the shape of the form factors, when using a $1/r$ potential, is much improved over the harmonic-oscillator case.

The absolute normalization in the $1/r$ case, however, is much too small for most of the resonances. (See Table V and Figs. 1–5.) (The only resonance having form factors that agree with experiment[9,10] is the 1236 resonance.) The small normalization factors come about, roughly speaking, because of the energy-level dependence of the exponential e^{-br} in a Coulomb potential: The constant b is inversely porportional to n, where n is the label of the energy level of the excited quark. Thus for large n (high-lying resonances) the exponential does not damp as strongly, and to normalize the wave function one must divide by a larger number. This effect was not present in the harmonic-oscillator case, and agreement for small q^2 was obtained. Thus we conclude that the magnitude of the form factors, as well as their shape, depends on the potential chosen.

We also note that whereas for a harmonic-oscillator well the form factors are all proportional to the elastic form factors,[3] this is no longer true for a Coulomb potential. Finally, we present in Table VI the quark-model predictions for various photoproduction amplitudes.[11] The predicted magnitudes are in general too large, but the signs (when $M_q=\frac{1}{3}m_{\text{proton}}$) agree with experiment. The agreement is better (when $M_q=\frac{1}{3}m_{\text{proton}}$) for a $1/r$ potential than for the harmonic-oscillator well.

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Theory of Currents, σ Model, and the Spherical Top in the Internal Space*

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A Lagrangian field theory is constructed which gives a canonical realization of the recently proposed theory of currents. It is very similar to Gell-Mann and Levy's σ model, but with some crucial differences. It is the second-quantized theory of the spherical top in the internal space, thus implying some connection the strong-coupling theory.

1. INTRODUCTION

RECENTLY a simple nontrivial model field theory in which only currents appear as the coordinates was proposed.[1] The vector and axial-vector currents were taken to satisfy the algebra of fields implied by the massive Yang-Mills theory.[2] Then the energy-momentum tensor was given in terms of these currents:

$$\theta_{\mu\nu}=(1/2C)[\{V_\mu{}^iV_\nu{}^i+V_\nu{}^iV_\mu{}^i-g_{\mu\nu}(V_\rho{}^iV_\rho{}^i)\}+(V\rightarrow A)]. \quad (1)$$

This form of $\theta_{\mu\nu}$ determines the theory completely and it was shown that the theory does not contain any internal inconsistencies. In this theory we do not have canonical variables explicitly. The reason for this was studied by Bardakci, Frishman, and Halpern.[3] It turned out that this theory is a peculiar limit of the Yang-Mills theory. Nevertheless, we might still be able to find some canonical realization of the theory.

We indeed found a Lagrangian field theory which is equivalent to the original theory of currents, at least when the internal symmetry is SU_2 or $SU_2\times SU_2$. A very important feature of this Lagrangian theory is that, although we have canonical variables in it, we cannot attach particles directly to them because of their transformation property in the internal space. Actually, the theory is quite similar to the "σ model" of Gell-Mann and Lévy[4] except for the difference in the isospin rotation. Thus our theory is very much like the currently popular phenomenological Lagrangian theory,[5] at least in appearance. We can easily extend

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this σ model to the $SU_2\times SU_2$ case and incorporate the PCAC in a certain sense.

We can also consider the SU_3 theory without much difficulty, but we leave this to another publication.

In any case, we have examined the case of SU_2 more extensively and have rewritten the σ-model Lagrangian in terms of the Eulerian angle in the isospace. It turns out that this is a second-quantized version of the spherical top. In the first-quantized case (Schrödinger equation) we know that it gives the following energy spectrum:

$$E=aI(I+1). \tag{2}$$

In our field theory it is not so simple as this. In the case of the harmonic oscillator the spectrum is the same both in the Schrödinger equation and in the field theory because of the linearity of the equation. The highly nonlinear character of our equation prevents us from making the analogous discussion.

