An Exact Solution for the Scattering of Electromagnetic Waves from Conductors of Arbitrary Shape. I. Case of Cylindrical Symmetry*

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The problem of the scattering of an electromagnetic plane wave, incident along the axis of symmetry on a cylindrically symmetric, though otherwise arbitrarily shaped conductor, is solved exactly by means of a perturbation-expansion technique developed for this purpose. The solution obtained is an exact analytical solution, equally valid in the near and far zones, as well as over the entire frequency range, including the resonance region. The general solution is obtained, and several special cases are treated in detail. The term-by-term agreement of the perturbation-series solution with the known exact solution is demonstrated analytically for the case of a sphere. The form of the solution is particularly well suited for methodical numerical evaluation by machine calculation.

I. INTRODUCTION

THE theoretical calculation of the scattering of electromagnetic waves by objects of arbitrarily specified shape and electromagnetic structure is a problem which has many and varied practical applications. Mathematically, this problem is represented by an exceedingly complicated boundary-value problem. Despite the prodigious amount of effort which has been expended in its analysis, an exact solution for the general case remains yet to be found. This is true even for the special case where the scattering object is a perfect conductor, which is the immediate concern of this paper.[1]

Analytically rigorous solutions to this problem have generally been based on the method of separation of variables and on the expansion of the general solution of the vector wave equation in terms of appropriate orthogonal functions.[2,3] However, in order for this method to be applicable, two requirements must be fulfilled. To begin with, the vector wave equation must be separable in some suitable coordinate system, and the resulting differential equations must be analytically solvable. Secondly, the relevant boundary conditions must have a simple form in the coordinate system selected, which generally requires that the scattering object in question must constitute a complete coordinate surface in the coordinate system chosen for the separation of variables. As a consequence of these limitations, *exact* solutions to the three-dimensional problem of the scattering of electromagnetic waves have been obtained by this method only for the cases of a sphere and an infinite cylinder.[4]

Because of the analytical complexity of the over-all boundary-value problem, a large number of approximate methods have been developed for dealing with the scattering problem. Foremost among these are various variational techniques.[5,6] However, variational principles can be developed for only a few of the physical quantities of interest; moreover, their success depends to a large extent on the ingenuity shown in choosing a suitable trial function. For the extreme cases where the wavelength of the incident radiation is either very large or very small compared to the characteristic dimension of the scatterer, satisfactory results are obtained by the methods of the Rayleigh scattering approximation[7] and by the methods of geometric and physical optics,[8] respectively. However, when these approximate methods are pushed beyond the extreme ranges of the wavelength, their physical basis becomes cloudy, and the reliability of the corresponding numerical results is generally uncertain. They are intrinsically incapable of yielding meaningful results in the important resonance region.

In light of the foregoing remarks, it is evident that there still exists a need for a straightforward method for treating the scattering from conductors of arbitrary shape which is of sufficient generality to encompass both a great variety of shapes and a broad range of the frequency spectrum including the resonance region. The present work is intended to provide such a method. It should be emphasized from the outset that the solution obtained here represents an *exact analytical solution* of the scattering problem, valid for *all* values of the relevant variables and physical parameters, for which the series solution converges.

The spirit of the method described here is to return to the original mathematical boundary-value problem in its full complexity, and to approach it analytically by means of a boundary-perturbation technique specially designed for this problem.[9] In an earlier paper,[10] the

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[1] The extension of any general method valid for conductors to the case of a constant refractive index is mathematically trivial in most cases.

[2] R. King and T. T. Wu, *The Scattering and Diffraction of Waves* (Harvard University Press, Cambridge, Mass., 1959).

[3] C. J. Bouwkamp, Rept. Progr. Phys. **17**, 35 (1954).

[4] For the simpler two-dimensional case, the class of shapes accesible to this method is slightly larger.

[5] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953).

[6] H. Levine and J. Schwinger, *Theory of Electromagnetic Waves* (Interscience Publications, Inc., New York, 1951).

