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# Invariant Amplitudes for Photon Processes

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A method for constructing independent invariant amplitudes for photon processes which satisfy gauge invariance and are free from both kinematic singularities and zeros (constraints) is proposed. The method is based on the use of a simple projection operator which automatically ensures gauge invariance at every stage. The invariant amplitudes for pion photoproduction, pion Compton scattering, and nucleon Compton scattering are found. Previously known results in the first two cases are reproduced and their interpretation is clarified. For nucleon Compton scattering the result is new. Comparisons are made between these invariant amplitudes, the Hearn and Leader amplitudes, and the regularized helicity amplitudes. The Mandelstam representation may be applied to these kinematic-singularity-free and zero-free amplitudes without any *ad hoc* subtractions. All known low-energy theorems on photon processes can be derived most directly from these invariant amplitudes.

## I. INTRODUCTION

RECENTLY, much attention has been directed toward the understanding of kinematic singularities and zeros of scattering amplitudes. Aside from the usual reason of isolating dynamics from pure kinematic effects, it is recognized that in many cases, a detailed understanding of the kinematic structure of the scattering amplitude imposes constraints on the dynamics. These constraints often are of considerable physical interest.

The kinematic-singularity structure of the two-body helicity amplitude has been extensively studied in the past two years.[1–3] It is shown that regularized helicity amplitudes can be defined which are free of kinematic singularities in both of the invariant variables. However, these amplitudes are not always independent; they must satisfy certain "constraint equations"[2,3] at values of their variables corresponding to the thresholds and pseudothresholds and possibly also at zero center-of-mass energy. These constraints (or kinematic zeros, as we shall refer to them henceforth) are consequences of Lorentz invariance, analyticity, and crossing and must be respected by any dynamical theory. This fact makes the use of these regularized helicity amplitudes in dynamical theories rather cumbersome.

It is well known, however, for scattering of particles with spin and nonzero mass, invariant amplitudes can be defined which are free of kinematic singularities and zeros.[4] These invariant amplitudes are completely independent from each other as far as Lorentz transformation and crossing are concerned. (They are related by unitarity.) They have the same analytic properties as those of spinless-particle scattering. The most familiar examples are the $A$ and $B$ amplitudes for $\pi N$ scattering. When some of the particles participating in the scattering process are massless, the existence of these kinematic-singularity-free and zero-free invariant amplitudes has not been clearly demonstrated. The reason for this is that the additional constraints imposed by gauge invariance render the usual techniques for finding these amplitudes inadequate.[5] Thus even for the physically interesting case of nucleon Compton scattering, such amplitudes have not been found. The need for finding these kinematic-singularity- and zero-free amplitudes (if they exist) is made more urgent by recent developments in the understanding of the pure kinematic nature of the old and new low-energy theorems of photon processes. All these results should follow most directly from the kinematic-singularity-free and zero-free invariant amplitudes which are also the most suitable ones to write down Mandelstam representations without *ad hoc* subtractions.

It is the purpose of this paper to propose a general method to construct such kinematic-singularity-free and zero-free amplitudes for photon processes. The principles on which our method is based are the same as the usual ones. However, with the help of an extremely simple projection operator we can automatically take care of all the requirements of gauge invariance. This not only enables us to find these simple invariant amplitudes for various photon processes but also allows us to see how the close connection between gauge invariance and Lorentz-transformation properties of massless particles plays an essential role in determining the analytic structure of the scattering amplitudes.

In Sec. II, we formulate the problem in more detail than described above. In Sec. III, we introduce the projection operator mentioned before and outline the general procedure to obtain kinematic-singularity-free and zero-free amplitudes for any photon process. In

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[1] Y. Hara, Phys. Rev. **136**, B507 (1964); L. L. C. Wang, *ibid.* **142**, 1187 (1966).

[2] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 111 (1968); J. P. Ader, M. Capdeville, and H. Navelet (unpublished).

[3] H. Stapp, Phys. Rev. **160**, 1251 (1967); H. Stapp (unpublished).

[4] A. C. Hearn, Nuovo Cimento **21**, 333 (1961).

[5] A. C. Hearn (Ref. 4). To remedy this situation Hearn proposed an alternative method of obtaining invariant amplitudes for massless particles. His method yields amplitudes which are free of kinematic singularities but must satisfy complicated constraint equations on the boundary of the physical region. Another drawback of his method is that it can only be applied to processes with an even number of photons.

Secs. IV and V we work out two illustrative examples ($\gamma N \to \pi N$, $\gamma\pi \to \gamma\pi$) where the results can be compared with previous work. The interpretation of these previous results are clarified. In Sec. VI we treat the interesting case of nucleon Compton scattering. A set of invariant amplitudes are found which are completely free from kinematic singularities and zeros (constraints). The relation between these amplitudes and those of Hearn and Leader and also the regularized helicity amplitudes are displayed. The crossing property and asymptotic behavior of these invariant amplitudes are studied. Double spectral representations for these amplitudes are written down. An interesting feature which emerges as a consequence of gauge invariance (or charge conservation) is that the Born terms contribute to the double spectral functions (as products of $\delta$ functions) as well as to the single spectral functions. This point is illustrated in all three examples. Low-energy theorems for nucleon Compton scattering to second order in the photon energy are derived by mere inspection.

## II. FORMULATION OF PROBLEM

To help in formulating the problem, let us briefly review the basic principles involved. The physical scattering matrix is written as

$$\langle f|S|i\rangle = \langle f|i\rangle - i(2\pi)^4\delta^4(\textstyle\sum p_i - \sum p_f)\langle f|T|i\rangle, \quad (1)$$

$$\langle f|T|i\rangle = \textstyle\prod \bar{u}_f(p_f)M(p_f,p_i)\prod u_i(p_i), \quad (2)$$

where $\prod\bar{u}_f(p_f)$ $[\prod u_i(p_i)]$ represents the product of wave functions for the final (initial) particles with momenta $p_f$ ($p_i$). The spinor-tensor amplitude $M$ defined above shall be called the $M$ function throughout this paper. The $M$ function transforms under Lorentz transformation as simple tensors or spinors and is assumed to have only dynamical singularities[6,7] when considered as a function of the four-momentum components. (The mass-shell conditions are always satisfied.) The invariant functions $A_i$ are obtained by expanding $M$ in terms of a spinor-tensor basis $\{l_i\}$,

$$M(p_f,p_i) = \textstyle\sum l_i(p_f,p_i)A_i. \quad (3)$$

Under Lorentz transformations $\{l_i\}$ transforms as $M$ does; the invariant functions $\{A_i\}$ are therefore scalar functions of the scalar invariants of the problem.

If one or more $l_i$ in a given basis $\{l_i\}$ contain kinematic singularities, then the corresponding invariant amplitudes $A_i$ must develop kinematic zeros at the same points to cancel these singularities since $M$ is assumed not to have kinematic singularities. In order to avoid these singularities, one always chooses a tensor basis which consists of polynomials in the four-momentum components. Let us consider two such polynomial bases, $\{l_i\}$ and $\{l_i'\}$. They are related by a transformation matrix $\alpha$,

$$l_i(p_f,p_i) = \textstyle\sum_j \alpha_{ij}(p_f,p_i)l_j'(p_f,p_i).$$

If $\det\alpha(p_f,p_i)=0$ for some values of its variables, then $\{l_i\}$ are not linearly independent at these points. In other words, $\{l_i\}$ have kinematic zeros at these points, implying that the associated invariant amplitudes have kinematic singularities. Now, we define a polynomial tensor basis $\{l_i\}$ to be "minimal" if the determinant of the transformation matrix $\alpha$ *from* any other such basis $\{l_i'\}$ does not vanish anywhere for (on-mass-shell) complex values of the momentum components. Two tensor bases are said to be *equivalent*, or related by simple relabelling, if neither $\det\alpha$ nor $\det(\alpha^{-1})$ have zeros. It is clear that, by definition, the minimal-polynomial tensor basis is unique up to equivalence if it exists. The invariant amplitudes associated with a minimal-polynomial tensor basis are expected to be free of kinematic singularities and zeros.[8]

The existence of such spinor-tensor basis leading to kinematic-singularity-free and zero-free invariant amplitudes has been proved by Hepp[9] and Williams.[10] The latter also gave the explicit form of the particular basis used in the proof. However, his construction did not incorporate the discrete transformations. Space and time-reversal invariance impose complicated relations among his invariant amplitudes and make them unsuitable for practical use in physical problems. However, it is well known that for processes involving particles of nonzero mass, such a minimal tensor-basis can always be found either by inspection or by tedious, albeit straightforward, elimination.[4] Invariant amplitudes obtained this way are used extensively in the literature.

