-photon propagator $D^s{}_{\mu\nu}(x)$, being the Fourier transform of a distribution with compact support, is an entire analytic function of x, and in particular that it has no singularity on the light cone. For example, in the Lorentz gauge it is of the form $D^s{}_{\mu\nu}=g_{\mu\nu}D^s{}_F$, and, taking the limit $K^0\to\infty$ justified (except near $x^0=0$) by the first inequality in (2.6), we find explicitly

$$D^s{}_F(x)=\frac{i}{4\pi^2x^2}\times\left[1-e^{-iK|x^0|}\left(\cos K|\mathbf{x}|+i\frac{|x^0|}{|\mathbf{x}|}\sin K|\mathbf{x}|\right)\right]. \tag{2.8}$$

It will be convenient at this point to investigate the asymptotic behavior of a function which will play an important role in our later discussion. If p_i and p_j denote the momenta of particles with charges e_i and e_j, then we shall write

$$L_{ij}(x)=e_ie_j\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{(p_i\cdot k)(p_j\cdot k)}(e^{ik\cdot x}-1), \tag{2.9}$$

where $k^0=|\mathbf{k}|$ in the integrand, and consider the asymptotic behavior of this function for large values of x.

Let us define, for any two vectors p^μ, p'^μ, the function

$$L(p,p';x)=\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\frac{p\cdot p'}{(p\cdot k)(p'\cdot k)}(e^{ik\cdot x}-1). \tag{2.10}$$

Then in the Lorentz gauge we obviously have

$$L_{ij}{}^L(x)=e_ie_jL(p_i,p_j;x). \tag{2.11}$$

On the other hand, using (2.4), we find that in the radiation gauge

$$L_{ij}{}^R(x)=e_ie_j[L(p_i,p_j;x)-L(p_i,n;x)-L(n,p_j;x)+L(n,n;x)]. \tag{2.12}$$

Thus it is sufficient to examine the asymptotic behavior of the function (2.10). This is easily found to be of the form

$$L(p,p';x)\approx(1/8\pi^2)\Phi(u)\ln(K^2x^2)+f(\mathbf{x}/x^0), \tag{2.13}$$

where f as indicated depends only on the direction of

x, not on its magnitude, and where $\Phi(u)$ is a function of the relative velocity

$$u=\left[1-\frac{p^2p'^2}{(p\cdot p')^2}\right]^{1/2}, \tag{2.14}$$

given by

$$\Phi(u)=\frac{1}{2u}\ln\frac{1+u}{1-u}. \tag{2.15}$$

It follows that the asymptotic behavior of L_{ij} is

$$L_{ij}(x)\approx\tfrac{1}{2}\xi_{ij}\ln(K^2x^2)+f_{ij}(\mathbf{x}/x^0), \tag{2.16}$$

where the parameters ξ_{ij} are given by

$$\xi_{ij}{}^L=(e_ie_j/4\pi^2)\Phi(u_{ij}), \tag{2.17}$$

or

$$\xi_{ij}{}^R=(e_ie_j/4\pi^2)[\Phi(u_{ij})-\Phi(u_i)-\Phi(u_j)+1]. \tag{2.18}$$

Here u_{ij} is the relative velocity of particles i and j,

$$u_{ij}=\left[1-\frac{m_i^2m_j^2}{(p_i\cdot p_j)^2}\right]^{1/2}, \tag{2.19}$$

and u_i is the velocity of particle i,

$$u_i=|\mathbf{p}_i/p_i{}^0|. \tag{2.20}$$

We note, in particular, that for $i=j$ the Lorentz-gauge parameter

$$\xi_{ii}{}^L=e_i{}^2/4\pi^2 \tag{2.21}$$

is positive, while in the radiation gauge (the case of physical interest)

$$\xi_{ii}{}^R=(e_i{}^2/2\pi^2)[1-\Phi(u_i)] \tag{2.22}$$

is always negative, except for zero velocity.

3. STRUCTURE OF GREEN'S FUNCTIONS

The Green's functions of any field theory may be defined as functional derivatives of the vacuum-to-vacuum transition amplitude $\langle 0;\text{out}|0;\text{in}\rangle_j$ in the presence of external sources $j(x)$, according to the relation

$$G(x_1\cdots x_n)=(-i)^n\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\langle 0;\text{out}|0;\text{in}\rangle_j\bigg|_{j=0}, \tag{3.1}$$

in which we have for simplicity suppressed all the indices except the space-time label x. If there is no explicit dependence of the field operators $\phi(x)$ on the sources $j(x)$, then the Green's functions may also be identified with the vacuum expectation values of time-ordered products,

$$G(x_1\cdots x_n)=\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle. \tag{3.2}$$

In general, however, for fields of spin greater than one-half, there are additional terms due to the explicit dependence of certain field components on the external sources.

