Soft Photons and the Classical Limit

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The first two orders of the low-frequency limit of a bremsstrahlung amplitude are determined by the general principles of quantum field theory. The leading order is in obvious correspondence with the classical limit. We present a conceptual experiment which shows that the next-order term is also determined by the classical limit. In addition, a generating functional is constructed that provides the two leading terms of the bremsstrahlung amplitude for the emission and absorption of an arbitrary number of soft photons. This generating functional is used to indicate a classical correspondence to all powers of the electrical charge.

1. INTRODUCTION

THE leading term in the bremsstrahlung amplitude for the emission of a soft photon depends inversely on the photon frequency; it is of order ω^{-1}. This term is easily computed either classically or by the methods of quantum field theory. In the classical calculation, the electromagnetic energy radiated by the over-all motion of a charged particle that undergoes an essentially instantaneous collision is computed. In the quantum-field-theory calculation, the amplitude for the emission of a soft photon in the propagation of a charged particle external to the major scattering event is computed; in this order, the emission vertex of the soft photon depends only upon the electrical charge.[1] Low[2] has shown that the basic postulates of quantum field theory, particularly Lorentz covariance, gauge invariance, and simple analytic properties, enable one to compute not only the leading contribution to the soft-photon emission amplitude, but also the next-order term. This term, of order $\hbar\omega^0$, involves only the intrinsic properties of the charged particle, its electric charge and magnetic moment, and the mass-shell value of the elastic scattering amplitude. It is our purpose to demonstrate the connection between Low's theorem and the classical limit, and thereby relate it to the correspondence principle.[2a]

There is an apparent difficulty with this program: The leading term already gives the classical cross section for low-frequency radiation. The next term is of order $\hbar$ and would not be expected to be related to a classical limit. The resolution of this difficulty is to consider a different classical experiment, one in which the leading terms cancel and the next-order terms correspond to the classical result. Such a conceptual experiment is provided by the scattering of a charged particle on a neutral target in the presence of an external, classical radiation field.[3] If, for example, the average energy supplied to the particles is computed, then, since in lowest order stimulated emission[4] is as likely as absorption, the leading terms cancel and the next-order terms give precisely the classical power gain.

Since we are concerned primarily with the physical basis of Low's theorem, we shall restrict the discussion to the radiation emitted in the otherwise elastic scattering of two spinless particles only one of which is charged. We review Low's theorem in Sec. 2, using an on-mass-shell technique and a variable choice that are suggested by the classical correspondence. This method simplifies the proof of the theorem and is easily extended to the case of the two-photon amplitude. The low-frequency behavior of this amplitude of orders $(\omega\omega')^{-1}$, ω^{-1}, and ω'^{-1} can, as in the single-photon case, be expressed in terms of the mass-shell value of the elastic-scattering amplitude. Furthermore, those terms of order unity can also be computed in terms of the mass-shell elastic amplitude and the structure-dependent single-photon amplitude. We shall find that these terms of order unity cancel, in the classical limit, the structure-dependent terms in the single-photon-emission cross section.[5]

We begin our illustration of the classical limit by briefly reviewing, in Sec. 3, the well-known low-frequency limit of the ordinary bremsstrahlung cross

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[1] S. Weinberg [Phys. Rev. **135**, B1049 (1964)] has shown that Lorentz covariance requires that the bremsstrahlung amplitude be gauge-invariant, and that gauge invariance determines the soft-photon emission vertex uniquely as well as implying the conservation of electrical charge. (Weinberg also makes the important observation that Lorentz covariance similarly determines the structure of the zero-frequency gravitational coupling and requires the equivalence of gravitational and inertial mass.) A lucid discussion of these ideas is contained in the lecture by T. W. B. Kibble, in *High-Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965).

[2] F. E. Low, Phys. Rev. **110**, 974 (1958).

[2a] The relationship between the low-frequency theorem and the classical limit in Compton scattering has been discussed by M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

[3] This thought experiment is not entirely unrelated to the heating of a plasma by a radiation field. See, e.g., the review by S. C. Brown, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 22.

[4] We may, of course, neglect the spontaneous emission and attendant radiative reaction (virtual-photon corrections), for these processes are independent of the strength of the radiation field.

[5] A similar cancellation of structure-dependent terms between the real-photon-emission cross section and the virtual-photon radiative correction to the elastic-scattering cross section occurs in the infrared corrections to scattering processes. Thus these corrections can be computed to a higher order in the photon frequency than has been previously realized.

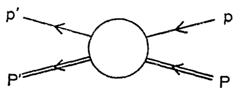

FIG. 1. Pictorial representation of the elastic-scattering process.

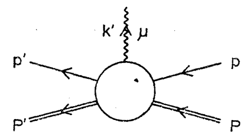

FIG. 2. Pictorial representation of the scattering process in which one photon is emitted or absorbed.

section. The quantum-mechanical description, to second order in the electric charge, of a scattering process occurring in the presence of a radiation field is presented in Sec. 4. In the low-frequency limit, all terms in the cross section of order $(\hbar\omega)^{-2}$ and $(\hbar\omega)^{-1}$ cancel, and the sum of the remaining terms does not depend upon the phase of the elastic-scattering amplitude. We show in Sec. 5 that the classical limit of the cross section may be computed in the approximation that the collision occurs instantaneously and at a point in space. The agreement between the two calculations shows the role that low-frequency theorems of quantum field theory play in the classical limit. The correspondence of the classical and quantum-mechanical expressions for the rate at which energy is transferred from the radiation field to the motion of the particles is a simple byproduct of the general discussion.

A generating functional is constructed for the collision amplitude involving the emission and absorption of an arbitrary number of soft photons in Sec. 6. It describes correctly the first two leading orders of the multiple soft-photon amplitudes. This generating functional is used in Sec. 7 to obtain heuristically the classical limit of the scattering cross section in the radiation field to all powers of the electrical charge.

2. LOW-FREQUENCY THEOREMS

We consider now the structure of the amplitudes describing the one- and two-photon radiation that accompanies an otherwise elastic collision of two spinless particles. The scattering process in the absence of radiation is depicted in Fig. 1: A spinless charged particle with initial four-momentum p and mass m scatters elastically on a neutral spinless target with initial four-momentum P and mass M. The final momenta of these particles are p' and P', respectively, with

$$p'+P'=p+P. \tag{1}$$

We denote the scattering amplitude for this elastic process by $T(\nu,t)$, where we have chosen a particular set of variables that will later prove convenient, namely, the variable ν which is proportional to the laboratory energy,[6]

$$\nu=-pP, \tag{2}$$

and t, the square of the four-momentum transfer,

$$t=-(P'-P)^2. \tag{3}$$

The process in which one photon is emitted or absorbed during the scattering is shown in Fig. 2, where the particle momenta are labeled as before, and the photon carries off four-momentum k', with

$$k'+p'+P'=p+P. \tag{4}$$

We consider the general case in which the photon may be virtual ($k^2\neq 0$), for such photons can be provided by external fields and, furthermore, virtual-photon amplitudes enter in the calculation of radiative corrections to scattering processes. The leading low-frequency terms in the amplitude T^μ for this process are those associated with graphs in which the photon is attached to an external particle line; these pole terms, displayed in Fig. 3, are of order ω^{-1}, where $\omega=k^0$ is the photon frequency. We define these pole terms in the dispersion-theory sense, which is to say that the residue of the pole, the elastic-scattering amplitude, is always kept at its physical, mass-shell value. This is a natural definition, since the low-frequency behavior is related to the classical limit in which no off-mass-shell processes occur. Indeed, if the pole terms were evaluated using the unphysical, off-mass-shell scattering amplitude, additional, nonsingular terms would appear in the low-frequency theorem that precisely cancel the off-mass-shell extrapolation.[7] Thus we define the pole terms as

$$\mathcal{P}^\mu=F(k'^2)\left(\frac{(2p'+k')^\mu}{2p'k'+k'^2}T(\nu,t)\right.$$
$$\left.+T(\nu+k'P,t)\frac{(2p-k')^\mu}{-2pk'+k'^2}\right), \tag{5}$$

where $T(\nu,t)$ is on the mass shell, $F(k'^2)$ is the mass-shell

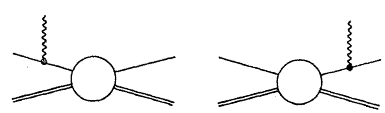

FIG. 3. Graphs representing the pole terms in scattering with one photon emitted or absorbed.