When some of the particles involved are massless, gauge invariance gives rise to nontrivial complications.[5] Let us consider a process involving only one external photon. The scattering is described by an $M$ function $M^\mu(k,p_1,p_2,\cdots)$, where $k$ is the four-momentum of the photon. We first ignore gauge invariance and expand the $M$ function in terms of a minimal polynomial-vector basis $\{l_i\}$ in the usual manner:

$$M^\mu(k,p_1,p_2,\cdots) = \sum_{i=1}^{m} l_i^\mu(k,p_1,p_2,\cdots)B_i. \quad (4)$$

The $B_i$'s are free of kinematic singularities by the usual argument. However, gauge invariance now imposes the

[6] H. Abarbanel and M. Goldberger, Phys. Rev. **165**, 1594 (1968); K. Bardakci and H. Pagels, *ibid.* **166**, 1783 (1968); S. R. Choudhury and D. Z. Freedman, *ibid.* **168**, 1739 (1968).

[7] H. P. Stapp, Phys. Rev. **125**, 2139 (1962). In field theory the $M$ function is closely related to the Fourier transform of the Green's function for the scattering process under consideration.

[8] The usual argument in support of this statement is based on Feynman graphs (Ref. 4). However, one does not have to rely on perturbation expansion. One can show, for instance, that if the $A_i$ have kinematic singularities, then the tensor basis $\{l_i\}$ cannot be minimal in the sense defined here. The conclusion therefore follows.

[9] K. Hepp, Helv. Phys. Acta **36**, 355 (1963).

[10] D. N. Williams (unpublished).

following condition on the amplitudes:

$$k_\mu M^\mu = \sum_{i=1}^{m} (k_\mu l_i^\mu) B_i = 0. \tag{5}$$

This means that the $B_i$'s are not all independent; certain linear combinations of them vanish. In our terminology, they have kinematic zeros. In principle one can solve Eq. (5) and obtain, say, $n$ ($n<m$), independent amplitudes. However, this turns out to be too complicated to handle for all but the simplest cases. Besides, even if one succeeds in doing this, he is still left with the problem of coping with kinematic zeros in the remaining $B_i$'s which is equally difficult to solve. What one needs is a way to find, directly, a tensor basis $\{\mathfrak{L}_{i\mu}\}$, $i=1, \cdots, n$, which is individually gauge-invariant, i.e.,

$$k_\mu \mathfrak{L}_i^\mu(k,p_1,p_2,\cdots)=0, \quad i=1, 2, \cdots, n \tag{6}$$

so that the corresponding expansion of the $M$ functions

$$M^\mu(k,p_1,p_2, \cdots) = \sum_{i=1}^{n} \mathfrak{L}_i^\mu A_i \tag{7}$$

is automatically gauge-invariant. If this set $\{\mathfrak{L}_i\}$ can also be easily adjusted to be minimal in the sense described earlier, the corresponding invariant amplitudes $A_i$ are free from kinematic singularities and zeros. A systematic method of accomplishing this is presented in Sec. III.

## III. SOLUTION OF PROBLEM

Our basic observation is that the gauge-invariance condition can be handled automatically by the use of a simple projection operator $I$,

$$I^{\mu\nu} = g^{\mu\nu} - [p^\mu k^\nu/(k\cdot p)], \tag{8}$$

where $p$ is any four-vector not identical to $k$, the photon momentum. In practice, $p$ is always chosen to be one of the $p_i$'s contained in $M^\mu(k,p_1,p_2,\cdots)$. $I^{\mu\nu}$ has the following basic properties:

$$I^{\mu\nu} M_\nu = M^\mu \tag{9}$$

if $M^\mu$ satisfies the gauge condition (5); and

$$k_\mu I^{\mu\nu} = 0. \tag{10}$$

Equation (9) states that when we contract $I$ with a gauge-invariant quantity, the latter is left unchanged. Equation (10) implies that when we contract $I^{\mu\nu}$ with an arbitrary vector, we get a gauge-invariant vector. These properties establish $I$ as a projection operator for gauge invariance.

Now, if we start from a minimal polynomial expansion without gauge invariance, say, Eq. (4), and contract with $I^{\mu\nu}$, we obtain

$$M^\mu = \sum (I^{\mu\nu} l_{i,\nu}) B_i; \tag{11}$$

the right-hand side of this equation is explicitly gauge-invariant because of Eq. (10). We observe that because of one additional property of $I^{\mu\nu}$, namely,

$$I^{\mu\nu} p_\nu = 0, \tag{12}$$

the terms in Eq. (4) proportional to $p^\mu$ are immediately eliminated and the sum in Eq. (11) involves fewer terms than in Eq. (4). In fact, as we shall see later, in most practical cases this step already (or almost already) eliminates all the linearly dependent amplitudes and simplifies the subsequent algebraic manipulations considerably. What this operation accomplishes is equivalent to solving all the equations implicit in Eq. (5) and substituting into Eq. (4) as one would do in the conventional method. Since the $B_i$'s in Eq. (11) are just a subset of those in Eq. (4), they are still free from kinematic singularities. The fact that they have kinematic zeros now emerges from the fact that $Il_i$ may be singular.[11] More explicitly,

$$I^{\mu\nu} l_{i\nu} = l_i^\mu - [(k\cdot l_i)/(k\cdot p)] p^\mu. \tag{13}$$

The second term is obviously singular when $k\cdot p=0$. Our task is then to remove these kinematic singularities in $Il_i$ in a minimal way. This can be achieved by observing that all the singular terms are proportional to $p^\mu/(k\cdot p)$ with coefficients $(k\cdot l_i)$. If some of these coefficients $(k\cdot l_i)$ are linearly dependent, modulus terms proportional to $(k\cdot p)$, i.e.,

$$\textstyle\sum_i \alpha_i (k\cdot l_i) = (k\cdot p) g, \tag{14}$$

where $\alpha_i$ and $g$ are constants or low-order polynomials in the momentum components, then clearly $\sum \alpha_i (Il_i)$ is a gauge-invariant basis vector free of singularities; in fact,

$$\textstyle\sum \alpha_i (Il_i)^\mu = \sum \alpha_i l_i^\mu - g p^\mu. \tag{15}$$

A particular case of (14) occurs when the left-hand side consists of only one term and the coefficient $\alpha$ is simply $(k\cdot p)$. Among the set of gauge-invariant singularity-free kinematic tensors constructed this way it is easy to select the ones which form a minimal set in the sense that all the others can be expressed in terms of these without introducing singularities. In practice, it suffices just to take those which are of lowest order in the momentum components. Let us denote such a minimal set by $\{\mathfrak{L}_i\}$, $i=1, 2, \cdots, n$; then by the usual argument the invariant amplitudes $\{A_i\}$, $i=1, 2, \cdots, n$, defined by

$$M^\mu(k,p_1,p_2,\cdots) = \sum_{i=1}^{n} \mathfrak{L}_i^\mu(k,p_1,p_2,\cdots) A_i, \tag{16}$$

are free of both kinematic singularities and zeros. It may seem that the above procedure depends on which vector $p$ one chooses in the definition for $I^{\mu\nu}$, Eq. (8). This is, however, not so. Different choices of $p$ lead to minimal tensor bases which are equivalent in the

[11] Since $M^\mu$ is free of kinematic singularities, Eq. (11) demands kinematic zeros in $B_i$ to cancel any kinematic singularities in $Il_i$.

sense defined before. This is indicated by the fact that $I^{\mu\lambda}(p)I_{\lambda}{}^{\nu}(p')=I^{\mu\nu}(p')$. We shall illustrate this point in the examples that follow.

The same procedure applies to processes involving two or more photons. A projection operator can be introduced for each photon and the above treatment carried out either one at a time or simultaneously. For instance, two-photon processes are described by an $M$ function $M^{\mu\nu}(k,k',p_1,p_2,\cdots)$, where $k$ and $k'$ are the photon momenta. $M^{\mu\nu}$ satisfies the gauge condition

$$k_\mu' M^{\mu\nu}=M^{\mu\nu}k_\nu=0. \quad (17)$$

In this case it is most convenient to introduce a single projection operator

$$I^{\mu\nu}=g^{\mu\nu}-[k^\mu k'^\nu/(k\cdot k')] \quad (18)$$

which can be used for both photons. Thus

$$I^{\mu\lambda}M_\lambda{}^\nu=M^{\mu\lambda}I_\lambda{}^\nu=I^{\mu\lambda}M_{\lambda\sigma}I^{\sigma\nu}=M^{\mu\nu} \quad (19)$$

and

$$k_\nu' I^{\nu\mu}=I^{\mu\nu}k_\nu=0. \quad (20)$$

In the following sections we shall treat several examples in detail and demonstrate how this procedure actually works.