[7] A. F. Stevenson, J. Appl. Phys. **24**, 1134 (1953).

[8] V. A. Fock, J. Phys. (USSR) **10**, 130 (1946); **10**, 399 (1946); J. B. Keller, J. Opt. Soc. Am. **52**, 102 (1962).

[9] For a review of other boundary-perturbation methods in mathematical physics, see Ref. 5.

[10] V. A. Erma, J. Math. Phys. **4**, 1517 (1963).

author was successful in developing a particularly simple and straightforward boundary-perturbation technique for treating the electrostatic problem (scalar problem) for irregularly shaped conductors, which yielded an analytical expression for the capacitance of such conductors, valid to all orders in the perturbation parameter. The possibility of generalizing this approach to the much more complicated case of the vector wave equation was first suggested by the author, who also obtained an explicit first-order solution for the special case of cylindrically symmetric conductors, with the incident electromagnetic wave along the axis of symmetry.[11] A number of other steps in this direction have been reported by various authors,[12] in particular by Yeh.[13,14] However, these were limited to results valid only to the first order in the perturbation, or were otherwise restricted. Inasmuch as the work of Yeh[13] closely resembles the approach of the present paper, it warrants more detailed discussion. Yeh treats the case of a scatterer of constant refractive index. However, as noted earlier, the generalization from the case of a perfect conductor to this case is mathematically trivial. He obtains a formal solution for a general form of the perturbed boundary, but calculates an explicit solution only for the restricted case of cylindrical symmetry, with the incident wave along the axis of symmetry, which corresponds to the case considered here and in Ref. 11. More important is, however, that Yeh developed the method only to first order in the perturbation parameter, which restricts its practical usefulness to shapes differing only minimally from that of a sphere. Furthermore, due partly to the particular analytic form chosen to represent the perturbation, the resulting analytical expressions are needlessly complicated.[15] Accordingly, while his claim that his method may be extended to higher orders is true "in principle," it is virtually impossible, or in any case much too cumbersome to be practically useful, to do so "in practice." Beyond the very first few orders, the analytical work required to determine the contribution of each term to a given order becomes precipitously prohibitive.

In the present paper, we restrict our considerations to the case of a cylindrically symmetric perfect conductor, with the incident wave traveling along the axis of symmetry. The analytical solution obtained here is exact to all orders of the perturbation parameter, and is valid at all points of space (both near and far zone), and for any arbitrary frequency of the incident wave.

The problem to be treated and the method of approach are formulated in Sec. II. In Sec. III we obtain the general analytic series solution, valid to *all orders* in the perturbation parameter. Section IV is devoted to the consideration of some special cases of the general solution. Thus, to begin with, we obtain the solution for a sphere of radius $a(1+\epsilon)$ (considered as a perturbation of a sphere of radius a) and show that it agrees with the known exact solution for this case. As an illustration of the basic simplicity of the formalism, the special case of the first-order solution is presented explicitly, and applied to the calculation of the scattering from an oblate spheriod. Finally, Sec. V encompasses various concluding remarks, as well as indications of future work.

II. FORMULATION OF THE PROBLEM

The problem we shall consider here consists of the scattering of an electromagnetic plane wave, incident along the axis of symmetry on a cylindrically symmetric perfect conductor. Specifically, we shall consider conductors, whose boundary surface can be described in spherical coordinates by an equation of the form

$$r=r_s(\theta)=a[1+\epsilon f(\theta)]. \tag{1}$$

Here a is a constant, representing the radius of the "unperturbed sphere," ϵ is a constant "smallness parameter," and $f(\theta)$ is a function which must obey the restriction

$$|\epsilon f(\theta)|<1, \quad 0\leq\theta\leq\pi \tag{2}$$

but is otherwise arbitrary.