We shall find it convenient to treat separately the soft-photon part of the external electromagnetic current, which we denote by $J_\mu(x)$. Thus $J_\mu(x)$ is a function whose Fourier transform vanishes outside Ω^s. We denote all other external sources, including the hard-photon part of the external electromagnetic current, collectively by $j(x)$. It is convenient to make a functional Taylor-series expansion in $j(x)$, but to retain the dependence on $J_\mu(x)$. Thus we consider the functions

$$G(x_1\cdots x_n|J)=(-i)^n\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\langle 0;\text{out}|0;\text{in}\rangle_{j,J}\bigg|_{j=0}, \tag{3.3}$$

or their Fourier transforms

$$G(p_1\cdots p_n|J)=\int dx_1\cdots dx_n\exp(-i\sum_{j=1}^{n}p_j\cdot x_j)G(x_1\cdots x_n|J). \tag{3.4}$$

We are interested particularly in the behavior of these functions for values of the momenta p_j close to their mass shells $p_j{}^2=-m_j{}^2$.

The reason for considering these quantities as functionals of J, rather than expanding also in powers of J, is that we thereby obtain greater generality. The operators $U(f)$, for example, may be defined formally as exponential functions of the soft-photon field operators, but we wish to be able to consider cases in which the expansion of the exponential is not permissible.

We shall sometimes make the identification (3.2) and write

$$G(x_1\cdots x_n|J)=\langle 0;\text{out}|T[\phi_1(x_1)\cdots\phi_n(x_n)]|0;\text{in}\rangle_J, \tag{3.5}$$

without explicitly indicating the additional terms which must appear for higher-spin fields, and in particular for the hard-photon part of the electromagnetic field. It is to be understood that (3.5) is really a formal expression for (3.3).

In computing the Green's functions (3.3) or (3.4) we encounter the usual problem of ultraviolet divergences, which are not our immediate concern. Since the mass and charge renormalization constants are not infrared-divergent, we may assume that these renormalizations have already been carried out. We shall denote by m_j and e_j the physical mass and charge of the particles annihilated by the field ϕ_j, and work only with these quantities, not with the bare masses or charges. However, the wave-function renormalization constants of charged fields are in general infrared-divergent. Indeed, in a physical gauge they normally vanish, reflecting the fact that the probability of creating a real charged particle alone, with no accompanying soft photons, is zero. Thus the fields ϕ_j appearing in (3.5) must be *un*renormalized fields. Later we will have to find an alternative to the usual formalism of wave-function renormalization.

The wave-function renormalization constant of the photon is of course identical with the charge renormalization constant. Thus along with the charge renormalization we may assume that the photon wave-function renormalization has been carried out, so that near $k^2=0$ the photon propagator is equal to the free propagator $\gamma_{\mu\nu}(k)/(k^2-i\epsilon)$. This means in particular that no self-energy corrections need be inserted on the internal soft-photon lines.

Isolation of Soft-Photon Contributions

Now let us consider a calculation of (3.4) by the standard methods of perturbation theory. In any Feynman diagram which contributes to it, we may identify a *core* diagram obtained by removing all internal soft-photon lines, and setting $J=0$ (which amounts to removing also the external soft-photon lines). We may then recover the full set of diagrams by reinserting the internal soft-photon lines, and the interactions with the external current J, into the core diagrams in all possible ways.

In fact, however, we need consider only a restricted set of soft-photon insertions. Only those contributions which are sufficiently singular at $k=0$ to yield a non-negligible result when integrated over the small volume Ω^s need be included. If we were concerned only with values of the momenta far from their mass shells, there would be no such contributions, other than the self-interaction terms involving the external current J. For the insertion of an internal soft-photon line introduces two extra charged-particle propagators, a photon propagator, and a four-dimensional integration over Ω^s, and unless the charged-particle propagators can become singular, this contribution is of the order K^2, and therefore negligible. However, we are interested in the behavior of the Green's functions for values of the momenta near their mass shells. If the vertices in question lie on the external lines of the core diagrams, then as the momenta approach their mass shells, the charged-particle propagators develop $1/k$ singularities. Hence the only internal soft-photon lines we need consider are those with both ends attached to the external lines of the core diagrams. Similarly, in the case of interactions with the external current J_μ, the role of one of the charged-particle propagators is played by the current $J_\mu(k)$, so that if the singularity of $J_\mu(k)$ at $k=0$ is no worse than $1/k$ (as we shall assume), then we again need to consider only insertions in the external lines of the core diagrams.

The insertion process may conveniently be described by the action of a functional differential operator. Let us introduce a classical external soft-photon field $A_\mu(k)$, and consider first the insertion of interactions with A_μ into the core diagrams. We denote by $G^h(p_1\cdots p_n|A)$ the sum of all diagrams including interactions with this external field, but with no internal soft-photon lines. (By an extension of our earlier notation, the superscript h denotes the absence of internal soft-photon lines.) Then we may write

$$G(p_1\cdots p_n|J) =\exp\left(-\tfrac{1}{2}i\int dxdy\,\frac{\delta}{\delta A_\mu(x)}D^s{}_{\mu\nu}(x-y)\frac{\delta}{\delta A_\nu(y)}\right) \times\exp\left(i\int dx\,A_\mu(x)J^\mu(x)\right)G^h(p_1\cdots p_n|A)\Big|_{A=0}. \quad (3.6)$$

In effect, the exponential factor on the right introduces vertices at which the external current J^μ appears, and the other factor then connects these, and the vertices inserted in the core diagrams, by soft-photon lines in all possible ways.