## IV. PHOTOPRODUCTION OF PIONS

As the first example, we shall consider the familiar case of photoproduction of pions off nucleons where the results can be easily compared with previous works.[12,13] We shall denote the four-momenta of the photon, the pion, and the two nucleons by $k$, $q$, $p$, and $p'$, respectively. We also use $P=\frac{1}{2}(p+p')$. The Mandelstam invariants are defined as usual:

$$s=(k+p)^2\,,\quad t=(k-q)^2\,,\quad u=(k-p')^2. \quad (21)$$

The $M$ function, before the gauge-invariance condition is imposed, can be written

$$M(k,q,P)=\sum l_iB_i=i\gamma^5\gamma^\mu[B_1+(\gamma k)B_2]+i\gamma^5P^\mu \times[B_3+(\gamma k)B_4]+i\gamma^5q^\mu[B_5+(\gamma k)B_6]. \quad (22)$$

Now we introduce the projection operator

$$I^{\mu\nu}=g^{\mu\nu}-[q^\mu k^\nu/(q\cdot k)].$$

Clearly, if we contract $I^{\mu\nu}$ with $M$, the last two terms on the right-hand side of Eq. (22) do not contribute and we are left with four independent amplitudes (which are just of the right number for pion photoproduction).[14] According to the method described in the previous section, we now proceed to eliminate the singularities in $Il_i$. To this end we examine the quantities $k\cdot l_i$:

$$\begin{aligned} k\cdot l_1&=i\gamma^5(\gamma k)\,,\\ k\cdot l_2&=0\,,\\ k\cdot l_3&=i\gamma^5(k\cdot P)\,,\\ k\cdot l_4&=i\gamma^5(k\cdot P)(\gamma k). \end{aligned} \quad (23)$$

We see that $l_2$ is already gauge-invariant. Since $k\cdot l_4$ is a simple multiple of $k\cdot l_1$, the singularity in $Il_4$ can be eliminated by taking the linear combination $I[l_4-(k\cdot P)l_1]$. The remaining singularities in $Il_1$ and $Il_3$ can only be removed by multiplying by $(k\cdot q)$. A gauge-invariant minimal polynomial tensor basis therefore consists of

$$\begin{aligned} \mathfrak{L}_1&=Il_2=l_2=i\gamma^5\gamma^\mu(\gamma k)\,,\\ \mathfrak{L}_2&=(k\cdot q)Il_3=i\gamma^5[(k\cdot q)P^\mu-(k\cdot P)q^\mu]\,,\\ \mathfrak{L}_3&=(k\cdot q)Il_1=i\gamma^5[(k\cdot q)\gamma^\mu-(\gamma\cdot k)q^\mu]\,,\\ \mathfrak{L}_4&=-Il_4+(k\cdot P)Il_1=i\gamma^5[(k\cdot P)\gamma^\mu-(\gamma\cdot k)P^\mu]. \end{aligned} \quad (24)$$

It is not hard to see that had we used $P$ instead of $q$ in the definition of $I^{\mu\nu}$, we would have arrived at the same set $\{\mathfrak{L}_i\}$ except that the roles of $\mathfrak{L}_3$ and $\mathfrak{L}_4$ would be interchanged.

The set $\{\mathfrak{L}_i\}$ are equivalent to the basis vectors written down by Chew, Goldberger, Low, and Nambu.[12] The properties of the invariant amplitudes $\{A_i\}$ associated with these are discussed by Ball in some detail.[13] His results are often given the interpretation that the amplitude $A_2$ has a kinematic pole at $t=\mu^2$. This would seem to contradict our assertion that these $A_i$'s are free from kinematic singularities and zeros. We shall show in the following that this is not the case. Our result is perfectly consistent with Ball's analysis, and we shall try to explain why the usual interpretation of his result is in fact a misinterpretation. The problem involved is a very interesting one and is a distinctive feature of gauge-invariant amplitudes, as we shall see again in the following sections where we treat the Compton scattering of pions and nucleons.

Ball based his analysis on the invariant amplitudes[15] $B_i$ (which are not gauge-invariant). He showed that the $B_i$'s are free of kinematic singularities. Gauge invariance imposes two conditions on the $B_i$'s; one of these is

$$(s-u)B_3=2(t-\mu^2)B_5. \quad (25)$$

[As defined in Eq. (22), our $B_3$ and $B_5$ correspond to Ball's $B_2$ and $B_3$, respectively.] Now, $A_2$ is related to these amplitudes by

$$A_2=2B_3/(t-\mu^2)=4B_5/(s-u). \quad (26)$$

$A_2$ cannot have a kinematic singularity at $t=\mu^2$ be-

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[13] J. Ball, Phys. Rev. **124**, 2014 (1961).

[14] The correct number of independent amplitudes for any process can be found most easily by counting helicity amplitudes.

[15] Ball has eight $B$ amplitudes as compared to our six defined in Eq. (22). The reason for this difference is that he has not used the condition $\epsilon_\mu k^\mu=0$ to eliminate his $B_4$ and $B_7$, which are physically irrelevant.

cause if it did, $B_5$ would also have a kinematic singularity at $t=\mu^2$. This is not possible as the $B_i$'s are free of kinematic singularities. (A similar argument shows that $A_2$ does not have a kinematic singularity at $s=u$.) Thus either $A_2$ is finite at $t=\mu^2$ or it has a *dynamical* singularity at that point. This means $B_3$ has a *kinematic zero* at $t=\mu^2$ (which, however, may be cancelled by a dynamical singularity at the same point). To determine precisely what happens one therefore has to look at the pole terms. There are three such diagrams corresponding to the nucleon pole in the $s$ and $u$ channels and the pion pole to the $t$ channel. Ball found that the contribution of these diagrams to the $B_i$'s are of the form

$$B_i^{(B)}=\frac{R_s^i}{s-m^2}+\frac{R_t^i}{t-\mu^2}+\frac{R_u^i}{u-m^2}, \tag{27}$$

while the contribution to $A_2$ is of the form

$$A_2^{(B)}=\frac{g}{t-\mu^2}\left(\frac{1}{s-m^2}\pm\frac{1}{u-m^2}\right). \tag{28}$$

Thus the poles in the different channels appear in $B_i$ as a sum of separate terms in the familiar manner as in hadron dynamics. In contrast, these pole terms appear in $A_2$ in the form of products which looks suspicious. However, it is not hard to see that this is actually very natural and, indeed, necessary. It is well known that the pole diagrams are not separately gauge-invariant; only the sum of all three is. These pole terms are properly correlated in $A_2$ because the latter is a gauge-invariant amplitude. On the other hand, the $B_i$'s are not gauge-invariant and the way the pole terms appear there is artificial. One must resort to gauge conditions like Eq. (25) to enforce the required correlation; the end result is the same as before. In this connection we mention the close relation between gauge invariance and the Lorentz transformation properties of massless particles as demonstrated by Weinberg.[16] We emphasize that the individual terms in Eq. (27) which are not gauge-invariant cannot possibly have the correct kinematic behavior demanded by Lorentz transformation. Indeed, this remark also applies to the full $B_i$ amplitudes; thus, although they are free of kinematic singularities, they must have kinematic zeros to satisfy gauge invariance. Only the $A_i$'s are free of both kinematic singularities and zeros. The fact that the pole terms enter $A_i$ in the correlated form (28) is a clear manifestation of gauge invariance (or charge conservation) and is a new feature of scattering amplitudes for massless particles. We shall go into more details on this point in a later section when we discuss the Mandelstam representation for these $A_i$'s.

An independent check on the kinematic-singularity-free and zero-free nature of the $\{A_i\}$ can be obtained by comparing them with the results of Ader, Capdeville, and Navelet[2] on the kinematic singularities and zeros of the helicity amplitudes. To this end we calculate the $t$-channel helicity amplitudes in terms of the $A_i$'s. We obtain[17]

$$\begin{aligned}
(f^t_{\frac12\frac12,10}+f^t_{-\frac12-\frac12,10})\sin^{-1}\psi&=\frac{i(t-m_\pi^2)(t-4m^2)^{1/2}}{\sqrt{2}t^{1/2}}\\
&\quad\times[-A_1-\tfrac12 tA_2-mA_4],\\
(f^t_{\frac12\frac12,10}-f^t_{-\frac12-\frac12,10})\sin^{-1}\psi&=\frac{i}{\sqrt{2}}(t-m_\pi^2)[A_1]\\
\left(\frac{f^t_{\frac12-\frac12,10}}{1+\cos\psi}+\frac{f^t_{-\frac12\frac12,10}}{1-\cos\psi}\right)&=\frac{i}{\sqrt{2}}(t-m_\pi^2)(t-4m^2)^{1/2}[A_3],\\
\left(\frac{f^t_{\frac12-\frac12,10}}{1+\cos\psi}-\frac{f^t_{-\frac12\frac12,10}}{1-\cos\psi}\right)&=i\frac{\sqrt{2}(t-m_\pi^2)}{\sqrt{t}}\\
&\quad\times[mA_1-\tfrac14(t-4m^2)A_4],
\end{aligned} \tag{29}$$

where $\psi$ is the $t$-channel c.m. scattering angle. The quantities appearing in the square brackets on the right-hand side correspond precisely to the regularized helicity amplitudes found by the authors mentioned above. Furthermore, one may notice when $t\to 4m^2$, the right-hand side of the second and the fourth line become proportional; similarly, when $t\to 0$, the first and the fourth lines become proportional. This implies that the regularized helicity amplitudes must satisfy constraint equations at these points. In fact, these are precisely the constraint equations found by the same authors.

## V. PION COMPTON SCATTERING

Let us denote by $k$, $k'$ and $q$, $q'$ the initial and final photon and pion momenta, respectively. We define $K=\frac12(k+k')$ and $Q=\frac12(q+q')$.