Two remarks should be made concerning the representation of arbitrary boundary surfaces by means of Eqs. (1) and (2). To begin with, it would appear at first sight that, in view of restriction (2), Eq. (1) is capable only of describing irregular surfaces which do not deviate excessively from a spherical shape. However, we must recall that both the value a of the radius of the unperturbed sphere, and the location of the center of the spherical coordinate system may be chosen arbitarily. Hence, it is clear that all cylindrically symmetric irregular shapes, for which it is possible to locate the center of the coordinate system in such a way that the radius vector to all points on the surface is single-valued,[16] can be described by Eqs. (1) and (2). For example, all simply connected, convex shapes fall into this class. The question of the optimum choice of the unperturbed sphere for any given conductor is considered in more detail in Appendix A.

Secondly, the above analytic form for the equation of the irregular surface, considered as a perturbation of

[11] V. A. Erma, Plasmadyne Corporation Report Nos. PTL-2-607 and PTL-2-607* 1963 (unpublished).

[12] P. C. Clemmon and V. H. Weston, Proc. Roy. Soc. (London) **A264**, 246 (1961); Lu, Acta Phys. Sinica, Peking **22**, 223 (1966); **21**, 1798 (1965); T. Oguchi, J. Radio Res. Lab. (Japan) **7**, 467 (1960); **11**, 19 (1964); M. L. Burrows, Can. J. Phys. **45**, 1729 (1967); C. J. Marcinkowski and L. B. Filsen, J. Res. Natl. Bur. Std. **66D**, 699 (1962); **66D**, 707 (1962).

[13] C. Yeh, Phys. Rev. **135**, A1193 (1964).

[14] C. Yeh, J. Math. Phys. **6**, 2008 (1964).

[15] In this connection, it should also be noted that the paper of Yeh (Ref. 13) contains errors of sign and normalization. Thus, Eq. (8) of Ref. 13 is incorrect in sign; this error persists throughout all subsequent equations derived therefrom, and presumably also affects the validity of the numerical results presented. We may also point out that the integrals given in the Appendix of Ref. 13, which are there evaluated by numerical machine computation, can very readily be evaluated analytically.

[16] These are known in mathematical parlance as surfaces which admit of a "radial single-valued explicit representation."

a sphere, differs from that used earlier by the author,[10] as well as by Yeh.[13] In both cases, the irregular surface was expressed in the form

$$r=r_s(\theta)=a[1+\epsilon f_1(\theta)+\epsilon^2 f_2(\theta)+\cdots], \quad |\epsilon f_1(\theta)+\epsilon^2 f_2(\theta)+\cdots|<1, \quad 0\leq\theta\leq\pi. \tag{3}$$

It is clear, however, that any surface which can be described by Eq. (3) can also be expressed in the simpler form (1), with a suitable choice of ϵ and $f(\theta)$. While form (3) was particularly well suited to the scalar-electrostatic problem,[10] it leads to needless complications in the vector problem of electromagnetic scattering, and, in fact, makes it impossible to obtain a single analytic expression valid to all orders in the perturbation.

Assuming a time dependence of $e^{-i\omega t}$ for the electromagnetic fields, the scattering problem is then defined by the boundary-value problem consisting of the vector Helmholtz equation for the fields outside the scatterer, and the boundary condition that the total tangential electric field must vanish on the surface of the conductor.

This boundary-value problem will here be attacked by means of a perturbation technique. Unlike in the more familiar perturbation methods of quantum mechanics, where the partial differential equation is perturbed, we are here dealing with perturbed boundaries. Since in this case the partial differential equation remains unchanged, it is not necessary to solve the vector Helmholtz equation anew for each particular shape. Instead, we must match boundary conditions at the perturbed boundary. As in the scalar case,[10] this may be accomplished by expanding the boundary condition in a Taylor series, which in effect transforms the boundary conditions at the perturbed boundary into a succession of boundary conditions at the unperturbed boundary. This is tantamount to replacing the single "necessary" boundary condition by an infinite set of "sufficient" boundary conditions. That this is indeed a consistent procedure may be demonstrated by means of the uniqueness theorem for the solutions of the relevant partial differential equation.