[6] We use a metric such that $pP\equiv p^\mu P_\mu\equiv \mathbf{p}\cdot\mathbf{P}-p^0P^0$ and employ, for the most part, natural units, $\hbar=c=1$. We shall display Planck's constant $\hbar$ explicitly in later sections when we make contact with the classical limit.

[7] The freedom in the definition of the pole term and the utility of its dispersion-theory definition were apparently not recognized by Low (Ref. 2) nor by subsequent authors: S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966); J. Pestieau, *ibid.* **160**, 1555 (1967); T. H. Burnett and N. M. Kroll, Phys. Rev. Letters **20**, 86 (1968). Burnett and Kroll have shown that the first two terms in the soft-photon bremsstrahlung cross section for the scattering of an unpolarized spin-$\frac{1}{2}$ charged particle are independent of its magnetic moment and depend only upon the unpolarized elastic cross section. This result, that the soft-photon-emission cross section does not involve phase information on the elastic amplitude but depends only upon the cross section, appears natural in view of the connection of the soft-photon theorem with the classical limit.

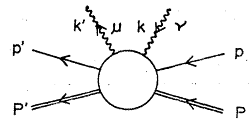

FIG. 4. Pictorial representation of the scattering process in which two photons are emitted or absorbed.

form factor of the charged particle, ν is the laboratory energy of the initial charged particle, Eq. (2), and t is the square of the four-momentum transfer of the neutral target particle, Eq. (3). We have used the momentum transfer of the neutral target, which is independent of the photon momentum, as a variable, rather than that of the charged particle which depends upon the photon momentum and differs for the two terms of Eq. (5). Furthermore, we have employed the laboratory-energy variable $\nu=-pP$ rather than the center-of-mass energy $s=-(p+P)^2$, for the former depends linearly upon the charged-particle momentum, while the latter involves p^2, which can go off the mass shell. Such off-mass-shell terms are essentially spurious, since they do not occur in the classical limit, where all particles remain on the mass shell.[8] Any set of invariant variables could be used, of course; however, the set that we employ is a natural one in view of the classical limit and enables the photon-emission amplitude to be written in a particularly simple form.

Gauge invariance requires that the single-photon amplitude be divergence-free, or

$$k_\mu' T^\mu = 0. \tag{6}$$

The pole contribution alone is not gauge-invariant, for

$$k_\mu' \mathcal{P}^\mu = F(k'^2)[T(\nu,t)-T(\nu+k'P,t)]. \tag{7}$$

However, we can easily construct a nonsingular quantity $\mathcal{S}^\mu$ which compensates for this lack of gauge invariance,

$$\mathcal{S}^\mu = P^\mu (k'P)^{-1} F(k'^2)[T(\nu+k'P,t)-T(\nu,t)], \tag{8}$$

for

$$k_\mu'(\mathcal{P}^\mu+\mathcal{S}^\mu)=0. \tag{9}$$

Thus we may write the entire one-photon amplitude as

$$T^\mu = \mathcal{P}^\mu + \mathcal{S}^\mu + \mathcal{R}^\mu, \tag{10}$$

with the remainder $\mathcal{R}^\mu$ nonsingular and gauge-invariant,

$$k_\mu' \mathcal{R}^\mu = 0. \tag{11}$$

If we differentiate this equation with respect to k' and then set $k'=0$, we learn that

$$\mathcal{R}^\mu(k'=0)=0. \tag{12}$$

[8] To be more explicit, we note that the emission or absorption of a photon by a particle, $p\to p-k'$, corresponds roughly to the classical oscillatory motion in an external field where the initial constant momentum is replaced with a proper-time-dependent function $p\to p(\tau)=p-f(\tau)$. Now under these replacements $\nu\to\nu+k'P$ or $\nu\to\nu+f(\tau)P$. On the other hand, since $p(\tau)^2=p^2=-m^2$ always remains on the mass shell, $s\to s+2k'P+k'(2p-k')$, while, classically, $s\to s+2f(\tau)P$. The correspondence between the two replacements $p\to p-k'$ and $p\to p-f(\tau)$ is exhibited in Secs. 4 and 5.

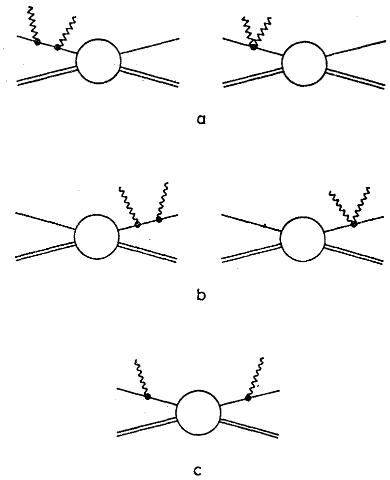

FIG. 5. Graphs representing the double-pole and "seagull" terms in scattering with the emission or absorption of two photons.

This is the content of Low's theorem: The single-photon amplitude T^μ is determined completely by the corresponding physical elastic amplitude $T(\nu,t)$ up to terms of order ω in the soft-photon limit. We note that, with our choice of invariants, the contact term $\mathcal{S}^\mu$ is proportional to the initial momentum of the target particle P^μ. Thus if the bremsstrahlung amplitude is evaluated in the laboratory-frame radiation gauge, the photon-polarization vector ϵ^μ is orthogonal to P^μ, $\epsilon_\mu P^\mu=0$, and the contact term $\mathcal{S}^\mu$ does not contribute. In this gauge, the soft-photon limit is given entirely by the dispersion-theory pole term, Eq. (5).

The method that we have used is easily extended to the case of the two-photon amplitude $T^{\mu\nu}$ shown in Fig. 4. We have chosen the photon with polarization ν and momentum k to be incoming and the photon with polarization μ and momentum k' to be outgoing, so that the energy-momentum balance reads

$$k'+p'+P'=k+p+P. \tag{13}$$

We decompose the two-photon amplitude $T^{\mu\nu}$ into a part that contains its double poles $\mathcal{P}^{\mu\nu}$, a part $\mathcal{S}^{\mu\nu}$ that combines with the double-pole term to make a gauge-invariant quantity, and the gauge-invariant remainder $\mathcal{R}^{\mu\nu}$,

$$T^{\mu\nu}=\mathcal{P}^{\mu\nu}+\mathcal{S}^{\mu\nu}+\mathcal{R}^{\mu\nu}. \tag{14}$$

The double-pole terms which give the leading contribution in the low-frequency limit of order $(\omega\omega')^{-1}$ are described by the graphs of Fig. 5. We have included with the double-pole graphs single-pole, "seagull" graphs so that the sum would be gauge-invariant if the elastic amplitude had no energy dependence. Thus we write the double-pole terms as

$$\mathcal{P}^{\mu\nu}=A^{\mu\nu}T(\nu,t)+T(\nu+k'P-kP,t)B^{\mu\nu} +C^{\mu\nu}T(\nu-kP,t)+\tilde{C}^{\mu\nu}T(\nu+k'P,t), \tag{15}$$

where the $A^{\mu\nu}$, $B^{\mu\nu}$ terms correspond to the graphs in

Figs. 5(a) and 5(b), respectively, and the $C^{\mu\nu}$, $\tilde{C}^{\mu\nu}$ terms correspond to the graph in Fig. 5(c). They are given by

$$A^{\mu\nu}=F(k'^2)F(k^2)\left(\frac{(2p'+k')^\mu(2p'+2k'-k)^\nu}{2p'k'+k'^2}+\frac{(2p'-2k+k')^\mu(2p'-k)^\nu}{-2p'k+k^2}-2g^{\mu\nu}\right)\frac{1}{2p'(k'-k)+(k'-k)^2},\tag{16a}$$

$$B^{\mu\nu}=F(k'^2)F(k^2)\left(\frac{(2p+2k-k')^\mu(2p+k)^\nu}{2pk+k^2}+\frac{(2p-k')^\mu(2p+k-2k')^\nu}{-2pk'+k'^2}-2g^{\mu\nu}\right)\frac{1}{2p(k-k')+(k-k')^2},\tag{16b}$$

$$C^{\mu\nu}=F(k'^2)F(k^2)\frac{(2p'+k')^\mu}{2p'k'+k'^2}\frac{(2p+k)^\nu}{2pk+k^2},\tag{16c}$$

and

$$\tilde{C}^{\mu\nu}=F(k'^2)F(k^2)\frac{(2p-k')^\mu}{-2pk'+k'^2}\frac{(2p'-k)^\nu}{-2p'k+k^2}.\tag{16c$'$}$$