As mentioned earlier, for Compton scattering the most convenient projection tensor to use is

$$I^{\mu\nu}=g^{\mu\nu}-(k^\mu k'^\nu/k\cdot k'). \tag{30}$$

The $M$ function, before imposing gauge invariance, can be written

$$\begin{aligned}M^{\mu\nu}=\sum l_iB_i=g^{\mu\nu}B_1+K^\mu K^\nu B_2+Q^\mu Q^\nu B_3&\\+(K^\mu Q^\nu+Q^\mu K^\nu)B_4.&\end{aligned} \tag{31}$$

[16] S. Weinberg, Phys. Rev. **134**, B882 (1964); **135**, B1049 (1964); **138**, B988 (1964). See also *Lectures on Particles and Field Theory* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965).

[17] The helicity amplitudes are normalized such that the c.m. differential cross section is given by

$$\frac{d\sigma}{d\Omega}=\left(\frac{1}{8\pi W}\right)^2\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}\sum_{\{\lambda\}}|f_{\{\lambda\}}|^2,$$

where $W$ is the total c.m. energy and $|\mathbf{p}_f|$ and $|\mathbf{p}_i|$ are the magnitudes of the 3-momentum in the final and initial channels, respectively. In calculating the helicity amplitudes we have followed the phase convention of Ref. 2. The only difference between this and the Jacob-Wick convention is that the factor $(-1)^{s-\lambda}$ associated with "particle 2" in each channel is ignored.

Now we operate on both sides by $I$:

$$M^{\mu\nu}=I^{\mu\lambda}M_{\lambda\sigma}I^{\sigma\nu}=I^{\mu\nu}B_1+(IQQI)^{\mu\nu}B_3. \quad (32)$$

The terms containing $K^\mu$ vanish because

$$I^{\mu\lambda}K_\lambda=K^\mu-(k'\cdot K/k\cdot k')k^\mu=\tfrac{1}{2}k'^\mu$$

and this, when contracted with the photon polarization vector $\epsilon'(k')$, vanishes. Similarly, $K_\sigma I^{\sigma\nu}$ does not contribute. We are left with only two independent amplitudes. This is just the correct number for pion Compton scattering. Next, we have to remove the singularities contained in $I$ and $(IQQI)$. We note

$$(IQQI)^{\mu\nu}=Q^\mu Q^\nu-\frac{(K\cdot Q)}{(k\cdot k')}[k^\mu Q^\nu+Q^\mu k'^\nu]+\frac{(K\cdot Q)^2}{(k\cdot k')^2}k^\mu k'^\nu.$$

It is easily seen that the most singular term in this last expression can be removed by taking a linear combination with $I^{\mu\nu}$, while the remaining singularities can only be removed by multiplication by $(k\cdot k')$. In this way we obtain the minimal gauge-invariant polynomial expansion

$$M^{\mu\nu}=\mathfrak{L}_1A_1+\mathfrak{L}_2A_2, \quad (33)$$

with

$$\mathfrak{L}_1=(k\cdot k')I^{\mu\nu}=(k\cdot k')g^{\mu\nu}-k^\mu k'^\nu,$$
$$\mathfrak{L}_2=I^{\mu\lambda}[(k\cdot k')Q_\lambda Q_\sigma+(K\cdot Q)^2g_{\lambda\sigma}]I^{\sigma\nu} \quad (34)$$
$$=(k\cdot k')Q^\mu Q^\nu+(K\cdot Q)^2g^{\mu\nu}-(K\cdot Q)(k^\mu Q^\nu+Q^\mu k'^\nu).$$

This expansion has been written down before by various authors using conventional methods.

## VI. NUCLEON COMPTON SCATTERING

Since we shall go into more details in this physically more interesting case of nucleon Compton scattering, it is desirable to spell out the kinematics more explicitly. The photon momenta are denoted by $k$, $k'$ and the nucleon momenta by $p$, $p'$. We consider the $s$-channel reaction

$$\gamma+N\rightarrow\gamma+N$$

and the $t$-channel reaction

$$\gamma+\gamma\rightarrow\bar{N}+N.$$

We define three independent vectors by the $s$-channel relations

$$\Delta=k-k'=p'-p,$$
$$K=\tfrac{1}{2}(k+k'), \quad (35)$$
$$P=\tfrac{1}{2}(p+p').$$

When going to the $t$ channel, the signs of $k'$ and $p$ are reversed. The Mandelstam invariants are defined as usual.

$$s=(k+p)^2=m^2+2P\cdot K-\tfrac{1}{2}\Delta^2,$$
$$t=(k-k')^2=\Delta^2, \quad (36)$$
$$u=(k-p')^2=m^2-2P\cdot K-\tfrac{1}{2}\Delta^2.$$

We also have

$$K^2=-\tfrac{1}{4}t,\quad P^2=-\tfrac{1}{4}t+m^2,$$
$$P\cdot K=\tfrac{1}{4}(s-u), \quad (37)$$

and

$$P\cdot\Delta=K\cdot\Delta=0.$$

From the previous example we see that in writing down the minimal polynomial expansion for $M^{\mu\nu}$, terms containing $k^\mu$ or $k'^\nu$ are immediately eliminated by the projection operator. We shall not bother to write them down in this case (there are six terms of this kind). We have

$$M^{\mu\nu}=\sum l_i^{\mu\nu}B_i$$
$$=I^{\mu\lambda}M_{\lambda\sigma}I^{\sigma\nu}=\sum(Il_iI)^{\mu\nu}B_i, \quad (38)$$

where $I^{\mu\nu}$ is defined by (30) and

$$l_1=g^{\mu\nu},$$
$$l_2=g^{\mu\nu}(\gamma K),$$
$$l_3=P^\mu P^\nu,$$
$$l_4=P^\mu(\gamma K)P^\nu, \quad (39)$$
$$l_5=P^\mu\gamma^\nu+\gamma^\mu P^\nu,$$
$$l_6=\tfrac{1}{2}[\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu],$$
$$l_7=\tfrac{1}{2}[\gamma^\mu(\gamma K)\gamma^\nu-\gamma^\nu(\gamma K)\gamma^\mu].$$

Since there should be only six independent amplitudes for nucleon Compton scattering, the seven terms listed above are not completely independent. It is not easy to see which six among these comprise the minimal set before the singularities associated with them are removed. The appropriate thing to do is, therefore, first eliminate the singularities in a minimal way for all seven tensors and then remove the redundant one at the end. We see that

$$(Il_iI)^{\mu\nu}=l_i^{\mu\nu}-(1/k\cdot k')[k^\mu(k'\cdot l_i)^\nu+(l_i\cdot k)^\mu k'^\nu]$$
$$+[1/(k\cdot k')^2](k'\cdot l_i\cdot k)k^\mu k'^\nu, \quad (40)$$

where $(k'\cdot l_i)^\nu=k_\mu'\cdot l_i^{\mu\nu}$, $(l_i\cdot k)^\mu=l_i^{\mu\nu}k_\nu$, and $(k'\cdot l_i\cdot k)=k_\mu'l_i^{\mu\nu}k_\nu$. For $l_1$ and $l_2$ the last term in (40) can be combined with the previous one. For the rest this last term can be easily removed by taking the following combination:

$$I[l_i+((k'\cdot l_i\cdot k)/(k\cdot k'))l_1]I=l_i^{\mu\nu}-[1/(k\cdot k')]$$
$$\times[k^\mu(k'\cdot l_i)^\nu+(l_i\cdot k)^\mu k'^\nu-(k'\cdot l_i\cdot k)g^{\mu\nu}]. \quad (41)$$

The problem is therefore reduced to a form similar to the photoproduction case. To remove the remaining singularity, we tabulate the relevant quantities $(k'\cdot l_i)^\nu$, $(l_i\cdot k)^\mu$, and $(k'\cdot l_i\cdot k)$, which appear in the singular term (Table I). By inspection of the table we see that there can only be one singularity-free combination among the last five lines (since there are four independent vectors, namely, $P$, $(\gamma K)P$, $\gamma$, $[(\gamma K),\gamma]$ in each of the first two columns). This combination is $I[l_4-\tfrac{1}{2}(P\cdot K)l_5+\tfrac{1}{2}(P\cdot K)l_7]I$. For the rest, the only way to remove the singularity is to multiply by the factor $(k\cdot k')$. We then obtain six more singularity-free tensors, namely, $(k\cdot k')I$, $(k\cdot k')(\gamma K)I$, and $I[(k\cdot k')l_i+(k'\cdot l_i\cdot k)l_1]I$, $i=3, 5, 6, 7$.