The point of departure for the solution for an irregular shape is provided by the known exact solution for the problem of the scattering of a plane wave from a perfectly conducting sphere. For later reference, we briefly review some of the well-known results[17] in connection with this problem. For the case of a cylindrically symmetric scatterer with the incident plane wave along the axis of symmetry (chosen to be the z axis), the general solution of the vector Helmholtz equation can be expanded in terms of two so-called "unit fields" $\mathbf{M}_n$ and $\mathbf{N}_n$, which are given by (the time dependence $e^{-i\omega t}$ has been suppressed *ab initio*)

$$\mathbf{M}_n=(\sin\theta)^{-1}z_n(\rho)P_n{}^1(x)\cos\varphi\,\mathbf{e}_\theta-z_n(\rho)\times(dP_n{}^1(x)/d\theta)\sin\varphi\,\mathbf{e}_\varphi, \tag{4}$$

$$\mathbf{N}_n=n(n+1)\rho^{-1}z_n(\rho)P_n{}^1(x)\cos\varphi\,\mathbf{e}_r+\rho^{-1}(d/d\rho)[\rho z_n(\rho)](dP_n{}^1(x)/d\theta)\cos\varphi\,\mathbf{e}_\theta-(\rho\sin\theta)^{-1}(d/d\rho)[\rho z_n(\rho)]P_n{}^1(x)\sin\varphi\,\mathbf{e}_\varphi. \tag{5}$$

Here $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ are unit vectors along the directions of the increasing spherical coordinates r, θ, and φ, respectively; $\rho=kr$ $(k=\omega/c)$; $x=\cos\theta$; $z_n(\rho)$ is an appropriate spherical cylinder function; $P_n{}^1(x)$ is an associated Legendre function; and n is a summation index: $n=1, 2, 3, \cdots$.

If we choose the incident wave to be a plane-polarized plane wave travelling in the direction of the positive z axis with its electric vector along the x axis, the incident electric field $\mathbf{E}^i$ of this wave can be expanded in the form

$$\mathbf{E}^i=e^{ikz}\mathbf{e}_x=\sum_{n=1}^{\infty}\nu_n[\mathbf{M}_n{}^j-i\mathbf{N}_n{}^j], \tag{6}$$

where

$$\nu_n=i^n(2n+1)/n(n+1), \quad i^2=-1, \tag{7}$$

and the superscript j on the unit fields denotes the choice $z_n(\rho)=j_n(\rho)$ for the spherical Bessel function. Finally, we have taken the incident field to be of unit magnitude.

The electric field $\mathbf{E}^s$ of the scattered wave can likewise be expanded in an infinite series of the unit fields $\mathbf{M}_n$ and $\mathbf{N}_n$, with undetermined coefficients. In order to satisfy the boundary condition at infinity, we must choose $z_n(\rho)=h_n{}^{(1)}(\rho)$. If we let the superscript s denote this particular choice for the spherical Bessel function, the scattered field may then be written in the form

$$\mathbf{E}^s=\sum_{n=1}^{\infty}\nu_n[a_n\mathbf{M}_n{}^s-ib_n\mathbf{N}_n{}^s], \tag{8}$$

where a_n and b_n are the undetermined "scattering coefficients."

The boundary condition to be satisfied for a perfect conductor is that the total tangential electric field must vanish at every point of the surface. For a sphere, the two independent tangential components may be chosen along the mutually perpendicular unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$, so that the boundary condition may be written in the form

$$E_\theta{}^i+E_\theta{}^s|_{r=a}=0, \tag{9}$$

$$E_\varphi{}^i+E_\varphi{}^s|_{r=a}=0. \tag{10}$$

If we then substitute Eqs. (6) and (8), with the unit fields obtained from Eqs. (4) and (5), into the boundary conditions (9) and (10), it may readily be verified that the latter are satisfied if, and only if,

$$a_n=-j_n(\rho_0)/h_n{}^{(1)}(\rho_0), \tag{11}$$

$$b_n=-\left\{\frac{d}{d\rho}[\rho j_n(\rho)]\right\}_{\rho=\rho_0}\Big/\left\{\frac{d}{d\rho}[\rho h_n{}^{(1)}(\rho)]\right\}_{\rho=\rho_0}, \tag{12}$$

[17] See, for example, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Co., New York, 1941).

where $\rho_0 = ka$. Expressions (11) and (12) for the scattering coefficients a_n and b_n complete the solution for the case of a perfectly conducting sphere. As is well known, all physical quantitites of interest, such as the various scattering and absorption cross sections, can be calculated in terms of these coefficients.