As in the single-photon case, the pole terms are not separately gauge-invariant if the elastic amplitude is energy-dependent. It is not difficult to construct a contact term $\mathcal{S}^{\mu\nu}$ that compensates for this lack of gauge invariance. In the two-photon case, this part contains singe-pole terms, and one finds

$$\mathcal{S}^{\mu\nu}=F(k'^2)F(k^2)\Bigg[\frac{P^\mu}{k'P}\left([T(\nu+k'P,t)-T(\nu,t)]\frac{(2p'-k)^\nu}{-2p'k+k^2}+[T(\nu+k'P-kP,t)-T(\nu-kP,t)]\frac{(2p+k)^\nu}{2pk+k^2}\right)$$
$$+\left(\frac{(2p'+k')^\mu}{2p'k'+k'^2}[T(\nu,t)-T(\nu-kP,t)]+\frac{(2p-k')^\mu}{-2pk'+k'^2}[T(\nu+k'P,t)-T(\nu+k'P-kP,t)]\right)P^\nu/kP$$
$$+(P^\mu P^\nu/k'PkP)[T(\nu+k'P,t)+T(\nu-kP,t)-T(\nu,t)-T(\nu+k'P-kP,t)]\Bigg].\tag{17}$$

That this contact term can be written in a relatively simple closed form is due to our choice of ν and t as the invariant variables of the elastic amplitude. The sum of the double-pole term and this contact term are divergence-free,

$$k_\mu'(\mathcal{P}^{\mu\nu}+\mathcal{S}^{\mu\nu})=0=(\mathcal{P}^{\mu\nu}+\mathcal{S}^{\mu\nu})k_\nu,\tag{18}$$

and the gauge invariance of the complete two-photon amplitude,

$$k_\mu'T^{\mu\nu}=0=T^{\mu\nu}k_\nu,\tag{19}$$

requires that the remainder $\mathcal{R}^{\mu\nu}$ be divergence-free,

$$k_\mu'\mathcal{R}^{\mu\nu}=0=\mathcal{R}^{\mu\nu}k_\nu.\tag{20}$$

Note that $\mathcal{S}^{\mu\nu}$ always involves either a factor of P^μ or P^ν, so that it does not contribute to the double-photon bremsstrahlung cross section if the latter is calculated in the laboratory-frame radiation gauge.

Thus far, we have accounted for the double-pole terms that occur in the two-photon amplitude in a gauge-invariant manner. However, the two-photon amplitude has single-pole terms as well as double poles. For example, there is a single pole when $2pk+k^2=0$ with

$$T^{\mu\nu}\to T^\mu(2p+k)^\nu/(2pk+k^2).\tag{21a}$$

If we use the structure of the single-photon amplitude exhibited in Eq. (10) and the decomposition of the two-photon amplitude given in Eq. (14), we find that as $2pk+k^2\to 0$,

$$\mathcal{R}^{\mu\nu}\to\mathcal{R}^\mu(k')(2p+k)^\nu/(2pk+k^2).\tag{21b}$$

Here $\mathcal{R}^\mu(k')$ is the gauge-invariant remainder of the physical single-photon amplitude and is on the mass shell, but with the initial momentum p replaced with $p+k$. There is another kind of single pole which appears when, for example, $2p(k-k')+(k-k')^2\to 0$. In this limit,

$$T^{\mu\nu}\to T[2p(k-k')+(k-k')^2]^{-1}C^{\mu\nu}(k',k),\tag{22a}$$

where $C^{\mu\nu}(k',k)$ is the Compton scattering amplitude for the charged particle whose invariant amplitudes are evaluated on the mass shell. We may decompose this amplitude into pole terms, a "seagull" term, and a nonsingular remainder,

$$C^{\mu\nu}(k',k)=F(k'^2)F(k^2)\left(\frac{(2p+2k-k')^\mu(2p+k)^\nu}{2pk+k^2}+\frac{(2p-k)^\mu(2p+k-2k')^\nu}{-2pk'+k'^2}-2g^{\mu\nu}\right)+D^{\mu\nu}(k',k).\tag{22b}$$

Using this decomposition and that given in Eq. (14), we find that as $2p(k-k')+(k-k')^2\to 0$,

$$\mathcal{R}^{\mu\nu}\to T[2p(k-k')+(k-k')^2]^{-1}D^{\mu\nu}(k',k).\tag{22c}$$

For a physical Compton scattering process in which both the initial and final charged particles are on the mass shell, $C^{\mu\nu}$ must be gauge-invariant, and in the decomposition (22b) this requires that

$$k_\mu' D^{\mu\nu} = 0 = D^{\mu\nu} k_\nu . \tag{22d}$$

Since $D^{\mu\nu}$ is nonsingular, these two divergence conditions imply[9] that $D^{\mu\nu}$ is of order $\omega'\omega$ even when one of the charged-particle four-momenta is unphysical and this amplitude is no longer transverse.

We may take account of all the single-pole terms in the remainder $\mathcal{R}^{\mu\nu}$ if we write

$$\mathcal{R}^{\mu\nu} = {}^{(1)}\mathcal{R}^{\mu\nu} + {}^{(2)}\mathcal{R}^{\mu\nu} , \tag{23}$$

with

$$\begin{aligned} {}^{(1)}\mathcal{R}^{\mu\nu} = {} & \mathcal{R}^\mu(k') \frac{(2p+k)^\nu}{2pk+k^2} + \frac{(2p'-k)^\nu}{-2p'k+k^2} \mathcal{R}^\mu(k') \\ & + \frac{(2p'+k')^\mu}{2p'k'+k'^2} \mathcal{R}^\nu(-k) + \mathcal{R}^\nu(-k) \frac{(2p-k')^\mu}{-2pk'+k'^2} \\ & + T[2p(k-k')+(k-k')^2]^{-1} D^{\mu\nu}(k',k) \\ & + D^{\mu\nu}(k',k) T [2p'(k'-k)+(k'-k)^2]^{-1} , \end{aligned} \tag{24}$$

where we have suppressed the momentum variables that occur in $\mathcal{R}^\mu$, $D^{\mu\nu}$, and T other than the photon momentum. The quantity ${}^{(2)}\mathcal{R}^{\mu\nu}$ is entirely free of poles and its leading terms, of order ω, ω', are determined by the gauge invariance (20) of $\mathcal{R}^{\mu\nu}$. The two-photon amplitude $T^{\mu\nu}$ is therefore, in principle, determined to order ω, ω' in terms of the elastic, single-photon, and Compton amplitudes. We shall not, however, explicitly write this down, for we shall need the two-photon amplitude accurate only to constant terms in the soft-photon limit, and this is provided by $\mathcal{P}^{\mu\nu}$, $\mathcal{S}^{\mu\nu}$, and ${}^{(1)}\mathcal{R}^{\mu\nu}$. To this accuracy, the terms containing $D^{\mu\nu}$ do not contribute, the change of particle momenta resulting from photon emission or absorption may be neglected, and the initial and final particle momenta appearing in the $\mathcal{R}^\mu$ factors of $\mathcal{R}^{\mu\nu}$ may be taken to be p and p'. We note that the sum $\mathcal{P}^{\mu\nu}+\mathcal{S}^{\mu\nu}$ alone gives the first two leading orders, and that these are determined entirely by the physical elastic scattering amplitude.