As the last step we must remove the redundant tensor contained in this set and obtain the minimal gauge-invariant tensor basis. To do this it is necessary to remember that $M^{\mu\nu}$ is to be dotted into the polarization vectors, $\epsilon'(k')$ and $\epsilon(k)$. The space in which the tensors satisfy $\epsilon'\cdot M\cdot\epsilon\neq 0$, $k'\cdot M=M\cdot k=0$ is spanned by a basis which consists of direct products of the following two orthonormal vectors:

$$\hat{P}^\mu=[K^2P^\mu-(K\cdot P)K^\mu]\cdot[K^2((P\cdot K)^2-P^2K^2)]^{-1/2},$$
$$\hat{N}^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\Delta_\nu K_\lambda P_\sigma[K^2((P\cdot K)^2-P^2K^2)]^{-1/2}. \tag{42}$$

In fact, an independent basis set in terms of these vectors is known to be[18] $\hat{P}^\mu\hat{P}^\nu$, $\hat{N}^\mu\hat{N}^\nu$, $(\hat{P}^\mu\hat{N}^\nu-\hat{N}^\mu\hat{P}^\nu)i\gamma_5$, $\hat{P}^\mu\hat{P}^\nu(\gamma K)$, $\hat{N}^\mu\hat{N}^\nu(\gamma K)$, and $(\hat{P}^\mu\hat{N}^\nu+\hat{N}^\mu\hat{P}^\nu)i\gamma_5(\gamma K)$. If we dot the vectors $\hat{P}$ and $\hat{N}$ into the tensor basis set obtained in the last paragraph and express them in terms of this orthonormal set, we see immediately which ones are linearly dependent. Removing the one which can be expressed in terms of the others with polynomial coefficients, we obtain the following minimal gauge-invariant expansion of the $M$ function:

$$M^{\mu\nu}=\sum_{i=1}^{6}\mathfrak{L}_iA_i, \tag{43}$$

where

$$\begin{aligned}
\mathfrak{L}_1&=K^2Il_1I\\
&=K^2g^{\mu\nu}-2K^\mu K^\nu,\\
\mathfrak{L}_2&=K^2Il_7I\\
&=\tfrac{1}{2}K^2[\gamma^\mu(\gamma K)\gamma^\nu-\gamma^\nu(\gamma K)\gamma^\mu]-(P\cdot K)(K^\mu\gamma^\nu+\gamma^\mu K^\nu)\\
&\qquad+(\gamma K)(K^\mu P^\nu+P^\mu K^\nu),\\
\mathfrak{L}_3&=I[ml_2-(P\cdot K)l_1-K^2l_6]I\\
&=m(\gamma K)g^{\mu\nu}-(P\cdot K)g^{\mu\nu}-\tfrac{1}{2}K^2(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\\
&\quad+K^\mu\tfrac{1}{2}[(\gamma K),\gamma^\nu]+\tfrac{1}{2}[\gamma^\mu,(\gamma K)]K^\nu-m(K^\mu\gamma^\nu+\gamma^\mu K^\nu)\\
&\qquad+(K^\mu P^\nu+P^\mu K^\nu),\\
\mathfrak{L}_4&=I[K^2l_5-mK^2l_1+(P\cdot K)l_2]I\\
&=K^2(\gamma^\mu P^\nu+P^\mu\gamma^\nu)-(P\cdot K)(K^\mu\gamma^\nu+\gamma^\mu K^\nu)\\
&\quad-(\gamma K)(K^\mu P^\nu+P^\mu K^\nu)+(P\cdot K)g^{\mu\nu}(\gamma K)\\
&\qquad-mK^2g^{\mu\nu}+2mK^\mu K^\nu,\\
\mathfrak{L}_5&=I[K^2l_3-\tfrac{1}{2}(P^2K^2-(P\cdot K)^2)l_1]I\\
&=K^2P^\mu P^\nu-(P\cdot K)(K^\mu P^\nu+P^\mu K^\nu)\\
&\qquad-\tfrac{1}{2}(P^2K^2-(P\cdot K)^2)g^{\mu\nu}+P^2K^\mu K^\nu,\\
\mathfrak{L}_6&=I[l_4+\tfrac{1}{2}(P\cdot K)ml_1-\tfrac{1}{2}P^2l_2-\tfrac{1}{2}(P\cdot K)l_5+\tfrac{1}{2}mK^2l_6\\
&\qquad+\tfrac{1}{2}(P\cdot K)l_7]I\\
&=P^\mu P^\nu(\gamma K)-\tfrac{1}{2}(P\cdot K)(\gamma^\mu P^\nu+P^\mu\gamma^\nu)+\tfrac{1}{4}(P\cdot K)\\
&\quad\times[\gamma^\mu(\gamma K)\gamma^\nu-\gamma^\nu(\gamma K)\gamma^\mu]+\tfrac{1}{4}mK^2(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\\
&\quad+\tfrac{1}{2}m(P\cdot K)g^{\mu\nu}-\tfrac{1}{2}P^2g^{\mu\nu}(\gamma K)+K^\mu K^\nu(\gamma K)\\
&\quad+\tfrac{1}{2}m^2(K^\mu\gamma^\nu+\gamma^\mu K^\nu)-\tfrac{1}{2}m(K^\mu P^\nu+P^\mu K^\nu)\\
&\qquad-\tfrac{1}{4}mK^\mu[(\gamma K),\gamma^\nu]-\tfrac{1}{4}[\gamma^\mu,(\gamma K)]K^\nu.
\end{aligned}\tag{44}$$

In writing down the above expressions we have also made some inessential relabelling to make the comparison with previous work as simple as possible. The invariant amplitudes $A_i$ so defined are expected to be free of both kinematic singularities and zeros.

TABLE I. Quantities appearing in the singular term in Eq. (40).

| | $k'\cdot l_i$ | $l_i\cdot k$ | $k'\cdot l_i\cdot k$ |
|---|---|---|---|
| $l_1$ | $k'$ | $k$ | $(k\cdot k')$ |
| $l_2$ | $(\gamma K)k'$ | $k(\gamma K)$ | $(k\cdot k')(\gamma K)$ |
| $l_3$ | $(P\cdot K)P$ | $P(P\cdot K)$ | $(P\cdot K)^2$ |
| $l_4$ | $(P\cdot K)(\gamma K)P$ | $P(\gamma K)(P\cdot K)$ | $(P\cdot K)^2(\gamma K)$ |
| $l_5$ | $(P\cdot K)\gamma+(\gamma K)P$ | $\gamma(P\cdot K)+P(\gamma K)$ | $2(P\cdot K)(\gamma K)$ |
| $l_6$ | $\tfrac{1}{2}[(\gamma K),\gamma]-m\gamma+P$ | $\tfrac{1}{2}[\gamma,(\gamma K)]-m\gamma+P$ | $2[(P\cdot K)-m(\gamma K)]$ |
| $l_7$ | $(P\cdot K)\gamma-(\gamma K)P$ | $\gamma(P\cdot K)-P(\gamma K)$ | 0 |

Prange[18] and Hearn and Leader[19] had written down invariant amplitudes using the orthonormal basis set mentioned above (Prange did not normalize the two orthogonal vectors). The amplitudes of Prange have kinematic singularities as the basis tensors are not a *minimal* polynomial set while Hearn and Leader's amplitudes have kinematic zeros (constraints) as their basis tensors contain denominators that may vanish. Let us compare our amplitudes with those of Hearn and Leader. In order to distinguish the two sets we denote their amplitudes by $A_i'$. We have the relations

$$\begin{aligned}
\tfrac{1}{2}(A_1'+A_2')&=-K^2A_1+(P\cdot K)A_3,\\
A_3'&=(P\cdot K)A_3+mK^2A_2,\\
\tfrac{1}{2}(A_4'+A_5')&=mA_3;
\end{aligned}\tag{45}$$

$$\begin{aligned}
\tfrac{1}{2}(A_1'-A_2')&=-mK^2A_4+\tfrac{1}{2}[(P\cdot K)^2-K^2P^2]A_5\\
&\qquad+\tfrac{1}{2}m(P\cdot K)A_6,\\
\tfrac{1}{2}(A_4'-A_5')&=-(P\cdot K)A_4+\tfrac{1}{2}P^2A_6,\\
A_6'&=K^2A_4-\tfrac{1}{2}(P\cdot K)A_6;
\end{aligned}\tag{46}$$

and conversely

$$\begin{aligned}
A_1&=\frac{1}{K^2}\left[-\tfrac{1}{2}(A_1'+A_2')+\frac{(P\cdot K)}{m}\tfrac{1}{2}(A_4'+A_5')\right],\\
A_2&=\frac{1}{mK^2}\left[A_3'-\frac{(P\cdot K)}{m}\tfrac{1}{2}(A_4'+A_5')\right],\\
A_3&=\frac{1}{2m}(A_4'+A_5');
\end{aligned}\tag{47}$$

$$\begin{aligned}
A_4&=\frac{1}{[P^2K^2-(P\cdot K)^2]}[(P\cdot K)\tfrac{1}{2}(A_4'-A_5')+P^2A_6'],\\
A_5&=\frac{2}{[P^2K^2-(P\cdot K)^2]}[-mA_6'-\tfrac{1}{2}(A_1'-A_2')],\\
A_6&=\frac{2}{[P^2K^2-(P\cdot K)^2]}[K^2\tfrac{1}{2}(A_4'-A_5')+(P\cdot K)A_6'].
\end{aligned}\tag{48}$$

It is known that Hearn and Leader's amplitudes must

[18] R. Prange, Phys. Rev. 110, 240 (1958).