Formula (3.6) is exact except that it contains contributions from disconnected vacuum parts which should be removed, but which are in any case negligible. However, as we have seen, in computing $G^h(p_1\cdots p_n|A)$ we need only consider insertions of A vertices in the external lines of the core diagrams, and even for these we need only keep the leading terms in k.

Structure of Core Diagrams

We begin by examining the core diagrams themselves, represented by the function $G^h(p_1\cdots p_n)$ with $A=0$. Since it is no part of our present purpose to enquire into such questions as the convergence of the perturbation expansion or the finiteness or otherwise of the renormalization constants (aside from their infrared parts), we shall assume that the contributions of all core diagrams can be summed, and that this sum possesses all the properties to be expected in a well-behaved field theory. In particular, we assume the usual singularity structure and the usual form of the ultraviolet divergences. Thus all the ultraviolet divergences may be collected in a single over-all factor of the form

$$\prod_j [Z^h{}_j(p_j)]^{1/2},$$

where $Z^h{}_j(p_j)$ is the wave-function renormalization "constant" of the field ϕ_j with the soft-photon contributions to self-energy parts removed. Note that this "constant" is a function of p_j even in a covariant gauge because of the noncovariance of the separation between hard and soft photons.

In order to examine the mass-shell singularity structure of the Green's functions it will be necessary to separate the core diagrams into connected parts. Let us consider first a line which passes straight through the diagram without interactions, other than self-energy parts, say, from the external line labeled p_i to that labeled p_j. Such a line can of course appear only if the quantum numbers associated with ϕ_i and ϕ_j are those of a particle and antiparticle so that $\phi_j=\phi_i^c$. It will contribute a factor

$$-i(2\pi)^4\delta(p_i+p_j)G^h{}_i(p_i)C_{ij}\,, \tag{3.7}$$

where C_{ij} denotes a charge-conjugation matrix in the spin space and where $G^h{}_i(p_i)$ denotes the propagator function for the field ϕ_i with all soft-photon contributions to the self-energy parts removed. This function has a simple pole on the mass shell, and for p_i^2 close to $-m_i^2$ takes the form

$$G^h{}_i(p_i)=Z^h{}_i(p_i)\Lambda_i(p_i)/(m_i^2+p_i^2-i\epsilon)\,, \tag{3.8}$$

where $\Lambda_i(p_i)$ is an appropriate spin matrix, for example, $(m_i-i\gamma\cdot p_i)$. The subscripts i, j on C indicate that this matrix connects the spin indices associated with fields ϕ_i and ϕ_j. For consistency, it is necessary that (3.7) be unchanged by interchanging i and j, except for a change of sign in the case of Fermi fields. This is assured by the equality $Z^h{}_j(-p)=Z^h{}_i(p)$ together with the relations

$$C\tilde{\Lambda}(-p)=\Lambda(p)C\,,\qquad \tilde{C}=\pm C\,, \tag{3.9}$$

where the $\pm$ signs refer to Bose and Fermi fields, and the tilde denotes transposition.

We also note that, for $p_j^2=-m_j^2$, $\Lambda_j(p_j)$ is essentially a projection matrix. It satisfies the relation

$$\Lambda_j(p_j)\Lambda_j(p_j)\simeq N_j\Lambda_j(p_j)\,, \tag{3.10}$$

where N_j is some constant normalization factor ($=2m_j$ in the case of a Dirac field).

Now let us turn to the completely connected core diagrams. Their contribution has a simple pole in each momentum variable at its mass shell, and in that neighborhood has the form

$$\prod_{j=1}^{n}\left\{\frac{-i[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)}{m_j^2+p_j^2-i\epsilon}\right\}\times(2\pi)^4\delta(p_1+\cdots+p_n)M^h(p_1\cdots p_n)\,. \tag{3.11}$$

The function M^h denotes the connected part of the scattering amplitude (or, more precisely, the "M function") with all soft-photon contributions removed. It is free of ultraviolet divergences, and finite on the mass shells, except of course at the usual physical-region singularities.

For simplicity, to avoid having to write a sum over various classes of diagrams with different connectivity structures, let us assume that we are interested in a region of momenta in which only those core diagrams of one particular class contribute significantly. Then the indices $(1\cdots n)$ may be partitioned into N sets A_α such that all the lines in one set are attached to one connected piece of the core diagram. Let us suppose that r of these sets consist of two indices only, corresponding to straight-through lines, and choose these to be $(1,r+1)\cdots(r,2r)$. The total contribution to $G^h(p_1\cdots p_n)$ will contain r factors of the form (3.7) together with $(N-r)$ factors (3.11), one for each connected piece of the diagrams.