3. BREMSSTRAHLUNG AND THE CLASSICAL LIMIT

For the sake of completeness, we shall review briefly the well-known connection between the low-frequency behavior of the quantum-bremsstrahlung cross section and the classical limit. The correspondence arises, of course, because quantum mechanically the photon frequency appears in leading order as an energy $\hbar\omega$, and taking the limit $\hbar\omega \to 0$ is tantamount to taking the limit $\hbar \to 0$. The exact single-photon bremsstrahlung cross section is given by the absolute square of the emission amplitude multiplied by the volume of the final-particle phase space and divided by the initial-particle flux:

$$\begin{aligned} d\sigma^{(\gamma)} = {} & \frac{\hbar^3 (d\mathbf{k}')}{(2\pi)^3} \frac{1}{2\hbar k'^0} \frac{(d\mathbf{p}')}{(2\pi)^3} \frac{1}{2p'^0} \frac{(d\mathbf{P}')}{(2\pi)^3} \frac{1}{2P'^0} \\ & \times (2\pi)^4 \delta(\hbar k' + p' + P' - p - P) \\ & \times (e^2/\hbar) | \epsilon_\mu T^\mu |^2 / 4(\nu^2 - m^2 M^2)^{1/2} . \end{aligned} \tag{25}$$

In the low-frequency limit, the photon energy momentum $\hbar k'$ can be neglected in the energy-momentum-conserving δ function, and the emission amplitude can be approximated by its poles $\mathcal{P}^\mu$, with $\nu + \hbar k' P$ replaced with ν. Thus in the low-frequency limit the bremsstrahlung cross section appears as a photon-emission factor multiplying the elastic-scattering cross section,

$$d\sigma^{(\gamma)} = \frac{e^2}{\hbar} \frac{(d\mathbf{k}')}{(2\pi)^3} \frac{1}{2k'^0} \left| \epsilon_\mu \left(\frac{p'}{p'k'} - \frac{p}{pk'} \right)^\mu \right|^2 d\sigma^{(\mathrm{el})} , \tag{26}$$

with

$$\begin{aligned} d\sigma^{(\mathrm{el})} = {} & \frac{(d\mathbf{p}')}{(2\pi)^3} \frac{1}{2p'^0} \frac{(d\mathbf{P}')}{(2\pi)^3} \frac{1}{2P'^0} (2\pi)^4 \delta(p'+P'-p-P) \\ & \times |T(\nu,t)|^2 / 4(\nu^2 - m^2 M^2)^{1/2} . \end{aligned} \tag{27}$$

The low-frequency limit of the analogous classical process is obtained by approximating the true motion of the charged particle by that appropriate to an instantaneous collision, for the detailed nature of the collision, which is confined in space and time, has no influence on the long-wavelength low-frequency part of the radiation spectrum. In this approximation, the collision may be considered to take place at the coordinate origin, and the velocity $v^\mu(\tau)$ and position $z^\mu(\tau)$ of the charged particle as a function of its proper time τ are given by

$$\begin{aligned} v^\mu(\tau) &= p^\mu/m , \quad \tau < 0 \\ &= p'^\mu/m , \quad \tau > 0 \end{aligned} \tag{28}$$

and

$$\begin{aligned} z^\mu(\tau) &= (p^\mu/m)\tau , \quad \tau < 0 \\ &= (p'^\mu/m)\tau , \quad \tau > 0 . \end{aligned} \tag{29}$$

The energy dE radiated into a small wave-number interval $(d\mathbf{k}')$ is related to the Fourier transform of the associated current

$$\begin{aligned} j^\mu(k') &= \int (dx)\, e^{-ik'x} e \int_{-\infty}^{\infty} d\tau\, v^\mu(\tau) \delta[x - z(\tau)] \\ &= -ie(p'/p'k' - p/pk')^\mu \end{aligned} \tag{30}$$

by

$$dE = [(d\mathbf{k}')/(2\pi)^3]\tfrac{1}{2} | \epsilon_\mu j^\mu(k') |^2 . \tag{31}$$

If this radiated energy is partitioned into parcels of size $\hbar k'^0$ and these parcels are associated with individual

[9] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).

photons, the number of photons emitted within some wave-number interval becomes

$$dN=\frac{e^2}{\hbar}\frac{(d\mathbf{k}')}{(2\pi)^3}\frac{1}{2k'^0}\left|\epsilon_\mu\left(\frac{p'}{p'k'}-\frac{p}{pk'}\right)^\mu\right|^2. \qquad (32)$$

Now the classical probability that a photon be radiated is the product of dN and the probability that a collision resulting in the final momentum p' occurs. Hence the classical bremsstrahlung cross section is given by

$$d\sigma^{(\gamma)}{}_{\mathrm{cl}}=dN d\sigma^{(\mathrm{el})}, \qquad (33)$$

which agrees precisely with the low-frequency limit of the quantum-mechanical cross section, Eq. (26).

4. QUANTUM-MECHANICAL SCATTERING IN A RADIATION FIELD

We have just seen that the low-frequency limit of the quantum-bremsstrahlung cross section, which is given entirely by the leading pole terms, corresponds to the classical limit. The terms of order $\hbar\omega^0$ vanish in the classical limit of this cross section. They can be exhibited only in processes in which the leading terms cancel. Such a cancellation should be expected to occur in the scattering of a charged particle on a neutral particle in the presence of an external, classical radiation field. Here the leading terms in the cross section are of order $(\hbar\omega)^{-2}$, and a cancellation must occur if the low-frequency limit is to be finite. We shall find that such a cancellation does indeed happen, and that this process provides a correspondence between Low's theorem and the classical limit.

We begin our discussion of this correspondence by computing the scattering cross section in the presence of an external radiation field

$$A^\mu(x)=a^\mu e^{ikx}+a^{\mu*}e^{-ikx}, \quad k^2=0, \qquad (34)$$

including terms up to second order in the electric charge. We shall impose the general gauge condition

$$k_\mu a^\mu=0 \qquad (35)$$

in order to simplify the structure of the pole terms and require, furthermore, that a^μ be in the laboratory-frame radiation gauge,

$$P_\mu a^\mu=0, \qquad (36)$$

so that the contact terms $\mathcal{S}^\mu$ and $\mathcal{S}^{\mu\nu}$ may be omitted. Before proceeding further, we must consider the structure of the two-photon double-pole terms $A^{\mu\nu}$ and $B^{\mu\nu}$. For the case of real, collinear photon momenta, $k'^2=0=k^2$, $k'k=0$, these pole terms may be written in the form (with the omission of the gauge terms k'^μ, k^ν)

$$A^{\mu\nu}=-\frac{p'^\mu}{p'k'}\frac{p'^\nu}{p'k}-\frac{g^{\mu\nu}}{p'(k'-k)} \qquad (37a)$$

and

$$B^{\mu\nu}=-\frac{p^\mu}{pk'}\frac{p^\nu}{pk}-\frac{g^{\mu\nu}}{p(k-k')}. \qquad (37b)$$

The "seagull" contributions involving $g^{\mu\nu}$ are singular in the limit $k'\to k$. This singularity disappears when we consider the monochromatic field (34) to be the limit of a wave train of finite length. The result[10,11] of this limiting procedure is that the seagull contributions cause a change in the charged-particle momenta from their values outside the field (p',p) to new values within the field $(\bar{p}',\bar{p})$, with[12]

$$\bar{p}^\mu=p^\mu-e^2a^*ak^\mu/pk \qquad (38)$$

and

$$\bar{p}'^\mu=p'^\mu-e^2a^*ak^\mu/p'k. \qquad (38')$$

This effect persists to all orders in e. Thus the monochromatic limit is achieved by omitting the singular seagull contributions and replacing the charged-particle momenta with their values inside the field. The validity of this procedure is verified in Sec. 7, where the low-frequency cross section is computed to all orders in the electric charge.