In Sec. 2 we present a Lagrangian formalism which is a particular solution to the original model, restricting ourselves to the SU_2 case. In Sec. 3 we extend this to the $SU_2\times SU_2$ case and incorporate the PCAC. In Sec. 4 we discuss the spherical top and related problems.

2. REDUCTION TO A LAGRANGIAN THEORY

Let us start our discussion by writing the equations of motion in our model. In the case of SU_2 symmetry, that is, when only itotopic spin vector currents appear in the theory, the equations are[1]

$$\partial_\mu V_\nu{}^i-\partial_\nu V_\mu{}^i=(1/2C)\epsilon_{ijk}(V_\mu{}^jV_\nu{}^k+V_\nu{}^kV_\mu{}^j), \tag{3}$$

$$\partial_\mu V_\mu{}^i=0, \tag{4}$$

where i, j, k run from 1 to 3. Introducing 2×2 vector matrices $V_\mu=\tau^iV_\mu{}^i$ ($\tau^i=$Pauli matrices), Eq. (3) is simplified:

$$\partial_\mu V_\nu-\partial_\nu V_\mu=-(i/2C)[V_\mu,V_\nu]. \tag{5}$$

Note that c-number terms $[V_\mu{}^i(x),V_\nu{}^i(x)]$ vanish because of our assumption on the equal-time commutation relations.

From the form of Eq. (5) it is easy to show that $\tilde{V}_\mu=U^{-1}V_\mu U-2iCU^{-1}\partial_\mu U$ is also a solution of Eq. (5) when V_μ satisfies (5) and $U(x)$ is an arbitrary matrix. In particular,

$$V_\mu=-2iCU^{-1}\partial_\mu U \tag{6}$$

satisfies (5). We express U in terms of new fields, $\varphi^i(x)$ and $\sigma(x)$, whose properties will be specified below:

$$U=\tau^i\varphi^i-i\sigma. \tag{7}$$

To insure the Hermiticity of V_μ, we require U to be unitary.[6] Then

$$\varphi^i\varphi^i+\sigma^2=1,$$
$$[\varphi^i(x),\varphi^i(x)]=[\varphi^j(x),\sigma(x)]=0. \tag{8}$$

[6] This is, of course, except for the normalization.

Using these variables, the currents are

$$V_\mu=2C\tau^i(\epsilon_{ijk}\varphi^j\partial_\mu\varphi^k-\varphi^i\partial_\mu\sigma+\sigma\partial_\mu\varphi^i)$$

or

$$V_\mu{}^i=2C(\epsilon_{ijk}\varphi^j\partial_\mu\varphi^k-\varphi^i\partial_\mu\sigma+\sigma\partial_\mu\varphi^i). \tag{9}$$

In deriving this equation we used the fact that

$$[\partial_0\varphi^i(\mathbf{x},t),\varphi^i(\mathbf{y},t)]+[\partial_0\sigma(\mathbf{x},t),\sigma(\mathbf{y},t)]_{\mathbf{x}\to\mathbf{y}}$$

becomes a singular c-number term which can be removed from the expression for $V_\mu(x)$ without affecting the physical content of the theory. It can be rigorously justified only after we establish the commutation relations among φ^i and σ [Eqs. (14)].

So far we have considered Eq. (3). Equation (4) implies

$$\epsilon_{ijk}\varphi^j\Box\varphi^k-\varphi^i\Box\sigma+\sigma\Box\varphi^i=0. \tag{10}$$

Furthermore, condition (8) and the currents (9) have the same forms as the ones considered in the nonlinear σ model by Gell-Mann and Lévy.[4] Thus we are led to the following possible Lagrangian in our field theory of currents:

$$L=-(1/2C)V_\mu{}^iV_\mu{}^i. \tag{11}$$

Using Eqs. (8) and (9) and regarding the φ^i, σ as classical fields, this can be rewritten

$$L=-2C(\partial_\mu\varphi^i\partial_\mu\varphi^i+\partial_\mu\sigma\partial_\mu\sigma). \tag{12}$$