To express this contribution in a convenient form, we define the function

$$\Delta^0(p_1\cdots p_n;q_1\cdots q_n)=\prod_{j=1}^{n}\Delta^0{}_j(p_j,q_j)\,, \tag{3.12}$$

where $\Delta^0{}_j$ is the free-particle propagator for a scalar particle of mass m_j, namely,

$$\Delta^0{}_j(p_j,q_j)=-i(2\pi)^4\delta(p_j-q_j)/(m_j^2+p_j^2-i\epsilon)\,. \tag{3.13}$$

Then near the mass shells we may write

$$\begin{aligned}G^h(p_1\cdots p_n)=&\prod_{j=1}^{r}\{Z^h{}_j(p_j)\Lambda_j(p_j)C_{j,j+r}\}\\&\times\prod_{j=2r+1}^{n}\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\}\int\frac{dq_{2r+1}}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\\&\times\Delta^0(p_1\cdots p_rp_{2r+1}\cdots p_n;-p_{r+1}\cdots-p_{2r}q_{2r+1}\cdots q_n)\\&\times\prod_{\alpha=r+1}^{N}(2\pi)^4\delta(\sum_{j\in A_\alpha}q_j)M^h{}_\alpha(\{q_j|j\in A_\alpha\})\,.\end{aligned} \tag{3.14}$$

The purpose of writing G^h in this form is that its mass-shell singularities are now contained entirely in the function Δ^0. We shall be able to show that the effect of introducing soft-photon contributions is simply to modify this function.

Insertion of Soft-Photon Parts

We now turn to the function $G^h(p_1\cdots p_n|A)$ obtained by inserting interactions with the external soft-photon field $A_\mu(x)$ into the core diagrams.

Let us consider the effect of inserting a vertex at which the external field appears, with small momentum transfer k, into a charged-particle line of momentum p and charge e. As has been shown in detail by Weinberg,[5] the leading contribution in powers of k (which is all we need consider) is independent of the spin of the charged particle. The effect is to replace the denominator of the propagation function $(m^2+p^2-i\epsilon)^{-1}$ by

$$(m^2+p^2-i\epsilon)^{-1}2ep\cdot A(k)(m^2+p^2-2p\cdot k-i\epsilon)^{-1}\,, \tag{3.15}$$

together with additional spin-dependent terms which are less singular at $k=0$ in the limit as p^2 approaches $-m^2$. The term k^2 in the second denominator is negligible, and has been omitted.

Because of this spin independence, the entire effect of introducing all such insertions into the external lines of the core diagrams may be expressed as a modifi-

[5] S. Weinberg, Phys. Rev. **140**, B516 (1965).

fication of the function Δ^0 appearing in (3.14). Thus $G^h(p_1\cdots p_n|A)$ is given by this same formula (3.14) but with $\Delta^0(p_1\cdots p_n;q_1\cdots q_n)$ replaced by

$$\Delta^0(p_1\cdots p_n;q_1\cdots q_n|A)=\prod_{j=1}^{n}\Delta^0{}_j(p_j,q_j|A),\quad(3.16)$$

where $\Delta^0{}_j(p,q|A)$ is the propagator for a scalar particle of mass m_j in a soft-photon external field $A_\mu(x)$. This function is obtained from (3.13) by adding the contributions of all insertions of the form (3.15). Explicitly, it is given by

$$\Delta^0(p,q|A)=-i\int dy\frac{e^{-i(p+q)\cdot y}}{m^2+p^2-i\epsilon}\sum_{n=0}^{\infty}\int\frac{dk_1}{(2\pi)^4}\cdots\frac{dk_n}{(2\pi)^4}\times\prod_{r=1}^{n}\frac{2ep\cdot A(k_r)\exp(ik_r\cdot y)}{m^2+p^2-2p\cdot k_1-\cdots-2p\cdot k_r-i\epsilon}.$$

To find a more convenient expression for it we introduce the representation

$$\frac{1}{m^2+p^2-i\epsilon}=i\int_0^\infty d\tau\, e^{-i\tau(m^2+p^2)}$$

for each propagator. Then, introducing the new variables

$$\sigma_r=\tau_r+\tau_{r+1}+\cdots+\tau_n,$$

we obtain

$$\Delta^0(p,q|A)=\int dy\, e^{-i(p-q)\cdot y}\sum_{n=0}^{\infty}i^n\int\frac{dk_1}{(2\pi)^4}\cdots\frac{dk_n}{(2\pi)^4}\times\int_0^\infty d\sigma_0\int_0^{\sigma_0}d\sigma_1\cdots\int_0^{\sigma_{n-1}}d\sigma_n\times\prod_{r=1}^{n}[2ep\cdot A(k_r)\exp(ik_r\cdot y)]\times\exp[-i\sigma_0(m^2+p^2)+2i\sum_{r=1}^{n}p\cdot k_r\sigma_r].$$

Symmetrizing in the n variables $\sigma_i\cdots\sigma_n$, we can sum the series to an exponential form, and perform the integration over σ in the exponent. Hence, writing σ for σ_0, we obtain finally

$$\Delta^0(p,q|A)=\int dy\, e^{-i(p-q)\cdot y}\int_0^\infty d\sigma\, e^{-i\sigma(m^2+p^2)}\times\exp\left[\int\frac{dk}{(2\pi)^4}\frac{ep\cdot A(k)}{p\cdot k}e^{ik\cdot y}(e^{2i\sigma p\cdot k}-1)\right].\quad(3.17)$$

It is interesting to note that this expression can also be written in the form

$$\Delta^0(p,q|A)=\int dy\, e^{-i(p-q)\cdot y}\int_0^\infty d\sigma\, e^{-i\sigma(m^2+p^2)}\times\exp\left[i\int_0^\sigma d\sigma' 2ep\cdot A(y+2p\sigma')\right],\quad(3.18)$$

in which the exponent may be interpreted as the line integral of A_μ along the trajectory of the particle from proper time zero to $2m\sigma$.