We now turn to the problem of obtaining the corresponding solution for irregular conductors whose surface shape is described by Eqs. (1) and (2). We note that Eq. (1), with r and θ reinterpreted as plane polar coordinates, also represents the equation of the boundary curve of the cross section of the surface (1) in any plane passing through the symmetry axis.

As before, we shall assume that the incident wave is a plane wave traveling along the positive z axis, with its electric field polarized along the x axis. The incident electric field $\mathbf{E}^i$ (of unit magnitude) is then again given by Eq. (6). Similarly, the scattered field $\mathbf{E}^s$ is again represented by the general expansion (8), with a_n and b_n as unknown coefficients. We must now formulate the boundary condition that the total tangential electric field vanishes at each point on the surface of the irregular conductor. To do this, we must first find two mutually perpendicular tangent vectors at each point of the surface of the irregular conductor described by Eq. (1). One of these is clearly $\mathbf{e}_\varphi$. Another vector, tangent to the surface (1) and perpendicular to $\mathbf{e}_\varphi$, is provided by

$$\boldsymbol{\tau} = d\mathbf{r}/d\theta\,, \tag{13}$$

where $\mathbf{r} = r_s\mathbf{e}_r$. We note that the tangent vector $\boldsymbol{\tau}$ defined in this manner is not a unit vector; however, inasmuch as the corresponding tangential component of the electric field vanishes, only the direction of the tangent vector is of significance. Noting that $d\mathbf{e}_r/d\theta = \mathbf{e}_\theta$, and substituting from Eq. (1) for r_s, we find that the tangent vector $\boldsymbol{\tau}$ is given explicitly by

$$\boldsymbol{\tau} = r_s\mathbf{e}_\theta + a\epsilon f'(\theta)\mathbf{e}_r\,. \tag{14}$$

The boundary condition that the total tangential electric field vanishes at each point of the surface may then be represented by the equations

$$E_\varphi|_{r=r_s} = 0\,, \tag{15}$$

$$r_sE_\theta + a\epsilon f'(\theta)E_r|_{r=r_s} = 0\,, \tag{16}$$

where $\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s$ represents the total electric field which enters into Eqs. (15) and (16). The problem now consists of employing the above boundary conditions in order to determine the unknown scattering coefficients a_n and b_n. This problem will be solved for the general case in Sec. III.

III. GENERAL SOLUTION

We now turn to the problem of determining the scattering coefficients a_n and b_n for the case of an irregular conductor whose surface is described by Eq. (1). As noted earlier, the incident and scattered electric fields are given by Eqs. (6) and (8), respectively, and the appropriate boundary conditions by means of which we hope to determine the coefficients a_n and b_n are provided by Eqs. (15) and (16).

Thus, if we substitute Eqs. (6) and (8), together with Eqs. (4) and (5), into the boundary conditions (15) and (16), the latter take the explicit form[18]

$$\sum_{n=1}^{\infty} \nu_n \bigg\{ -[\rho_s j_n(\rho_s)]\frac{dP_n^1}{d\theta} + \frac{i}{\sin\theta} \times[\rho j_n(\rho)]'_{\rho_s}P_n^1 - a_n[\rho_s h_n^{(1)}(\rho_s)]\frac{dP_n^1}{d\theta} + \frac{ib_n}{\sin\theta}[\rho h_n^{(1)}(\rho)]'_{\rho_s}P_n^1\bigg\} = 0\,, \tag{17}$$