We can also start from the Lagrangian (12) with the constraint (8) and follow the conventional canonical formalism. The results are as follows: The equation of motion is

$$\varphi^i\Box\sigma-\sigma\Box\varphi^i=0, \tag{13}$$

and the canonical quantization is

$$[\varphi^i(x),\varphi^j(y)]_{x_0=y_0}=0,$$
$$[\varphi^i(x),\partial_0\varphi^j(y)]_{x_0=y_0}=(i/4C)(\delta_{ij}-\varphi^i\varphi^j)\delta^3(x-y),$$
$$[\varphi^i(x),\partial_0\sigma(y)]_{x_0=y_0}=-(i/4C)\varphi^i\sigma\delta^3(x-y),$$
$$[\partial_0\varphi^i(x),\partial_0\varphi^j(y)]_{x_0=y_0}$$
$$=-(i/4C)(\varphi^i\partial_0\varphi^j-\varphi^j\partial_0\varphi^i)\delta^3(x-y). \tag{14}$$

Our next important task is to check the commutation relations of the algebra of fields. This can be done straightforwardly, using Eqs. (14). We simply note that the σ term in Eq. (9) is essential to obtain the correce c-number Schwinger term which otherwise would be a q number.

Although the similarity of our model to the σ model is quite obvious, there are two important differences. First, φ^i and σ must have the same parity in our model. Second, and more important in the development of the theory, the transformation properties of φ^i and σ in SU_2 space are different from those in the σ model. In fact, defining the total isotopic spin I^i,

$$I^i=\int V_0{}^i(x)d^3x.$$

We obtain

$$[I^i,\varphi^j(x)]=\tfrac{1}{2}i\epsilon_{ijk}\varphi^k(x)-\tfrac{1}{2}i\delta_{ij}\sigma(x)\,,$$
$$[I^i,\sigma(x)]=\tfrac{1}{2}i\varphi^i(x)\,. \qquad (15)$$

These equations definitely indicate that we are treating fields

$$\psi=\begin{pmatrix}\varphi^1-i\varphi^2\\ -\varphi^3+i\sigma\end{pmatrix} \qquad (16)$$

which transforms like an isodoublet rather than an isotriplet φ^i. The point is that the σ term is contained in the isotopic spin current, whereas in the Gell-Mann–Lévy model the σ term gives an axial-vector current. In terms of ψ, the equations take a more symmetric form:

$$L=-C(\partial_\mu\psi^\dagger\partial_\mu\psi+\partial_\mu\psi\partial_\mu\psi^\dagger)\,, \qquad (17)$$

$$V_\mu{}^i=-\tfrac{1}{2}iC(\partial_\mu\psi^\dagger\tau^i\psi-\psi^\dagger\tau^i\partial_\mu\psi+\text{symmetrized terms})\,. \qquad (18)$$

The equation of motion (13) guarantees the conservation of the σ part of our vector current (axial-vector part in the Gell-Mann–Lévy model). It is clear that (13) is equivalent to (10) because $\epsilon_{ijk}\varphi^j\ \varphi^k=0$. The important thing here is that we have an extra conserved current in our theory. In fact this is obvious from our Lagrangian being O_4 symmetric. The restriction of the c-number Schwinger term forces us to choose this particular combination of the two currents. Because our original current theory does not imply this kind of symmetry at all, it implies that we are choosing a very special solution.

3. EXTENSION TO $SU_2\times SU_2$ AND PCAC

In order to incorporate the axial-vector current $A_\mu{}^i(x)$, we introduce another independent set of fields $\tilde{\psi}(x)$. The extended Lagrangian is (for simplicity, we do not symmetrize the formulas below)

$$L=-C(\partial_\mu\psi^\dagger\partial_\mu\psi+\partial_\mu\tilde{\psi}^\dagger\partial_\mu\tilde{\psi})\,, \qquad (19)$$

with the constraints

$$\psi^\dagger\psi=\tilde{\psi}^\dagger\tilde{\psi}=1\,. \qquad (20)$$

The definition of space inversion P is

$$P\psi(\mathbf{x},t)P^{-1}=\tilde{\psi}(-\mathbf{x},t)\,,$$
$$P\tilde{\psi}(\mathbf{x},t)P^{-1}=\psi(-\mathbf{x},t)\,. \qquad (21)$$