Since the field $A_\mu(k)$ appears only in the exponent of (3.17) or (3.18), we can immediately perform the functional differentiations in (3.6). Thus we see that $G(p_1\cdots p_n|J)$ again has the structure (3.14) but with $\Delta^0(p_1\cdots p_n;q_1\cdots q_n)$ replaced by a function $\Delta^s(p_1\cdots p_n;q_1\cdots q_n|J)$ which includes soft-photon internal lines. This function may be written in the form

$$\Delta^s(p_1\cdots p_n;q_1\cdots q_n|J)=\int dy_1\cdots dy_n\int_0^\infty d\sigma_1\cdots d\sigma_n\times\exp[-i\sum_{j=1}^{n}(p_j-q_j)\cdot y_j-i\sum_{j=1}^{n}\sigma_j(m_j^2+p_j^2)]\times\exp\left(\tfrac{1}{2}i\int_{\Omega^s}\frac{dk}{(2\pi)^4}I^\mu(k)^*\frac{\gamma_{\mu\nu}(k)}{k^2-i\epsilon}I^\nu(k)\right),\quad(3.19)$$

where

$$I^\mu(k)=J^\mu(k)+i\sum_{j=1}^{n}\frac{e_jp_j^\mu}{p_j\cdot k}\exp(-ik\cdot y_j)\times[\exp(-2i\sigma_jp_j\cdot k)-1].\quad(3.20)$$

We note that the real part of the integral in the exponent of (3.19) comes from the $\delta(k^2)$ term,

$$-\tfrac{1}{2}\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}I^\mu(k)^*\gamma_{\mu\nu}(k)I^\nu(k),$$

which is always negative in a physical gauge.

The structure of the Green's function is illustrated in Fig. 1.

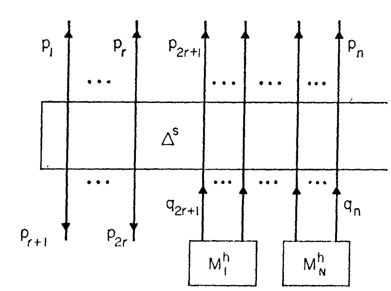

FIG. 1. Structure of the n-point Green's function: $M^h{}_\alpha$ are the connected pieces of the nonsoft-photon (core) diagrams, and Δ^s represents the soft-photon corrections to the external lines.

4. MASS-SHELL SINGULARITIES

We now wish to examine the nature of the mass-shell singularities implied by the structure obtained above for the special case in which the soft-photon external current J is set equal to zero. (A similar analysis has been given by Hagen[6] for the special case of the propagator function.) For simplicity, we shall assume that we are interested in a region of momenta in which only completely connected core diagrams can contribute significantly, so that we may set $r=0$ and $N=1$ in (3.14).

The function Δ^s given by (3.19) has a singularity when $p_j=q_j$, which is a modified form of the δ-function singularity in (3.12). The nature of this singularity is governed by the asymptotic behavior of the integrand for large y_j. The function falls off rapidly with increasing values of p_j-q_j, corresponding physically to the fact that the soft photons cannot transfer large amounts of momentum. Thus p_j-q_j may be treated as small. Now it is reasonable to suppose that the function $M^h(q_1\cdots q_n)$ is a slowly varying function of the momenta. Therefore, it is legitimate to replace it by $M^h(p_1\cdots p_n)$ and bring it outside the integral. Moreover, for $J=0$, the function Δ^s contains an over-all energy-momentum-conserving δ function $\delta(\sum p_j-\sum q_j)$, so that we may replace $\delta(\sum q_j)$ by $\delta(\sum p_j)$, and obtain

$$G(p_1\cdots p_n)=\prod_{j=1}^{n}\{[Z^h{}_j(p_j)]^{1/2}\Lambda_j(p_j)\}(2\pi)^4 \times\delta(p_1+\cdots+p_n)M^h(p_1\cdots p_n)\Gamma^s(p_1\cdots p_n)\,, \quad (4.1)$$

where

$$\Gamma^s(p_1\cdots p_n)=\int\frac{dq_1}{(2\pi)^4}\cdots\frac{dq_n}{(2\pi)^4}\Delta^s(p_1\cdots p_n;q_1\cdots q_n|0)\,. \quad (4.2)$$

Integrating over q_j in (3.19) has the effect of setting each $y_j=0$. Thus we obtain

$$\Gamma^s(p_1\cdots p_n)=\int_0^\infty d\sigma_1\cdots d\sigma_n\exp[-i\sum_{j=1}^{n}\sigma_j(m_j^2+p_j^2)] \times\exp\{\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}X_{ij}(\sigma_i,\sigma_j)\}\,, \quad (4.3)$$

where

$$X_{ij}(\sigma_i,\sigma_j)=ie_ie_j\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{k^2-i\epsilon} \times\left(\frac{\exp(2i\sigma_ip_i\cdot k)-1}{p_i\cdot k}\right)\left(\frac{\exp(-2i\sigma_jp_j\cdot k)-1}{p_j\cdot k}\right). \quad (4.4)$$

The function Γ^s given by (4.3) has a singularity in each momentum variable on the mass shell $p_j^2=-m_j^2$, which is governed by the asymptotic behavior of the integrand for large σ_j. It is the nature of these singularities which we wish to investigate in this section, since the remaining factors in (4.1) are nonsingular.