$$\sum_{n=1}^{\infty} \nu_n \bigg\{ \frac{1}{\sin\theta}[\rho_s j_n(\rho_s)]P_n^1 - i[\rho j_n(\rho)]'_{\rho_s}\frac{dP_n^1}{d\theta} + \frac{a_n}{\sin\theta}[\rho_s h_n^{(1)}(\rho_s)]P_n^1 - ib_n[\rho h_n^{(1)}(\rho)]'_{\rho_s}\frac{dP_n^1}{d\theta} - \epsilon\rho_0 f'(\theta)\bigg[\frac{in(n+1)}{\rho_s}j_n(\rho_s)P_n^1 + \frac{ib_n n(n+1)}{\rho_s}h_n^{(1)}(\rho_s)P_n^1\bigg]\bigg\} = 0\,, \tag{18}$$

where the subscript ρ_s denotes that the function in question is to be evaluated at $\rho = \rho_s = kr_s$, a prime denotes differentiation with respect to ρ, and the argument of the modified Legendre functions P_n^1 is understood to be $x = \cos\theta$. Equations (17) and (18) represent the final form of the boundary conditions which must now be used to determine the scattering coefficients.

At this stage it becomes necessary to introduce the perturbation technique which forms the basis of our method for determining the coefficients a_n and b_n. Accordingly, following the basic principle of any perturbation method, we write the coefficients a_n and b_n in the form

$$a_n = \sum_{p=0}^{\infty} \epsilon^p a_n^p\,, \tag{19}$$

$$b_n = \sum_{p=0}^{\infty} \epsilon^p b_n^p\,, \tag{20}$$

where a_n^p and b_n^p represent the pth-order corrections to the unperturbed scattering coefficients a_n^0 and b_n^0, given by Eqs. (11) and (12), respectively.

The next step is to expand each of the terms occurring in the boundary conditions (17) and (18) as a power series in ϵ. This may be accomplished by expanding each

[18] For the sake of analytical convenience, the boundary condition (15) was multiplied by the factor kr_s prior to substitution.

function containing ρ_s in a Taylor series about the point $\rho=\rho_0=ka$. Thus, for example, let us consider the first term of Eq. (18). In view of Eq. (1), the function $[\rho_s j_n(\rho_s)]$ may be expanded in a Taylor series of the form

$$[\rho_s j_n(\rho_s)]=\sum_{p=0}^{\infty}\frac{\alpha_n{}^p}{p!}\epsilon^p\rho_0{}^p f^p, \tag{21}$$

where we have defined

$$\alpha_n{}^p=[\rho j_n(\rho)]_{\rho_0}{}^{(p)}\equiv\left\{\frac{d^p}{d\rho^p}[\rho j_n(\rho)]\right\}_{\rho=\rho_0}, \tag{22}$$

and where $\rho=kr$, $\rho_s=kr_s$, $\rho_0=ka$. Accordingly, the first term of Eq. (18) can be written in the expanded form

$$\sum_{p=0}^{\infty}\epsilon^p\sum_{n=1}^{\infty}\frac{\nu_n}{\sin\theta}\frac{\rho_0{}^p f^p\alpha_n{}^p P_n{}^1}{p!}. \tag{23}$$

The second term of Eq. (18) may be similarly expanded and yields

$$-i\sum_{p=0}^{\infty}\epsilon^p\sum_{n=1}^{\infty}\nu_n\frac{\rho_0{}^p f^p}{p!}\alpha_n{}^{p+1}\frac{dP_n{}^1}{d\theta}. \tag{24}$$