Under the infinitesimal transformations with gauge parameters $\boldsymbol{\varepsilon}$, $\tilde{\boldsymbol{\varepsilon}}$,

$$\psi\rightarrow(1+\tfrac{1}{2}i\boldsymbol{\varepsilon}\cdot\boldsymbol{\tau})\psi\,,$$
$$\tilde{\psi}\rightarrow(1+\tfrac{1}{2}i\tilde{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\tau})\tilde{\psi}\,, \qquad (22)$$

the following two currents are generated, respectively, from Eq. (19):

$$V_\mu{}^i+A_\mu{}^i=-iC(\partial_\mu\psi^\dagger\tau^i\psi-\psi^\dagger\tau^i\partial_\mu\psi)\,,$$
$$V_\mu{}^i-A_\mu{}^i=-iC(\partial_\mu\tilde{\psi}^\dagger\tau^i\tilde{\psi}-\tilde{\psi}^\dagger\tau^i\partial_\mu\tilde{\psi})\,. \qquad (23)$$

It is easy to see that the vector and axial-vector currents given by Eqs. (23) satisfy the commutation relations of the albebra of fields, and the equation of motion given in Ref. 1. The fields which have definite parity are easily defined to be

$$\psi_S=\tfrac{1}{2}\sqrt{2}(\psi+\tilde{\psi})\ \text{for scalar field}\,,$$
$$\psi_P=\tfrac{1}{2}\sqrt{2}(\psi-\tilde{\psi})\ \text{for pseudoscalar field}\,.$$

Next, we try to incorporate PCAC, where we destroy only the conservation of the axial-vector current by introducing an additional term to the above Lagrangian. Writing the Lagrangian

$$L=L_{\text{inv}}+L'(\psi,\tilde{\psi})\,, \qquad (24)$$

with L_{inv} given by Eq. (19), we get the partial conservation under the same transformation as (22):

$$\partial_\mu(V_\mu{}^i+A_\mu{}^i)=i\frac{\delta}{\delta\epsilon^i}L'\,,$$
$$\partial_\mu(V_\mu{}^i-A_\mu{}^i)=i\frac{\delta}{\delta\tilde{\epsilon}^i}L'\,. \qquad (25)$$

One might think that the linear term proportional to $\sigma+\tilde{\sigma}$ in L' would give us PCAC in analogy with the σ model. However, this is not correct, because it also yields the breaking of vector-current conservation. Thus the simplest realization of PCAC is achieved by the following L':

$$L'=f(\varphi^i\varphi^i+\sigma\tilde{\sigma})$$
$$=f\psi^\dagger\tilde{\psi}\,, \qquad (26)$$

where f is some constant. We can easily check that

$$\partial_\mu V_\mu{}^i=0\,,$$
$$\partial_\mu A_\mu{}^i=if\psi^\dagger\tau^i\tilde{\psi}\,. \qquad (27)$$

Note that the effective pion field is proportional to $\psi^\dagger\tau^i\tilde{\psi}$.

The equations of motion (3) (and the similar equations for $A_\mu{}^i$) are satisfied because the structure of the currents is unchanged by the extra term in the Lagrangian.

Finally, since our $\theta_{\mu\nu}$ contains the term $fg_{\mu\nu}\psi^\dagger\tilde{\psi}$ explicitly in addition to the original current term, we can no longer claim that we have only vector and axial-vector currents in the theory. We are not particularly fond of this kind of theory with additional field. It would be very interesting to see if it is possible to incorporate the PCAC within a theory where only $V_\mu{}^i$ and $A_\mu{}^i$ appear.