[19] A. C. Hearn and E. Leader, Phys. Rev. 126, 789 (1962).

satisfy constraint equations at $K^2=0$ ($t=0$) and $P^2K^2-(P\cdot K)^2=0$ ($su-m^4=0$). This is most easily seen by observing the relation between $A_i'$ and the $s$-channel helicity amplitudes as given by Hearn and Leader[19] [their Eq. (2.21)]. The right-hand side of their equations for $\Phi_3$ and $\Phi_6$ fail to reproduce the factor $\sin^2(\frac{1}{2}\theta_s)$ demanded by conservation of angular momentum for forward scattering, and similarly those for $\Phi_4$ and $\Phi_5$ fail to reproduce the factor $\cos^2(\frac{1}{2}\theta_s)$ for backward scattering. The constraint equations can thus be read off from their equations and after some simplification we obtain[20]

$$m(A_1'+A_2')-\tfrac{1}{2}(s-m^2)(A_4'+A_5')=0\,,$$
$$\frac{s-m^2}{2m}(A_4'+A_5')-2A_3'=0\,,\quad \text{at } t=0 \tag{49}$$

and

$$\tfrac{1}{2}(s-m^2)(A_4'-A_5')+(s+m^2)A_6'=0\,,$$
$$\frac{1}{2m}(A_1'-A_2')+A_6'=0\,,\quad \text{at } su=m^4. \tag{50}$$

Upon noting that $P\cdot K=\frac{1}{2}(s-m^2)$ at $t=0$, and that $K^2=(1/4s)(s-m^2)^2$, $P^2=(1/4s)(s+m^2)^2$, and $P\cdot K=(1/4s)\times(s^2-m^4)$ at $su=m^4$, we can easily see the right-hand side of all the equations in (47) and (48) remain finite even when the denominators vanish. The reciprocal relations (45)–(48) then clearly indicate that our invariant amplitudes are free of both kinematic singularities and zeros.

It is also useful to establish the relation between our amplitudes $A_i$ and the $s$- and $t$-channel helicity amplitudes which are more directly related to physically measureable quantities. The comparison between the two also furnish us with an independent check on our results since the kinematic-singularity structure of the helicity amplitudes have recently been fully analyzed.[2] For the $s$-channel helicity amplitudes we obtain[17]

$$f^s{}_{-1-\frac{1}{2},1\frac{1}{2}}=\sin\tfrac{1}{2}\theta\,\frac{1}{4\sqrt{s}}\{-t(s+m^2)A_1+mt(s-m^2)A_2+2[(s-m^2)^2-(su-m^4)]A_3\}\,,$$
$$f^s{}_{-1\frac{1}{2},1\frac{1}{2}}=\sin^2(\tfrac{1}{2}\theta)\cos\tfrac{1}{2}\theta\,\frac{m(s-m^2)^2}{4s}\{2A_1+2A_3\}\,, \tag{51}$$
$$f^s{}_{-1\frac{1}{2},1-\frac{1}{2}}=\sin^3(\tfrac{1}{2}\theta)\,\frac{(s-m^2)^2}{4s^{3/2}}\{(s+m^2)A_1+m(s-m^2)A_2+2m^2A_3\}\,;$$

$$f^s{}_{1\frac{1}{2},1\frac{1}{2}}=\cos\tfrac{1}{2}\theta\times\tfrac{1}{4}\{2[(s-m^2)^2-m^2t]A_4-m\times(su-m^4)A_5-[(s-m^2)^2+m^2t]A_6\}\,,$$
$$f^s{}_{1-\frac{1}{2},1\frac{1}{2}}=\sin\tfrac{1}{2}\theta\cos^2(\tfrac{1}{2}\theta)\,\frac{(s-m^2)^2}{4\sqrt{s}}\times\{-2mA_4-\tfrac{1}{2}(s+m^2)A_5-mA_6\}\,, \tag{52}$$
$$f^s{}_{1-\frac{1}{2},1-\frac{1}{2}}=\cos^3(\tfrac{1}{2}\theta)\,\tfrac{1}{4}(s-m^2)^2\{2A_4+mA_5+A_6\}\,,$$

where $\theta$ is the c.m. scattering angle, $\cos\theta=1+2st/(s-m^2)^2$. On the right-hand side of each equation we have factored out the characteristic angular factor for the helicity amplitudes and an additional momentum factor such that inside the curly brackets the coefficients of $A_i$ are polynomials in $s$ and $t$. These quantities inside the curly brackets are free of kinematic singularities in both variables $s$ and $t$. It can be easily checked that these are precisely the regularized helicity amplitudes found by Ader *et al.*[2] In addition, the kinematic-constraint equations on the helicity amplitudes found by these authors are automatically satisfied if Eqs. (51) and (52) are used.

For the $t$-channel reaction, we find the comparison can be made simpler by writing down the "parity-conserving" helicity amplitudes $\tilde{f}_{\{\lambda\}}$ in terms of our $A_i$. We obtain

$$\tilde{f}^+{}_{\frac{1}{2}\frac{1}{2},11}=f^t{}_{\frac{1}{2}\frac{1}{2},11}+f^t{}_{-\frac{1}{2}-\frac{1}{2},11}=\tfrac{1}{2}t^{1/2}[-mtA_2+(s-u)A_3]\,,$$
$$\tilde{f}^-{}_{\frac{1}{2}\frac{1}{2},11}=f^t{}_{\frac{1}{2}\frac{1}{2},11}-f^t{}_{-\frac{1}{2}-\frac{1}{2},11}=\frac{t}{2(t-4m^2)^{1/2}}\times[-(t-4m^2)A_1-(s-u)A_3]\,, \tag{53}$$
$$\tilde{f}_{\frac{1}{2}-\frac{1}{2},11}=(\sin\tfrac{1}{2}\psi\cos\tfrac{1}{2}\psi)^{-1}f^t{}_{\frac{1}{2}-\frac{1}{2},11}=t[mA_3]\,;$$

$$\tilde{f}_{\frac{1}{2}\frac{1}{2},1-1}=(\sin\tfrac{1}{2}\psi\cos\tfrac{1}{2}\psi)^{-2}f^t{}_{\frac{1}{2}\frac{1}{2},1-1}=t(t-4m^2)^{1/2}\times[\tfrac{1}{8}(t-4m^2)A_5+mA_4]\,,$$
$$\tilde{f}^+{}_{\frac{1}{2}-\frac{1}{2},1-1}=(\sin\tfrac{1}{2}\psi\cos^3\tfrac{1}{2}\psi)^{-1}f^t{}_{\frac{1}{2}-\frac{1}{2},1-1}+(\sin^3\tfrac{1}{2}\psi\times\cos\tfrac{1}{2}\psi)^{-1}f^t{}_{-\frac{1}{2}\frac{1}{2},1-1}=-t(t-4m^2)[\tfrac{1}{2}A_6]\,, \tag{54}$$
$$\tilde{f}^-{}_{\frac{1}{2}-\frac{1}{2},1-1}=(\sin\tfrac{1}{2}\psi\cos^3\tfrac{1}{2}\psi)^{-1}f^t{}_{\frac{1}{2}-\frac{1}{2},1-1}-(\sin^3\tfrac{1}{2}\psi\cos\tfrac{1}{2}\psi)^{-1}\times f^t{}_{-\frac{1}{2}\frac{1}{2},1-1}=t^{3/2}(t-4m^2)^{1/2}[A_4]\,,$$

where $\psi$ is the c.m. scattering angle,

$$\cos\psi=[t(t-4m^2)]^{-1/2}(s-u)\,.$$

Now, the angular factors appear on the left-hand side as part of the definition of $\tilde{f}_{\{\lambda\}}$. On the right-hand side we again factor out a kinematic term such that the quantities inside the square brackets are free of kinematic singularities. These again are just the regularized $t$-channel helicity amplitudes found by Ader *et al.*[2] It is easy to verify that in terms of our $A_i$, the kinematic-constraint equations on the helicity amplitudes found by these authors are automatically satisfied. As a last comment on this comparison with helicity amplitudes, we notice that the amplitudes $A_1$, $A_2$, $A_3$ and $A_4$, $A_5$, $A_6$ are naturally separated into two groups in all the Eqs. (45)–(54). The first group is related to photon helicity $(\lambda,\lambda')=(1,-1)$ in the $s$ channel and $(\lambda,\lambda')$

[20] The sign of $A_3'$ given below is opposite to that of Hearn and Leader. In recalculating their formula (2.21) we obtain this sign difference and cannot attribute it to a difference in the definition of $A_3'$ without also changing the sign of other $A_i'$s. Our results (47)–(50) are self-consistent.

$=(1,1)$ in the $t$ channel, while the second group is related to $(\lambda,\lambda')=(1,1)$ in the $s$ channel and $(1,-1)$ in the $t$ channel.

The $s$-$u$ crossing symmetry properties for the $A_i$'s are easily inferred from the definition of the tensor basis associated with them. We find that $A_1$, $A_2$, $A_4$, $A_5$ are even while $A_3$ and $A_6$ are odd under $s$-$u$ crossing.