It is useful to note first certain symmetry properties of X_{ij}. By making the substitution $k\rightarrow -k$ and using (2.3) we see that $X_{ij}=X_{ji}$, and also that it is unaltered by simultaneously changing the signs of both p_i and p_j.

In the familiar case of a theory without massless particles, the mass-shell singularities of different external lines are essentially unrelated, and it makes no difference whether we consider them sequentially or simultaneously. This is far from being the case in the present theory, however, and we have to decide on the order in which to take the various limits involved. It will be most convenient to consider the limits sequentially, first allowing p_1^2 to approach $-m_1^2$, then p_2^2 to approach $-m_2^2$, and so on. Consequently, we need to study the asymptotic behavior of X_{ij} as first one and then the other of the variables σ_i and σ_j is allowed to approach infinity.

It is not hard to see that, for large values of σ_i and σ_j, X_{ij} behaves qualitatively like the function $\ln[K\sigma_i\sigma_j/(\sigma_i+\sigma_j)]$. That is to say, if we let either variable become infinite, σ_j say, the function remains finite in the limit. Then as $\sigma_i\rightarrow\infty$ it behaves logarithmically. (On the other hand, if we allow both variables to approach infinity together, keeping their ratio fixed, it also behaves logarithmically.)

Let us suppose that $j<i$, and consider first the limit $\sigma_j\rightarrow\infty$. (We treat the special case $i=j$ separately later.) In this limit X_{ij} remains finite, and takes the form

$$X_{ij}(\sigma_i,\infty)=-ie_ie_j\int_{\Omega^s}\frac{dk}{(2\pi)^4}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{k^2-i\epsilon} \times\left(\frac{\exp(2i\sigma_ip_i\cdot k)-1}{p_i\cdot k}\right)\frac{1}{p_j\cdot k-i\epsilon}. \quad (4.5)$$

Since X_{ij} is unchanged by the simultaneous change of sign of both p_i and p_j, there is no essential loss of generality in supposing that $p_i^0>0$. Then using the fact that the limit K^0 of the k^0 integration is large compared to K, we may complete the contour of the k^0 integral in the lower half k^0 plane, and evaluate it by contour integration.

In general, this process yields two terms, one from the pole at $k^0=|\mathbf{k}|$ and the other from $p_j\cdot k=0$. The latter contribution appears only when $p_j^0>0$, since otherwise this pole lies above the real axis and is excluded from the contour. Thus we can write

$$X_{ij}(\sigma_i,\infty)=X_{ij}{}^{(1)}(\sigma_i,\infty)+\theta(p_i^0p_j^0)X_{ij}{}^{(2)}(\sigma_i,\infty)\,, \quad (4.6)$$

where

$$X_{ij}{}^{(1)}(\sigma_i,\infty)=e_ie_j\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{(p_i\cdot k)(p_j\cdot k)} \times[\exp(2i\sigma_ip_i\cdot k)-1] \quad (4.7)$$

[6] C. R. Hagen, Phys. Rev. 130, 813 (1963).

and

$$X_{ij}^{(2)}(\sigma_i,\infty)=e_ie_j\int_{\Omega'}\frac{dk}{(2\pi)^4}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_j{}^\nu}{k^2}\times\left(\frac{\exp(2i\sigma_ip_i\cdot k)-1}{p_i\cdot k}\right)2\pi\delta(p_j\cdot k).\quad(4.8)$$

(We may drop the small imaginary part in the denominator k^2, since k is always spacelike in the region where the argument of the δ function vanishes.)

Let us examine first the function $X_{ij}^{(1)}$. It is easily seen to be precisely the function $L_{ij}(x)$ defined in (2.9) with $x^\mu=2p_i{}^\mu\sigma_i$. Hence, its asymptotic behavior is given by (2.16), or

$$X_{ij}^{(1)}(\sigma_i,\infty)\approx\xi_{ij}\ln(Km_i\sigma_i)+\text{constant},\quad(4.9)$$

where ξ_{ij} is the parameter defined in (2.17) or (2.18) as a function of the velocities.