The cross section for the scattering into a small region of final-particle phase space, including the effect of this momentum shift, is given by a sum of incoherent partial cross sections corresponding to the absorption or stimulated emission of an arbitrary number of photons into the external field

$$d\sigma=\int\frac{(d\mathbf{p}')}{(2\pi)^3}\frac{1}{2p'^0}\int\frac{(d\mathbf{P}')}{(2\pi)^3}\frac{1}{2P'^0}\times\sum_{n=-\infty}^{\infty}\delta(\bar{p}'+P'-\bar{p}-P-n\hbar k)\times|T^{(n)}(\bar{p}',P';\bar{p},P)|^2/4(\nu^2-m^2M^2)^{1/2}. \qquad (39)$$

Here $T^{(n)}$ is the amplitude for scattering with the absorption of n photons if n is positive, or with the stimulated emission of $|n|$ photons if n is negative. The integration region in the final-particle phase space may be small, but it must not vanish in the limit $\hbar k\to 0$, for if the resolution becomes arbitrarily precise, the classical

[10] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

[11] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965).

[12] This momentum shift has a simple classical origin. The radiation field is assumed to be adiabatically switched on and off over a long time interval $\mathbf{A}\sim 2\,\mathrm{Re}\,\mathbf{a}e^{-i\omega t-\gamma|t|}$, $\gamma\to 0$, and, to first approximation, the velocity of the charged particle follows the vector potential $\mathbf{v}\sim-(e/m)\mathbf{A}$. There is a force $e\mathbf{v}\times\mathbf{B}$ along the direction of propagation of the radiation field, and during the switching time this force produces an impulse

$$\Delta\mathbf{p}\sim-\mathbf{k}|a|^2\frac{4e^2}{m}\int_{-\infty}^0 dt\, e^{\gamma t}\cos\omega t\sin\omega t, \quad \gamma\to 0$$
$$\sim\hat{k}(e^2|a|^2/m),$$

which is the nonrelativistic limit of Eq. (38). A complete relativistic derivation of this effect is given in Sec. 5.

limit is not obtained.[13] If we consider only terms up to second order in the electrical charge, then only the amplitudes $T^{(0)}$ and $T^{(\pm 1)}$ appear in the sum. The low-frequency limit of these amplitudes is given in Sec. 2, and, bearing in mind the previous discussion of the seagull singularities of Eqs. (37), we have, in the laboratory-frame radiation gauge,

$$T^{(0)}=T(\bar{\nu},t)-e^2\Bigg[\left(\left|\frac{p'a}{p'\hbar k}\right|^2+\left|\frac{pa}{p\hbar k}\right|^2\right)T(\nu,t) -\frac{p'a}{p'\hbar k}T(\nu+\hbar kP,t)\frac{pa^*}{p\hbar k} -\frac{p'a^*}{p'\hbar k}T(\nu-\hbar kP,t)\frac{pa}{p\hbar k} +\left(\frac{p'a}{p'\hbar k}-\frac{pa}{p\hbar k}\right)\mathfrak{R}^\mu(k)a_\mu^* +\left(\frac{pa^*}{p\hbar k}-\frac{p'a^*}{p'\hbar k}\right)\mathfrak{R}^\mu(-k)a_\mu\Bigg], \quad (40)$$

$$T^{(+1)}=e\left(T(\nu-\hbar kP,t)\frac{pa}{p\hbar k}-\frac{p'a}{p'\hbar k}T(\nu,t)+\mathfrak{R}^\mu(-k)a_\mu\right), \quad (41a)$$

and

$$T^{(-1)}=e\left(\frac{p'a^*}{p'\hbar k}T(\nu,t)-T(\nu+\hbar kP,t)\frac{pa^*}{p\hbar k}+\mathfrak{R}^\mu(k)a_\mu^*\right), \quad (41b)$$

where

$$\bar{\nu}=-\bar{p}P=\nu+e^2|a|^2kP/pk. \quad (42)$$

To obtain the classical limit of the cross section (39), we must expand the energy-momentum-conserving δ function in powers of $n\hbar k$ and combine the various incoherent squared amplitudes that multiply the same derivative of the δ function. Since the leading terms of the squared amplitudes are of order $(\hbar k)^{-2}$, we must expand the δ function to order $(n\hbar k)^2$. Thus the classical limit should be achieved by writing

$$\sum_{n=-1}^{1}\delta(\bar{p}'+P'-\bar{p}-P-n\hbar k)|T^{(n)}|^2$$
$$\cong\delta(\bar{p}'+P'-\bar{p}-P)(|T^{(0)}|^2+|T^{(+1)}|^2+|T^{(-1)}|^2) +\frac{\partial}{\partial\bar{p}^\lambda}\delta(\bar{p}'+P'-\bar{p}-P)\hbar k^\lambda(|T^{(+1)}|^2-|T^{(-1)}|^2) +\frac{1}{2}\frac{\partial}{\partial\bar{p}^\lambda}\frac{\partial}{\partial\bar{p}^\sigma}\delta(\bar{p}'+P'-\bar{p}-P)\hbar^2k^\lambda k^\sigma \times(|T^{(+1)}|^2+|T^{(-1)}|^2). \quad (43)$$

We may make use of Eqs. (40) and (41) to compute the low-frequency limits

$$(|T^{(0)}|^2+|T^{(+1)}|^2+|T^{(-1)}|^2) =|T(\bar{\nu},t)|^2+e^2|pa|^2\left(\frac{kP}{pk}\right)^2\frac{\partial^2}{\partial\nu^2}|T(\nu,t)|^2, \quad (44)$$

$$\hbar(|T^{(+1)}|^2-|T^{(-1)}|^2) =2e^2\,\mathrm{Re}\left[\left(\frac{p'a}{p'k}-\frac{pa}{pk}\right)pa^*\right]\frac{kP}{pk}\frac{\partial}{\partial\nu}|T(\nu,t)|^2, \quad (45)$$

and

$$\tfrac{1}{2}\hbar^2(|T^{(+1)}|^2+|T^{(-1)}|^2) =e^2\left|\left(\frac{p'a}{p'k}-\frac{pa}{pk}\right)\right|^2|T(\nu,t)|^2. \quad (46)$$

We note that the terms of orders $\hbar^{-2}$ and $\hbar^{-1}$ have cancelled, as they must if we are to have a classical limit. Furthermore, in the low-frequency limit, the unknown structure-dependent terms $\mathfrak{R}^\mu$ which are of order ω^0 cancel.[5] This cancellation also must occur for the validity of the classical limit, for the low-frequency limit of the classical cross section can be computed exactly. Finally, we note that the low-frequency forms depend only upon the transition probability $|T(\nu,t)|^2$ and its derivatives and require no information on the phase of the scattering amplitude which would destroy the classical correspondence.

5. CLASSICAL SCATTERING IN A RADIATION FIELD

We must now show that the low-frequency limit of the quantum-mechanical scattering cross section calculated in Sec. 4 does in fact correspond to the classical limit. The classical cross section may be computed in the idealization that the collision of the charged particle with the target occurs instantaneously, with no spatial displacement; the error resulting from this approximation is of order $\omega\Delta t$ or $|k|\Delta r$, where Δt and Δr charac-

[13] It is a general feature of the correspondence principle that the experimental resolution must remain finite as $\hbar\to 0$ if the classical limit is to be attained. For example, in the high-energy scattering of a particle of momentum p on a hard sphere of radius a, there is a diffraction peak of width $\hbar/pa$ superimposed on an isotropic background. The diffraction peak contributes an amount πa^2 to the total cross section and the isotropic background contributes a similar amount, so that the total quantum-mechanical cross section is $2\pi a^2$. On the other hand, the classical differential cross section is, of course, isotropic, and the total cross section is πa^2. This apparent violation of the correspondence principle is removed when it is realized that the classical limit is achieved only in an experiment with a finite angular resolution, and in such experiments the diffraction peak merges with the unscattered beam in the limit $\hbar/pa\to 0$, and only the classical, isotropic scattering is observed.

terize the finite temporal and spatial extent of the collision, and this error is negligible in the low-frequency limit. With this approximation, the charged particle with initial four-momentum p enters the radiation field adiabatically, and its momentum becomes a function $p(\tau)$ of its proper time τ. The particle suffers a collision at $\tau=\tau_c$ and is scattered elastically with the momentum $p(\tau_c)$ changing into $p'(\tau_c)$ with a probability given by the differential elastic-scattering cross section multiplied by the incident flux at the instant of collision. The particle then moves out of the field adiabatically with its momentum changing from $p'(\tau_c)$ to p'. The classical cross section for the over-all process is given by the rate at which particles with initial momentum p are scattered into a final momentum p' divided by the initial flux, and averaged over the phase of the radiation field. This phase averaging removes any reference to the position and proper time of a particular collision.