To get some idea about the asymptotic behavior of these amplitudes, let us assume the helicity amplitudes are bounded by some common upper bound, say,

$$f^t{}_{\{\lambda\}}(s,t)\lesssim C(s) \quad \text{for } s\to\infty \text{ and fixed } t;$$

then

$$\begin{aligned} A_1,A_2&\lesssim C(s)\,,\\ A_3&\lesssim s^{-1}C(s)\,,\\ A_4,A_5,A_6&\lesssim s^{-2}C(s)\,. \end{aligned} \tag{55}$$

Similarly, if

$$f^s{}_{\{\lambda\}}(s,t)\lesssim C(t) \quad \text{for } t\to\infty \text{ and fixed } s\,,$$

then

$$\begin{aligned} A_1,A_3,A_4&\lesssim t^{-1/2}C(t)\,,\\ A_1+mA_2,A_5,(A_4+\tfrac{1}{2}A_6)&\lesssim t^{-3/2}C(t)\,. \end{aligned} \tag{56}$$

Recently, Mahoux and Martin[21] obtained Froissart bounds for helicity amplitudes. Their proof (using results from axiomatic quantum-field theory) does not cover the case of massless-particle scattering. If one conjectures that their results still apply, one should have

$$C(s)=Cs(\ln s)^{3/2}$$

and similarly for $C(t)$. We see that this is not enough to make any of the $A_i$'s superconvergent. To write down superconvergence relations for some of these amplitudes, it is necessary to make additional dynamical assumptions (like Regge asymptotic behavior).

## VII. MANDELSTAM REPRESENTATION

In the preceding sections we have demonstrated how one can construct kinematic-singularity-free and zero-free invariant amplitudes for photon processes. These invariant amplitudes *independently* have the analytic structure of a spinless amplitude. The simplest analytic structure would be for each amplitude to satisfy an unsubstracted Mandelstam representation of the form

$$\begin{aligned} A_i(s,t,u)=&\int ds'\frac{\rho_s{}^i(s')}{s'-s}+\int dt'\frac{\rho_t{}^i(t')}{t'-t}+\int du'\frac{\rho_u{}^i(u')}{u'-u}\\ &+\int ds'dt'\frac{\rho_{st}{}^i(s',t')}{(s'-s)(t'-t)}+\int dt'du'\frac{\rho_{tu}{}^i(t',u')}{(t'-t)(u'-u)}\\ &+\int du'ds'\frac{\rho_{us}{}^i(u',s')}{(u'-u)(s'-s)}. \end{aligned} \tag{57}$$

We have seen that the complication due to gauge invariance does not represent an essential obstacle in obtaining these invariant amplitudes. However, as already discussed briefly for the case of photoproduction, gauge invariance does have interesting implications as exemplified in the single-particle intermediate-state contribution (pole terms) to the scattering amplitudes. In the case of massive-particle scattering, the single-particle-exchange terms enter separately in the single-spectral functions. They are independent of each other. On the other hand, for photon processes, gauge invariance (or charge conservation) requires that the photon be coupled to all charged particles with the same strength. The charge couplings in the various channels are therefore correlated and cannot be treated independently. Only when all single-particle-exchange terms consistent with charge conservation are put together does one obtain the correct kinematic structure for the scattering amplitude (demanded by Lorentz invariance, etc.). As a consequence, we found the charge-coupling terms must contain a contribution to the *double-spectral functions* in addition to the single-spectral functions. We shall illustrate this point by examining the contribution of the single-particle-exchange terms to the invariant amplitudes introduced for the three photon processes considered in the previous sections.

### A. Photoproduction of Pion

In this case there are three single-particle diagrams. Their contribution to the $M$ function can be written as

$$\begin{aligned} M_\mu{}^{\alpha(B)}(qp',kp)=&(gi\gamma_5\boldsymbol{\tau}^\alpha)(\gamma p+\gamma k+m)\\ &\times(\mathbf{e}\gamma_\mu-(\boldsymbol{\mu}/2m)\sigma_{\mu\nu}k^\nu)(s-m^2)^{-1}+(\mathbf{e}\gamma_\mu-(\boldsymbol{\mu}/2m)\sigma_{\mu\nu}k^\nu)\\ &\times(\gamma p'-\gamma k+m)(gi\gamma_5\boldsymbol{\tau}^\alpha)(u-m^2)^{-1}+gi\gamma_5[\boldsymbol{\tau}^\alpha,\mathbf{e}]\\ &\times(2q-k)_\mu(t-m_\pi{}^2)^{-1}, \end{aligned} \tag{58}$$

where $\alpha$ is the isotopic spin index for the pion, $\boldsymbol{\tau}^\alpha$ are isotopic spin matrices for the nucleons, and the charge and magnetic moment matrices $\mathbf{e}$ and $\boldsymbol{\mu}$ are

$$\mathbf{e}=\begin{pmatrix} e & 0\\ 0 & 0\end{pmatrix},\quad \boldsymbol{\mu}=\begin{pmatrix} e\mu_p & 0\\ 0 & e\mu_n\end{pmatrix}.$$

We can obtain the contribution of these pole terms to the $A_i$'s by expanding $M_\mu{}^{\alpha(B)}$ in terms of the minimal tensor basis $\mathfrak{L}_i{}^\mu$. We find that these terms contribute to the single-spectral functions in $A_1$, $A_3$, $A_4$ and to the double-spectral function in $A_2$. The explicit results are[22]

$$\begin{aligned} A_1{}^{\alpha B}=&g\boldsymbol{\tau}^\alpha(\mathbf{e}+\boldsymbol{\mu})(m^2-s)^{-1}+g(\mathbf{e}+\boldsymbol{\mu})\boldsymbol{\tau}^\alpha(m^2-u)^{-1}\,,\\ A_3{}^{\alpha B}=&-g\boldsymbol{\tau}^\alpha(\boldsymbol{\mu}/2m)(m^2-s)^{-1}+g(\boldsymbol{\mu}/2m)\boldsymbol{\tau}^\alpha\\ &\times(m^2-u)^{-1}\,, \qquad (59)\\ A_4{}^{\alpha B}=&-g\boldsymbol{\tau}^\alpha(\boldsymbol{\mu}/2m)(m^2-s)^{-1}-g(\boldsymbol{\mu}/2m)\boldsymbol{\tau}^\alpha\\ &\times(m^2-u)^{-1}; \end{aligned}$$

[21] G. Mahoux and A. Martin (unpublished).

[22] The superscript $\alpha(+,-,0)$ is not the same as that of Ref. 12; it is defined in (58). Under $s$-$u$ crossing $A_2{}^+\leftrightarrow A_2{}^-$.

$$A_2^{+B}=-4g\mathbf{e}\boldsymbol{\tau}^+(m_\pi^2-t)^{-1}(m^2-u)^{-1}, \quad (\gamma n\to\pi^-p)$$
$$A_2^{0B}=4g\mathbf{e}(m^2-s)^{-1}(m^2-u)^{-1}, \quad (\gamma n\to\pi^0 n)\ (\gamma p\to\pi^0 p) \quad (60)$$
$$A_2^{-B}=-4g\boldsymbol{\tau}^-\mathbf{e}(m_\pi^2-t)^{-1}(m^2-s)^{-1}, \quad (\gamma p\to\pi^+ n).$$

The correlation between the charges of the particles involved and the particular combination of poles which appear in $A_2$ are only too apparent. This simple and natural manifestation of charge conservation is destroyed by using Ball's $B$ amplitudes[13] which are not gauge-invariant, resulting in the artificial interpretation of the $s$- and $u$-channel poles as dynamical poles and the $t$-channel pole as kinematic.

### B. Pion Compton Scattering

The single-particle-exchange contribution to the $M$ function is given by

$$M^{\mu\nu}(k',q';k,q)^{(B)}=e(2q'+k')^\mu e(2q+k)^\nu(s-m^2)^{-1} +e(2q'-k)^\nu e(2q-k')^\mu(u-m^2)^{-1}-2e^2g^{\mu\nu}. \quad (61)$$

Using the tensor basis of Eq. (34), we find the following contributions to the invariant amplitudes:

$$A_1^B=e^2[(m^2-s)^{-1}+(m^2-u)^{-1}],$$
$$A_2^B=8e^2(m^2-s)^{-1}(m^2-u)^{-1}. \quad (62)$$

The contribution to $A_2$ is again in the form of a double-spectral function.