Next we consider the function $X_{ij}^{(2)}$. By making the transformation $k\to-k$ in (4.8), we see that it is purely imaginary. It is also easy to verify that, like $X_{ij}^{(1)}$, it behaves logarithmically as $\sigma_i\to\infty$. Moreover, unlike ξ_{ij}, the coefficient of $\ln\sigma_i$ in $X_{ij}^{(2)}$ is independent of the choice of gauge, because the dominant contribution comes from the region near $p_i\cdot k=0$, so that the gauge-dependent terms proportional to $p_i\cdot k$ or $p_j\cdot k$ can be dropped. (Such terms are finite in the limit $\sigma_i\to\infty$.) Explicit calculation shows that

$$X_{ij}^{(2)}(\sigma_i,\infty)\approx-i\eta_{ij}\ln(Km_i\sigma_i)+\text{constant},\quad(4.10)$$

where η_{ij} is given in terms of the relative velocity u_{ij} by

$$\eta_{ij}=e_ie_j/4\pi u_{ij}.\quad(4.11)$$

Note that, unlike ξ_{ij}, this coefficient diverges in the limit of zero relative velocity. This suggests that the asymptotic behavior of the imaginary part for $i=j$ will be different, and we shall see later that this is indeed the case. The term $X_{ij}^{(2)}$ is in fact closely related to the formally divergent "Coulomb phase" which appears in the nonrelativistic Coulomb scattering amplitude, and which describes the asymptotic distortion of the Coulomb wave functions.

Combining both terms in (4.6), we obtain the asymptotic behavior

$$X_{ij}(\sigma_i,\infty)\approx\zeta_{ij}\ln(Km_i\sigma_i)+\text{constant},\quad(4.12)$$

where the complex coefficient ζ_{ij} is given by

$$\zeta_{ij}=\xi_{ij}-i\eta_{ij}\theta(p_i{}^0p_j{}^0).\quad(4.13)$$

For example, in the Lorentz gauge, using (2.15) and (2.17), we have

$$\zeta_{ij}{}^L=\xi_{ij}{}^L-i\eta_{ij}\theta(p_i{}^0p_j{}^0)=\frac{e_ie_j}{4\pi^2u_{ij}}\left(\tfrac{1}{2}\ln\frac{1+u_{ij}}{1-u_{ij}}-i\pi\theta(p_i{}^0p_j{}^0)\right),\quad(4.14)$$

while in the radiation gauge we have the extra terms given by (2.18).

It is interesting to note that (4.14) can be written in a form in which the real and imaginary parts appear naturally together. Let us define

$$\chi_{ij}=p_i\cdot p_j/m_im_j\quad(4.15)$$

and note that $p_i{}^0p_j{}^0>0$ means $\chi_{ij}<0$. We introduce the function $\Psi(\chi)$ defined by

$$\Psi(\chi)=[\chi/(\chi^2-1)^{1/2}]\ln[\chi+(\chi^2-1)^{1/2}],\quad(4.16)$$

together with the stipulation that the principal branch of the logarithm is to be chosen for $\chi>1$. Then it is easy to verify that (4.14) may be written in the form

$$\zeta_{ij}{}^L=(e_ie_j/4\pi^2)\Psi(\chi_{ij}-i\epsilon).\quad(4.17)$$

Now let us return to the special case $i=j$ that was excluded from the above discussion. This case must be treated separately because we cannot any longer take the limits $\sigma_j\to\infty$ and $\sigma_i\to\infty$ sequentially. If we distort the k^0 contour to pass above or below the points where $p_i\cdot k=0$, then we can separate the various terms in (4.4). Changing the sign of k in two of the four terms, to obtain a form similar to (4.5), we find

$$X_{ii}(\sigma_i)=-ie_i^2\int_{\Omega'}\frac{dk}{(2\pi)^4}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_i{}^\nu}{k^2-i\epsilon}[\exp(2i\sigma_ip_i\cdot k)-1]\times\left(\frac{1}{(p_i\cdot k+i\epsilon)^2}+\frac{1}{(p_i\cdot k-i\epsilon)^2}\right).\quad(4.18)$$

We can now again complete the k^0 contour in the lower half k^0 plane and obtain two terms. The term which arises from the pole at $k^0=|\mathbf{k}|$ is similar to (4.7) but with an extra factor of 2, whose origin may be traced to the fact that there are now two nonoscillatory terms in (4.4) instead of just one. The second term, which arises from the pole at $p_i\cdot k=0$, is similar to (4.8), but instead of being logarithmic in σ_i, it is actually linear. It is given correctly by letting p_j approach p_i in (4.8), and is

$$X_{ii}^{(2)}(\sigma_i)=2ie_i^2\sigma_i\int_{\Omega'}\frac{dk}{(2\pi)^4}\frac{p_i{}^\mu\gamma_{\mu\nu}(k)p_i{}^\nu}{k^2}2\pi\delta(p_i\cdot k)=-i\frac{e_i^2m_i^2K\sigma_i}{\pi^2p_i{}^0}\Phi(u_i).\quad(4.19)$$

Formally, this term represents the soft-photon contribution to the mass renormalization, as may be seen from the fact that its effect would be to shift the position of the singularity in p_i^2 away from $-m_i^2$. However, it is in any case of order K and therefore negligible, as it must be since the mass renormalization constant has no infrared-divergent part.

Because this imaginary term is negligible, we are left only with the term $X_{ii}^{(1)}$, so that

$$X_{ii}(\sigma_i)=2e_i^2\int_{\Omega^s}\frac{d\mathbf{k}}{(2\pi)^3 2k^0}\frac{p_i^\mu\gamma_{\mu\nu}(k)p_i^\nu}{(p_i\cdot k)^2}\times[\exp(2i\sigma_i p_i\cdot k)-1], \quad (4.20)$$

which agrees with (4.7) except for the additional factor of 2. Its asymptotic behavior is therefore again given by (4.9):

$$X_{ii}(\sigma_i)\approx 2\xi_{ii}\ln(Km_i\sigma_i)+\text{constant}. \quad (4.21)$$

This factor of 2 may be rather surprising, but as we shall see in subsequent papers, it plays an essential role in maintaining the consistency of the theory.