4. SPHERICAL TOP IN THE INTERNAL SPACE

As we have seen above, the simplest version of our theory of currents in the SU_2 case is equivalent to the

σ model:

$$L=-2C(\partial_\mu\varphi^i\partial_\mu\varphi^i+\partial_\mu\sigma\partial_\mu\sigma)\,, \tag{12}$$

$$\varphi^i\varphi^i+\sigma^2=1\,, \tag{8}$$

where

$$\begin{pmatrix}\varphi^1-i\varphi^2\\ -\varphi^3+i\sigma\end{pmatrix}$$

transforms like a doublet under isospin rotation.

We introduce Eulerian angles in the internal space by the following definition:

$$\begin{pmatrix}\varphi^1-i\varphi^2\\ -\varphi^3+i\sigma\end{pmatrix}=\begin{pmatrix}\sin\frac{1}{2}\theta\; e^{i(\varphi-\psi)/2}\\ \cos\frac{1}{2}\theta\; e^{i(\varphi+\psi)/2}\end{pmatrix}. \tag{28}$$

Then it is easy to see that condition (8) is satisfied automatically. We can rewrite our σ-model Lagrangian in terms of these variables. The result is

$$L=-\tfrac{1}{2}C[(\partial_\mu\theta)^2+(\sin\theta\;\partial_\mu\varphi)^2+(\partial_\mu\psi+\cos\theta\;\partial_\mu\varphi)^2]. \tag{29}$$

The reader may immediately notice the similarity of this Lagrangian to that of the symmetrical top in free space, C being the moment of inertia. The essential difference is that we have the top in the internal space attached to every point in the external space. Moreover, we have space-derivative terms which are obviously absent in the ordinary top Lagrangian.

The quantization of this system is quite conventional:

$$\pi_\theta=\delta L/\delta\partial_0\theta=C\partial_0\theta\,, \tag{30}$$

$$\pi_\varphi=C(\partial_0\varphi+\cos\theta\;\partial_0\psi)\,, \tag{31}$$

$$\pi_\psi=C(\partial_0\psi+\cos\theta\;\partial_0\varphi)\,. \tag{32}$$

We can check that this choice of canonical variables is consistent with the commutation relations among φ^i and $\partial_0\varphi^j$, and therefore with the original current-commutation relations.

In terms of θ, φ, and ψ the current densities are

$$\begin{aligned}J_\mu{}^1&=C(\sin\psi\;\partial_\mu\theta-\sin\theta\cos\psi\;\partial_\mu\varphi)\,,\\ J_\mu{}^2&=C(-\cos\psi\;\partial_\mu\theta-\sin\theta\sin\psi\;\partial_\mu\varphi)\,,\\ J_\mu{}^3&=C(\partial_\mu\psi+\cos\theta\;\partial_\mu\varphi)\,.\end{aligned} \tag{33}$$

In particular, $J_0{}^3(x)=\pi_\psi(x)$. This means that in order to diagonalize the isospin we have to diagonalize $\int\pi_\psi(x)d^3x$.

It is well known[7] that the first-quantized spherical top is a soluble problem. In fact, it appeared in particle physics through the strong-coupling theory.[8] But our second-quantized theory does not seem to be so easy to solve. The perturbation method is incorrect here because θ, φ, and ψ do not correspond to real particles though they are canonical. The reason is, of course, that they do not have definite isospin. The problem is to find out the vacuum state which has the required property: Lorentz invariance and isospin invariance. We have not succeeded in solving this problem.

We have tried to apply the method used in solving the Ising and Heisenberg models for ferromagnetism.[9] However, because we have space components of the currents in our Hamiltonian, we could not go through with it. Perhaps a more sophisticated version of the method would work, at least to get a reasonable approximation.

Finally, the fact that the Schwinger constant C is the moment of inertia could be guessed immediately if one looks at the original Hamiltonian in terms of current densities:

$$H=\frac{1}{2C}\int[J_0{}^i(x)J_0{}^i(x)+J_a{}^i(x)J_a{}^i(x)]d^3x. \tag{34}$$

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