### C. Nucleon Compton Scattering

The two single-nucleon-exchange diagrams give the following contribution to the $M$ function:

$$M^{\mu\nu}(k'p',k,p)^{(B)}=\left(\mathbf{e}\gamma^\mu+\frac{\boldsymbol{\mu}}{2m}\sigma^{\mu\tau}k_\tau'\right)(\gamma p'+\gamma k'+m) \times\left(\mathbf{e}\gamma^\nu-\frac{\boldsymbol{\mu}}{2m}\sigma^{\nu\lambda}k_\lambda\right)(s-m^2)^{-1}+\left(\mathbf{e}\gamma^\nu-\frac{\boldsymbol{\mu}}{2m}\sigma^{\nu\lambda}k_\lambda\right) \times(\gamma p'-\gamma k+m)\left(\mathbf{e}\gamma^\mu+\frac{\boldsymbol{\mu}}{2m}\sigma^{\mu\tau}k_\tau'\right)(u-m^2)^{-1}. \quad (63)$$

The contribution of these terms to the invariant amplitudes can again be evaluated by expanding the above expression in terms of the minimal tensor basis $\mathfrak{L}_i$, Eq. (44). We explicitly exhibit their contributions to the single- and double-spectral functions by writing

$$A_i^B(s,t,u)=R_i^+[(m^2-s)^{-1}+(m^2-u)^{-1}] +R_i^-[(m^2-s)^{-1}-(m^2-u)^{-1}] +R_i^{su}(m^2-s)^{-1}(m^2-u)^{-1}, \quad (64)$$

and the $R_i$'s are given in Table II.

TABLE II. The Born term residues $R_i$ in Eq. (64).

| $A_i$ | $R_i^{su}$ | $R_i^+$ | $R_i^-$ |
|---|---|---|---|
| $A_1$ | $4m\mathbf{e}^2$ | $-(2\boldsymbol{\mu}\mathbf{e}+\boldsymbol{\mu}^2)/2m$ | 0 |
| $A_2$ | $4\mathbf{e}(\mathbf{e}+\boldsymbol{\mu})$ | $(2\boldsymbol{\mu}\mathbf{e}+\boldsymbol{\mu}^2)/2m^2$ | 0 |
| $A_3$ | 0 | 0 | $(2\boldsymbol{\mu}\mathbf{e}+\boldsymbol{\mu}^2)/2m$ |
| $A_4$ | $4(\mathbf{e}+\boldsymbol{\mu})\mathbf{e}$ | $\boldsymbol{\mu}^2/2m^2$ | 0 |
| $A_5$ | $-8\boldsymbol{\mu}\mathbf{e}/m$ | 0 | 0 |
| $A_6$ | 0 | 0 | $-\boldsymbol{\mu}^2/m^2$ |

In addition to the nucleon-exchange terms required by charge conservation, there are also pseudoscalar meson poles in the $t$ channel, for instance, $\pi^0$ and $\eta$. These pole terms individually satisfy the gauge-invariance condition. The $M$ function for these terms is given by

$$M^{\mu\nu}(k'p';k,p)^{(P)}=(g_{N\pi}i\gamma_5\boldsymbol{\tau}^0)\epsilon^{\mu\rho\nu\sigma}k_\rho k_\sigma'F_\pi(m_\pi^2-t)^{-1} +(g_{N\eta}i\gamma_5)\epsilon^{\mu\rho\nu\sigma}k_\rho k_\sigma'F_\eta(m_\eta^2-t)^{-1}, \quad (65)$$

where $g_{N\pi}$, $g_{N\eta}$ are the meson-nucleon coupling constants and $F_\pi$, $F_\eta$ are the meson-decay constants. In terms of the invariant amplitudes, these pole terms only contribute to $A_2$. We have

$$A_2^{(P)}=(2/m)g_{N\pi}\boldsymbol{\tau}^0F_\pi(m_\pi^2-t)^{-1}+(2/m)g_{N\eta}F_\eta \times(m_\eta^2-t)^{-1}. \quad (66)$$

## VIII. LOW-ENERGY THEORIES

Recently, there has been renewed interest in low-energy theorems for photon processes.[6,23] It is shown[6] that these theorems follow simply from detailed analysis of the kinematic structure of the scattering amplitudes. As such, these low-energy theorems should be most easily derived from our invariant amplitudes because they are free from all kinematic singularities and zeros (in other words, all the kinematic structure are explicitly displayed in the tensor basis $\mathfrak{L}_i$). We shall see that this is indeed the case. Let us consider the physically more interesting case of nucleon Compton scattering. The invariant amplitudes can be separated into two parts: the Born term as given in the previous section and the continuum contribution:

$$A_i=A_i^B+A_i^C. \quad (67)$$

It is obvious from previous discussions that this separation is gauge-invariant and the two parts separately are free from kinematic singularities and zeros. Since $A_i^C$ do not contain any dynamical singularity in the limit of zero photon energy, the continuum contributions must remain finite. In addition, the invariant functions $A_3$ and $A_6$ being odd under $s$-$u$ crossing contain a factor $(s-u)$ which vanishes in the fixed angle, low-energy limit. Thus we conclude

$$A_3^C,A_6^C\sim\text{const}\times(s-u)\to 0,$$
$$A_1^C,A_2^C,A_4^C,A_5^C\sim\text{const}, \quad \text{at fixed } \theta \text{ and } s\to m^2. \quad (68)$$

[23] V. Singh, Phys. Rev. Letters **19**, 730 (1967); Phys. Rev. **165**, 1532 (1968); A. Pais, Phys. Rev. Letters **19**, 544 (1967); Nuovo Cimento **53**, 433 (1968).

The derivation of the low-energy theorems then consists in using Eq. (68) in Eqs. (51) and (52) to see to which order in the photon energy do these unknown continuum terms contribute to the physical-scattering amplitudes. All terms which are of lower order than these are then precisely determined by the known Born terms. For this purpose, the convenient variables to use are the $s$-channel scattering angle $\theta$, the magnitude $p$ of the three-momenta of the particles, and the nucleon energy $E$. These are related to $s$ and $t$ by

$$p=\frac{1}{2\sqrt{s}}(s-m^2)\,,\quad E=(p^2+m^2)^{1/2}=\frac{1}{2\sqrt{s}}(s+m^2)\,,\qquad \cos\theta=1+\frac{2st}{(s-m^2)^2}. \tag{69}$$

In terms of these variables, the $s$-channel helicity amplitudes are

$$\begin{aligned}
f^s_{-1-\frac{1}{2},1\frac{1}{2}}&=\sin\tfrac{1}{2}\theta\times\tfrac{1}{2}p^2[-E(1-\cos\theta)A_1+mp(1-\cos\theta)A_2\\
&\qquad+s^{1/2}(3+\cos\theta)A_3]\,,\\
f^s_{-1\frac{1}{2},1\frac{1}{2}}&=\sin^2\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta\; mp^2[A_1+A_3]\,,\\
f^s_{-1\frac{1}{2},1-\frac{1}{2}}&=\sin^3\tfrac{1}{2}\theta\,\frac{p^2}{\sqrt{s}}[s^{1/2}EA_1+ms^{1/2}pA_2+m^2A_3]\,,\\
f^s_{1\frac{1}{2},1\frac{1}{2}}&=\cos\tfrac{1}{2}\theta\times\tfrac{1}{2}p^2[(4s^{1/2}p+m^2(1+\cos\theta))A_4\\
&\qquad+\tfrac{1}{2}ms(1+\cos\theta)A_5\\
&\qquad-\tfrac{1}{2}(4s^{1/2}E-m^2(1+\cos\theta))A_6]\,,\\
f^s_{1-\frac{1}{2},1\frac{1}{2}}&=\sin\tfrac{1}{2}\theta\,\cos^2\tfrac{1}{2}\theta\,s^{1/2}p^2[-mA_4-\tfrac{1}{2}s^{1/2}EA_5-\tfrac{1}{2}mA_6]\,,\\
f^s_{1-\frac{1}{2},1-\frac{1}{2}}&=\cos^3\tfrac{1}{2}\theta\; sp^2[A_4+\tfrac{1}{2}mA_5+\tfrac{1}{2}A_6].
\end{aligned} \tag{70}$$

Using Eq. (68), it is immediately seen that to order $p^2$ the continuum contributions only enter through two constants, namely, $(A_1{}^C)_{s=m^2,t=0}$ and $(A_4{}^C+\frac{1}{2}mA_5{}^C)_{s=m^2,t=0}$. Thus for fixed $\cos\theta$, all the six helicity amplitudes are determined to order $p^2$ by the Born terms $A_i{}^{(B)}$ plus these two constants. This result has been obtained by Choudhury and Freedman using helicity amplitudes and crossing.[6]

It is quite obvious that the same method can be applied to derive low-energy theorems in other photon processes. However, we do not intend to do this in this paper.

In conclusion, we have seen that invariant amplitudes for photon processes which satisfy gauge invariance and are free of all kinematic singularities and zeros do exist and the techniques we proposed to find these amplitudes are rather general and effective. The invariant amplitudes thus found have very distinctive advantages over the regularized helicity amplitudes and the various types of invariant amplitudes previously proposed to describe photon processes. In particular, one may write down Mandelstam representations for these amplitudes without *ad hoc* substractions. Thus, these amplitudes should prove to be useful in the dispersion theoretical analysis of low- and medium-energy scattering data for photon processes.

*Note added in proof.* After this article was submitted, a paper on nucleon Compton scattering by Kurio Yamamoto appeared in Phys. Rev. **169**, 1353 (1968). He obtained a minimal tensor basis by direct algebraic reduction of the Prange basis.[18] His results are equivalent (in the sense defined in Sec. II) to those given in Eq. (44).


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