We may now return to the formula (4.3) and determine the nature of the singularities in the function Γ^s, and therefore in the Green's functions.

Let us first consider the behavior as p_1^2 approaches $-m_1^2$. The nature of this singularity is governed by the asymptotic behavior of the integrand as $\sigma_1\rightarrow\infty$ with all other σ_i held fixed. The only term in the exponent which does not remain finite in this limit is $\frac{1}{2}X_{11}$, whose asymptotic behavior is described by (4.21). Thus, omitting factors which remain finite as $\sigma_1\rightarrow\infty$, the integral we have to consider is

$$\int_0^\infty d\sigma_1\exp[-i\sigma_1(m_1^2+p_1^2)](Km_1\sigma_1)^{\xi_{11}}=\frac{\xi_{11}!}{m_1K}\left(\frac{-im_1K}{m_1^2+p_1^2-i\epsilon}\right)^{1+\xi_{11}}.$$

Hence we find that, as $p_1^2\rightarrow -m_1^2$, Γ^s behaves like

$$\Gamma^s(p_1\cdots p_n)\approx -i\Gamma^s{}_{(1)}(p_1\cdots p_n)\times\frac{\xi_{11}!}{m_1^2+p_1^2-i\epsilon}\left(\frac{-im_1K}{m_1^2+p_1^2-i\epsilon}\right)^{\xi_{11}}, \quad (4.22)$$

where $\Gamma^s{}_{(1)}$ is nonsingular at $p_1^2=-m_1^2$. Note that in a physical gauge ξ_{11} is always negative [for example, in the radiation gauge, it is given by (2.22)], so that the Green's functions are *less* singular than in a theory without massless particles, as we should expect physically.

Next let us examine the behavior of $\Gamma^s{}_{(1)}$ as $p_2^2\rightarrow -m_2^2$. This is governed by the asymptotic behavior of the integrand as $\sigma_2\rightarrow\infty$ with $\sigma_3\cdots\sigma_n$ held finite. The limit $\sigma_1\rightarrow\infty$ has already been taken, so that we now have two logarithmic terms in the exponent, $\frac{1}{2}X_{22}$ and X_{21}. Thus the integral we have to consider is now

$$\int_0^\infty d\sigma_2\exp[-i\sigma_2(p_2^2+m_2^2)](Km_2\sigma_2)^{\xi_{22}+\zeta_{21}}.$$

Hence we find that the behavior of $\Gamma^s{}_{(1)}$ as $p_2^2\rightarrow -m_2^2$ is of the form

$$\Gamma^s{}_{(1)}(p_1\cdots p_n)\approx -i\Gamma^s{}_{(12)}(p_1\cdots p_n)\times\frac{(\xi_{22}+\zeta_{21})!}{m_2^2+p_2^2-i\epsilon}\left(\frac{-im_2K}{m_2^2+p_2^2-i\epsilon}\right)^{\xi_{22}+\zeta_{21}}, \quad (4.23)$$

where again $\Gamma^s{}_{(12)}$ is nonsingular at $p_2^2=-m_2^2$.

We may continue in this way. In general, the exponent which appears in the asymptotic behavior of $\Gamma^s{}_{(1\cdots j-1)}$ as $p_j^2\rightarrow -m_j^2$ is $\xi_{jj}+\zeta_{jj-1}+\cdots+\zeta_{j1}$.

Finally, for future reference, we note one special case. In the radiation gauge the parameter ξ_{11}^R vanishes when $\mathbf{p}_1=\mathbf{0}$. Thus, when the spatial momentum $\mathbf{p}_1$ is zero, the Green's functions have a simple pole at $p_1^2=-m_1^2$, that is, at $p_1^0=\pm m_1$. This pole may be identified with the contribution of a true one-particle state. More generally, if $\mathbf{p}_j=\mathbf{0}$, then all the parameters ξ_{ij}^R vanish, as may be seen from (2.44). However, the imaginary parts η_{ij} do not vanish. In that case, the singularity of the Green's functions is of the form $(m_j^2+p_j^2-i\epsilon)^{-1+i\eta}$. Physically, this corresponds to the fact that although the real soft photons disappear for $\mathbf{p}=\mathbf{0}$, the Coulomb phases remain.

5. CONCLUSIONS

We have shown that the soft-photon contribution to an arbitrary Green's function may be isolated in a single function Δ^s, given by (3.19). This function replaces the function Δ^0, of (3.17), which appears in the expression (3.14) for the Green's function G^h with soft-photon contributions removed. The mass-shell singularities of the Green's functions are governed by the behavior of this function Δ^s. They were shown to be branch points with the structure described by (4.22) and (4.23).

In the following paper, we shall investigate the nature of the asymptotic states implied by this singularity structure. Instead of choosing the space of asymptotic states *a priori*, we shall allow the theory itself to determine them.

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