Our first task is to compute the classical motion of the charged particle within the radiation field. The position $z^\mu(\tau)$ and momentum $p^\mu(\tau)$ of the particle are related by

$$\frac{d}{d\tau}mz^\mu(\tau)=p^\mu(\tau)\,, \tag{47}$$

with

$$\frac{d}{d\tau}p^\mu(\tau)=(e/m)F^{\mu\nu}[z(\tau)]p_\nu(\tau)\,. \tag{48}$$

This equation of motion may be solved exactly for the case of a plane-wave field: a field of arbitrary spectral composition and polarization properties, but which is characterized by a unique propagation direction specified by the null vector n^μ:

$$n^2=0\,,\quad n^0>0\,. \tag{49}$$

In this situation

$$F_{\mu\nu}=-\frac{d}{dy}[n_\mu A_\nu(y)-n_\nu A_\mu(y)]\,, \tag{50}$$

in which A_μ is an arbitrary function of the variable

$$y=-nz \tag{51}$$

constrained by the condition

$$n^\mu A_\mu(y)=0\,. \tag{52}$$

Since we now have

$$\frac{d}{d\tau}np(\tau)=0\,, \tag{53}$$

the variable $y(\tau)$ is given by

$$y(\tau)=-nz(\tau)=-\tau np/m\,, \tag{54}$$

where we have chosen the coordinate frame such that $y(0)=0$. Thus we may use y rather than τ to parametrize the trajectory of the particle. In terms of this variable, the canonical momentum $(p+eA)$ satisfies an equation of motion

$$\frac{d}{dy}(p+eA)^\mu=\frac{e}{np}n^\mu\frac{dA^\nu}{dy}(p+eA)_\nu-\frac{e^2}{2np}n^\mu\frac{dA^2}{dy}\,, \tag{55}$$

which has the immediate solution

$$p^\mu(y)+eA^\mu(y)=p_{(0)}{}^\mu+n^\mu I_p(y)\,, \tag{56}$$

where $p_{(0)}{}^\mu$ is an integration constant and

$$I_p(y)=(1/2np)[2ep_{(0)}A(y)-e^2A(y)^2]\,. \tag{57}$$

We can now apply this general result. The momentum of the initial charged particle is subject to a boundary condition in the remote past

$$y\to-\infty:\quad p(y)\to p\,, \tag{58a}$$

while, after it is elastically scattered, it must satisfy a boundary condition in the distant future

$$y\to+\infty:\quad p'(y)\to p'\,. \tag{58b}$$

These boundary conditions are obeyed by

$$p^\mu(y,\phi)=p^\mu-eA^\mu(y)+n^\mu I_p(y) \tag{59a}$$

and

$$p'^\mu(y,\phi)=p'^\mu-eA^\mu(y)+n^\mu I_{p'}(y)\,. \tag{59b}$$

We have exhibited the dependence of the momenta on the over-all phase ϕ of the radiation field which we must average over later. Finally, in the monochromatic limit of interest, we have

$$A^\mu(y)=a^\mu e^{-i(\omega y-\phi)}+a^{\mu*}e^{i(\omega y-\phi)}\,, \tag{60}$$

with

$$k^\mu=\omega n^\mu\,. \tag{61}$$

It is clear that the final phase averaging will remove the y dependence of the momenta, and we may now omit this variable.

The rate at which particles are scattered from momentum p to momentum p', at a given phase ϕ, is proportional to the differential cross section for the elastic scattering $p(\phi)\to p'(\phi)$ multiplied by the incident flux at the instant of collision:

$$d\sigma^{(\mathrm{el})}4[\nu(\phi)^2-m^2M^2]^{1/2}\,, \tag{62}$$

with

$$\nu(\phi)=-p(\phi)P\,. \tag{63}$$

We may use the general form (27) of the elastic cross section to write this rate in a manner that makes explicit its kinematical structure:

$$d\sigma^{(\mathrm{el})}4[\nu(\phi)^2-m^2M^2]^{1/2}$$
$$=\int\frac{(d\mathbf{p}'(\phi))}{(2\pi)^3}\frac{1}{2p'^0(\phi)}\int\frac{(d\mathbf{P}')}{(2\pi)^3}\frac{1}{2P'^0}\times(2\pi)^4\delta(p'(\phi)+P'-p(\phi)-P)\,|T(\nu(\phi),t)|^2\,. \tag{64}$$

Here the integration region of the final-target-particle momentum $\mathbf{P}'$ is that appropriate to the given experimental arrangement. On the other hand, the integration region of the charged-particle momentum immediately after scattering $p'(\phi)$ is that which evolves as $\tau \rightarrow \infty$ into the experimentally detected region of final-particle momentum p'. This is an awkward state of affairs, for it is a difficult geometrical problem to relate a momentum region of $p'(\phi)$ to the corresponding region of p'. However, we do not need to do this explicitly, since the phase-space integral is Lorentz-invariant and the motion of $p'(\phi)$ to p' can be described by a Lorentz transformation. Although this Lorentz transformation depends upon p, it is not difficult to prove, using Eq. (59b), that the Jacobian is unity;

$$\frac{\partial(p'(\phi))}{\partial(p')} \equiv \det{}_4\left(\frac{\partial p_\mu'(\phi)}{\partial p_\nu'}\right) = 1\,, \tag{65}$$

for the determinant may be written as unity plus a sum of traces of $n^\mu n^\nu$ and $n^\mu a^\nu$ which vanish. Hence

$$\int \frac{(d\mathbf{p}'(\phi))}{(2\pi)^3} \frac{1}{2p'^0(\phi)} = \int \frac{(d\mathbf{p}')}{(2\pi)^3} \frac{1}{2p'^0}\,, \tag{66}$$

and the integration region over the final charged-particle momentum is now that which is directly given by the experimental detection resolution. The rate divided by the initial flux and averaged over the phase of the radiation field is the classical cross section, and we obtain

$$d\sigma = \int_0^{2\pi} \frac{d\phi}{2\pi} \int \frac{(d\mathbf{p}')}{(2\pi)^3} \frac{1}{2p'^0} \int \frac{(d\mathbf{P}')}{(2\pi)^3} \frac{1}{2P'^0} \times (2\pi)^4 \delta(p'(\phi) + P' - p(\phi) - P) \times |T(\nu(\phi),t)|^2 / 4(\nu^2 - m^2 M^2)^{1/2}. \tag{67}$$

This is the exact low-frequency limit of the classical cross section accurate to all orders in the electrical charge.

In order to make contact with the previous quantum-mechanical low-frequency limit, we must expand the classical cross section (67) in powers of the electrical charge and retain terms to order e^2. To the accuracy required, and in the laboratory-frame radiation gauge, we have

$$p'^\mu(\phi) - p^\mu(\phi) = \bar{p}'^\mu - \bar{p}^\mu + n^\mu 2e \operatorname{Re}[(p'a/p'n - pa/pn)e^{i\phi}] \tag{68}$$

and

$$\nu(\phi) = \bar{\nu} - 2e(Pk/pk) \operatorname{Re}(pa\, e^{i\phi})\,. \tag{69}$$

The expansion of Eq. (67) in powers of the electrical charge to order e^2 produces terms multiplying $\delta(\bar{p}' + P' - \bar{p} - P)$ and its first and second derivatives, and we arrive at precisely the structure of the quantum-mechanical result exhibited at the end of Sec. 4. Thus the role of the photon momentum $\hbar k$ as the expansion parameter of the energy-momentum-conserving δ function in the quantum case is replaced with the electric charge e in the classical case.[8] It is a simple matter to verify explicitly that the coefficients of the δ function and its first two derivatives are identical with the coefficients (44)–(46) calculated at the end of Sec. 4. This completes the proof of the classical correspondence of the low-frequency limit for scattering in a radiation field.

The rate at which energy is transferred from the radiation field to the motion of the particles within the field may be found classically by multiplying the integrand of the expression for the classical cross section (67) by $(\bar{p}' + P' - \bar{p} - P)^0$, expanding the δ function in e, and performing the integrals. The quantum-mechanical rate may be similarly obtained by multiplying the integrand of Eq. (39) by $n\hbar\omega$, with only the term (45) contributing in the soft-photon limit. It is not difficult to verify that the two calculations agree in the low-frequency limit.

6. A GENERATING FUNCTIONAL

In order to establish the connection between the quantum-mechanical and classical scattering cross sections to all powers of the electrical charge, we shall employ an approximate generating functional that yields the two leading orders of all the multiple soft-photon amplitudes. We turn now to the construction of this generating functional.

We alter our notation slightly by defining all photon momenta to be incoming. The exact amplitude describing the absorption of n photons in the otherwise elastic-scattering process,

$$T^{\mu_1 \cdots \mu_n}(p',P';p,P;k_1 \cdots k_n)\,,$$

is obtained by the functional differentiation of the generating functional

$$\mathcal{T}[p,'P';p,P;A] = \sum_{n=0}^{\infty} \frac{e^n}{n!} \int \frac{(dk_1)}{(2\pi)^4} \cdots \frac{(dk_n)}{(2\pi)^4} A_{\mu_1}(k_1) \cdots A_{\mu_n}(k_n) \times (2\pi)^4 \delta(p' + P' - p - P - k_1 \cdots - k_n) \times T^{\mu_1 \cdots \mu_n}(p',P';p,P;k_1 \cdots k_n)\,. \tag{70}$$

The leading soft-photon contributions to this functional come from the absorption of photons during the propagation of the charged particle exterior to the major scattering event. The totality of these contributions form initial- and final-state wave functions $\Phi_p{}^{\text{in}}$ and $\Phi_{p'}{}^{\text{out}*}$. We shall use exact on-mass-shell vertices, so that the wave functions satisfy a Klein-Gordon equation

$$[(i^{-1}\partial - e\mathcal{A}(x))^2 + m^2]\Phi[x;\mathcal{A}] = 0\,, \tag{71}$$

where the effective potential

$$\mathcal{A}^\mu(x)=\int\frac{(dk)}{(2\pi)^4}\mathcal{A}^\mu(k)e^{ikx} \tag{72}$$

is given by

$$\mathcal{A}^\mu(k)=F(k^2)A^\mu(k). \tag{73}$$

We note that, according to Eq. (71), a gauge variation

$$\mathcal{A}_\mu(x)\rightarrow\mathcal{A}_\mu(x)+\partial_\mu\lambda(x) \tag{74}$$

produces

$$\Phi[x;\mathcal{A}+\partial\lambda]=e^{ie\lambda(x)}\Phi[x;\mathcal{A}]. \tag{75}$$

The initial- and final-state wave functions are the solutions to the Klein-Gordon equation (71) fixed by the boundary conditions $x^0\rightarrow-\infty$:

$$\Phi_p{}^{\text{in}}[x;\mathcal{A}]\rightarrow e^{ipx}, \tag{76a}$$

and $x^0\rightarrow+\infty$:

$$\Phi_{p'}{}^{\text{out}}[x;\mathcal{A}]^*\rightarrow e^{-ip'x}. \tag{76b}$$

In the leading order, we may neglect the changes in the energy variable ν of the elastic amplitude that arises from the absorption of soft photons by the charged particle, and we have

$$\mathcal{T}[p',P';p,P;A]\approx T(\nu,t)\int(dx)\,e^{-iP'x}\times\Phi_{p'}{}^{\text{out}}[x;\mathcal{A}]^*\Phi_p{}^{\text{in}}[x;\mathcal{A}]e^{iPx}. \tag{77}$$

It follows from Eq. (75) that this approximate form is gauge-invariant. Although this approximation gives the over-all momentum conservation dictated by the δ functions occurring in Eq. (70), it does not produce the exact residues of the multiple poles [involving $T(\nu-k_1P-\cdots-k_nP,t)$], but rather ones [involving $T(\nu,t)$] in which the photon momenta are neglected.

We may correct this approximation and account for the alterations in the energy variable ν produced by photon absorption during the external propagation if we write the elastic amplitude in terms of its Fourier transform,

$$T(\nu,t)=\int d\zeta\,T(\zeta,t)e^{-i\nu\zeta}. \tag{78}$$

Since that part of the incoming wave function involving the absorption of n photons has the space-time behavior

$$[\Phi_p{}^{\text{in}}]_n\sim e^{i(p+k_1+\cdots+k_n)x}, \tag{79}$$

and since

$$\int d\zeta\,T(\zeta,t)e^{i(p+k_1+\cdots+k_n)P\zeta}=T(\nu-k_1P-\cdots-k_nP,t), \tag{80}$$

the structure

$$\int d\zeta\int(dx)\,e^{-iP'x}\Phi_{p'}{}^{\text{out}}[x;\mathcal{A}]^*T(\zeta,t)\times\Phi_p{}^{\text{in}}[x+P\zeta;\mathcal{A}]e^{-iPx} \tag{81}$$

not only gives the leading soft-photon contributions to the generating functional, but also correctly accounts for momentum conservation. However, since the coordinates of the wave functions differ, it is not gauge-invariant, for under a gauge transformation there appears the factor

$$\exp\{ie[\lambda(x+P\zeta)-\lambda(x)]\}. \tag{82}$$

Thus we should construct a function that compensates for this gauge change. This function must not be a global functional of $\mathcal{A}$ if we are to preserve the locality properties of the multiphoton amplitudes. An obvious candidate that preserves locality is

$$\exp\left(-ie\int_x^{x+P\zeta}d\xi_\mu\,\mathcal{A}^\mu(\xi)\right), \tag{83}$$

in which the line integral runs along a straight path from x to $x+P\zeta$. Accordingly,

$$\mathcal{T}[p',P';p,P;A]\cong\int d\zeta\int(dx)\,e^{-iP'x}\times\Phi_{p'}{}^{\text{out}}[x;\mathcal{A}]^*T(\zeta,t)\Phi_p{}^{\text{in}}[x+P\zeta;\mathcal{A}]e^{iPx}\times\exp\left(-ie\int_x^{x+P\zeta}d\xi_\mu\,\mathcal{A}^\mu(\xi)\right) \tag{84}$$

is a gauge-invariant momentum-conserving generating functional that represents the leading soft-photon contributions. It is interesting to note that the phase

$$-ie\int_x^{x+P\zeta}d\xi_\mu\,\mathcal{A}^\mu(\xi)=-ie\int_0^1 d\lambda\,\zeta P_\mu\mathcal{A}^\mu(x+\lambda\zeta P)=e\int\frac{(dk)}{(2\pi)^4}F(k^2)\frac{P_\mu A^\mu(k)}{kP}[e^{ikx}-e^{ik(x+\zeta P)}] \tag{85}$$

vanishes in the laboratory-frame radiation gauge.

An indication of the accuracy of this generating functional is obtained if it is expanded in powers of the electric charge. The first-order terms in e produced by the wave functions are easily seen to agree with the pole terms $\mathcal{P}^\mu$ of Eq. (5). It is a simple matter to verify that the gauge phase (85) corresponds identically to the contact term $\mathcal{S}^\mu$ of Eq. (8). Thus, to first order in e, the generating functional is identical with the sum $\mathcal{P}^\mu+\mathcal{S}^\mu$ and accounts for the first two orders of the low-frequency limit. It is also not difficult to establish that the terms of order e^2 correspond precisely to the double-photon terms $\mathcal{P}^{\mu\nu}+\mathcal{S}^{\mu\nu}$, and hence they also provide the two leading orders in the soft-photon limit. In the general case, the difference between the n-photon amplitude given by the approximate generating functional and the exact n-photon amplitude can have terms containing the product of, at most, $n-1$ poles, for our generating functional gives the correct nth-order pole terms. If this difference had a piece of order $\omega^{-(n-1)}$, where ω stands for any of the photon frequencies, this piece would therefore be the sum of terms of the form

$$\sum \frac{Q_1^{\mu_1}\cdots Q_n^{\mu_n}}{K_1Q_1'\cdots K_{n-1}Q_{n-1}'}, \tag{86}$$

where the Q_a and Q_a' are linear combinations of the particle momenta and the K_a are linear combinations of the photon momenta. The gauge invariance of the approximate and correct amplitudes implies that the contraction of these terms with any $k_a{}^{\mu_a}$, $a=1,\cdots,n$, must vanish. This can occur only if in every term in (86) there is a factor k_aQ_a in the denominator; but, since there are n photon momenta and only $n-1$ factors in the denominators, this can not happen for all of the k_a and there is a contradiction. Consequently, the two leading orders in photon frequency to all orders in e are correctly given by the approximate generating functional.

7. EXACT CORRESPONDENCE

We can now make use of the generating functional to establish heuristically the correspondence between the quantum and classical scattering in the radiation field to all orders of the electrical charge. It is readily verified that the wave functions[14] in a plane-wave radiation field (50)

$$\Phi_p{}^{\text{in}}(x)=e^{ipx}\exp\left(-i\int_{-\infty}^{y}dy'\,I_p(y')\right), \tag{87a}$$

and

$$\Phi_{p'}{}^{\text{out}}(x)^*=e^{-ip'x}\exp\left(-i\int_{y}^{\infty}dy'\,I_{p'}(y')\right) \tag{87b}$$

satisfy the Klein-Gordon equation (71) along with the appropriate boundary conditions (76a) and (76b). In the monochromatic limit, the combination that occurs in the generating functional has the structure

$$\begin{aligned}\Phi_{p'}{}^{\text{out}}(x)^*\Phi_p{}^{\text{in}}(x+P\zeta)\exp\left(-ie\int_x^{x+P\zeta}d\xi_\mu\,A^\mu(\xi)\right)\\ =\exp[-i\bar{p}'x+i\bar{p}(x+P\zeta)]\\ \times\exp\{i[f_{p'}(kx+\phi)-f_p(kx+kP\zeta+\phi)\\ +g(kx+\phi)-g(kx+kP\zeta+\phi)]\},\end{aligned} \tag{88}$$

[14] These wave functions are derived in Ref. 10. Their relationship to the WKB approximation is also discussed in this reference.

where, omitting a physically irrelevant infinite phase factor,

$$f_p(\phi)=ie\left(\frac{pa}{pk}e^{i\phi}-\frac{pa^*}{pk}e^{-i\phi}\right)-(ie^2/4pk)(a^2e^{2i\phi}-a^{*2}e^{-2i\phi}), \tag{89}$$

and

$$g(\phi)=-ie\left(\frac{Pa}{Pk}e^{i\phi}-\frac{Pa^*}{Pk}e^{-i\phi}\right). \tag{90}$$

These functions are periodic in kx. Hence we may write

$$\begin{aligned}\Phi_{p'}{}^{\text{out}}(x)^*\Phi_p{}^{\text{in}}(x+P\zeta)\exp\left(-ie\int_x^{x+P\zeta}d\xi_\mu\,A^\mu(\xi)\right)\\ =e^{-i\bar{p}'x+i\bar{p}(x+P\zeta)}\sum_{n=-\infty}^{\infty}C_n(\zeta)e^{in(kx+\phi)},\end{aligned} \tag{91}$$

in which

$$C_n(\zeta)=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-in\phi}\exp\{i[f_{p'}(\phi)-f_p(\phi+kP\zeta)+g(\phi)-g(\phi+kP\zeta)]\}. \tag{92}$$

The approximate generating functional (84) thus has the form

$$\begin{aligned}\mathcal{T}[p',P';p,P;A]\\ =\int d\zeta\sum_{n=-\infty}^{\infty}(2\pi)^4\delta(\bar{p}'+P'-\bar{p}-P-nk)\\ \times C_n(\zeta)T(\zeta,t)e^{i\zeta\bar{p}P},\end{aligned} \tag{93}$$

and the n-photon amplitudes that enter in the quantum cross section (39) are given by

$$T^{(n)}=\int d\zeta\,C_n(\zeta)T(\zeta,t)e^{i\zeta\bar{p}P}. \tag{94}$$

If we introduce an integral representation for the energy-momentum-conserving δ function that occurs in the quantum cross section,

$$\begin{aligned}(2\pi)^4\delta(\bar{p}'+P'-\bar{p}-P-nk)\\ =\int(dx)\exp[-i(\bar{p}'+P'-\bar{p}-P-nk)x],\end{aligned} \tag{95}$$

then we encounter the sum

$$\begin{aligned}\sum_n e^{inkx}|T^{(n)}|^2=\int d\zeta'd\zeta\,T(\zeta',t)^*e^{-i\zeta'\bar{p}P}\\ \times T(\zeta,t)e^{i\zeta\bar{p}P}\sum_n e^{inkx}C_n(\zeta')^*C_n(\zeta).\end{aligned} \tag{96}$$

We may use the integral representation (92) of the Fourier coefficients C_n along with the completeness relation

$$\sum_n e^{in\theta}=2\pi\sum_n \delta(\theta+2\pi n) \tag{97}$$

to secure

$$\sum_n e^{inkx}C_n(\zeta')^*C_n(\zeta)=\int_0^{2\pi}\frac{d\phi}{2\pi}$$
$$\times\exp\{i[f_{p'}(\phi)-f_{p'}(\phi-kx)-f_p(\phi+kP\zeta)$$
$$+f_p(\phi+kP\zeta'-kx)+g(\phi)-g(\phi-kx)$$
$$-g(\phi+kP\zeta)+g(\phi+kP\zeta'-kx)]\}. \tag{98}$$

Having done the summation over the multiple-photon partial probabilities, we can now take the low-frequency limit $k\to 0$. This is permissible because we are dealing with quantities that occur within an integral over a finite region of final-particle phase space,[13] and this integral effectively limits the range of variation of the parameter x so that $kx\to 0$ as $k\to 0$. We obtain, as $k\to 0$,

$$\sum_n e^{inkx}C_n(\zeta')^*C_n(\zeta)\to\int_0^{2\pi}\frac{d\phi}{2\pi}$$
$$\times\exp\{-i[p'(\phi)-\bar{p}'-p(\phi)+\bar{p}]x$$
$$+i(\zeta-\zeta')P[p(\phi)-\bar{p}]\}, \tag{99}$$

where $p'(\phi)$ and $p(\phi)$ are the classical momenta given in Eqs. (59). Accordingly, in the low-frequency limit,

$$\sum_n (2\pi)^4\delta(\bar{p}'+P'-\bar{p}-P-nk)|T^{(n)}|^2\to\int_0^{2\pi}\frac{d\phi}{2\pi}$$
$$\times\delta(p'(\phi)+P'-p(\phi)-P)\int d\zeta' d\zeta$$
$$\times T(\zeta',t)^*e^{-i\zeta' p(\phi)P}T(\zeta,t)e^{i\zeta p(\phi)P}$$
$$=\int_0^{2\pi}\frac{d\phi}{2\pi}\delta(p'(\phi)+P'-p(\phi)-P)|T(\nu(\phi),t)|^2, \tag{100}$$

which establishes a precise correspondence between the quantum scattering cross section (39) and the classical scattering cross section (67) to all orders in the electrical charge.

The foregoing is not a complete proof of the classical correspondence, because the final result is of order ω^0 in the photon frequency and, as was found in Sec. 4, there are structure-dependent terms of this order and indeed of lower order. The exact generating functional (70) has, in addition to the elastic amplitude $T(\nu,t)$, various irreducible structure-dependent terms $\mathcal{R}^\mu$, ${}^{(2)}\mathcal{R}^{\mu\nu},\cdots$, and irreducible Compton amplitudes $D^{\mu\nu}\cdots$ multiplying the elastic amplitude T sandwiched between the initial and final wave functions. The same cancellations which gave the final result (100) as a function of order ω^0 multiplying the elastic amplitude $|T(\nu,t)|^2$ should occur for the additional structure-dependent quantities. Since these are at least of order ω, they should not contribute to the low-frequency limit.

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