The third term of Eq. (18), to wit,

$$\sum_{n=1}^{\infty}\frac{\nu_n a_n}{\sin\theta}[\rho_s h_n{}^{(1)}(\rho_s)]P_n{}^1, \tag{25}$$

is more complicated in nature. The function containing ρ_s may again be expanded in a Taylor series, as follows:

$$[\rho_s h_n{}^{(1)}(\rho_s)]=\sum_{p=0}^{\infty}\frac{\beta_n{}^p\epsilon^p\rho_0{}^p f^p}{p!}, \tag{26}$$

where we have defined

$$\beta_n{}^p=[\rho h_n{}^{(1)}(\rho)]_{\rho_0}{}^{(p)}. \tag{27}$$

On the other hand, according to the ansatz (19), the parameter ϵ is also involved in the coefficient a_n. We thus have

$$a_n[\rho_s h_n{}^{(1)}(\rho_s)]=\left(\sum_{p=0}^{\infty}\epsilon^p a_n{}^p\right)\left(\sum_{p=0}^{\infty}\epsilon^p\frac{\beta_n{}^p\rho_0{}^p f^p}{p!}\right). \tag{28}$$

However, this expression may be written as a single power series in ϵ by making use of the following well-known theorem for the product of two infinite power series:

$$\left(\sum_{p=0}^{\infty}\epsilon^p\alpha_p\right)\left(\sum_{p=0}^{\infty}\epsilon^p\beta_p\right)=\sum_{p=0}^{\infty}\epsilon^p\left(\sum_{q=0}^{p}\alpha_q\beta_{p-q}\right). \tag{29}$$

When this theorem is applied to term (28), expression (25) which represents the third term of the boundary condition (18), may be written in the form

$$\sum_{p=0}^{\infty}\epsilon^p\sum_{n=1}^{\infty}\frac{\nu_n}{\sin\theta}\sum_{q=0}^{p}\frac{\beta_n{}^q\rho_0{}^q f^q a_n{}^{p-q}P_n{}^1}{q!}. \tag{30}$$

If the remaining terms of Eq. (18) are treated in a similar fashion, the expanded form of Eq. (18) may be written as

$$\sum_{p=0}^{\infty}\epsilon^p\sum_{n=1}^{\infty}\nu_n\Bigg\{\frac{\alpha_n{}^p\rho_0{}^p f^p}{p!}\frac{P_n{}^1}{\sin\theta}-\frac{i\alpha_n{}^{p+1}}{p!}\rho_0{}^p f^p\frac{dP_n{}^1}{d\theta}+\sum_{q=0}^{p}\frac{\beta_n{}^q}{q!}\rho_0{}^q f^q a_n{}^{p-q}\frac{P_n{}^1}{\sin\theta}-i\sum_{q=0}^{p}\frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q f^q b_n{}^{p-q}\frac{dP_n{}^1}{d\theta}$$
$$-in(n+1)\epsilon\rho_0 f'\left[\frac{\gamma_n{}^p}{p!}\rho_0{}^p f^p P_n{}^1+\sum_{q=0}^{p}\frac{\delta_n{}^q}{q!}\rho_0{}^q f^q b_n{}^{p-q}P_n{}^1\right]\Bigg\}=0, \tag{31}$$

where we have introduced the additional abbreviations

$$\gamma_n{}^p=[j_n(\rho)/\rho]_{\rho_0}{}^{(p)}, \tag{32}$$

$$\delta_n{}^p=[h_n{}^{(1)}(\rho)/\rho]_{\rho_0}{}^{(p)}. \tag{33}$$

Following the same procedure, Eq. (17) may be expanded in the form

$$\sum_{p=0}^{\infty}\epsilon^p\sum_{n=1}^{\infty}\nu_n\left(-\frac{\alpha_n{}^p}{p!}\rho_0{}^p f^p\frac{dP_n{}^1}{d\theta}+i\frac{\alpha_n{}^{p+1}}{p!}\rho_0{}^p f^p\frac{P_n{}^1}{\sin\theta}-\sum_{q=0}^{p}\frac{\beta_n{}^q}{q!}\rho_0{}^q f^q a_n{}^{p-q}\frac{dP_n{}^1}{d\theta}+i\sum_{q=0}^{p}\frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q f^q b_n{}^{p-q}\frac{P_n{}^1}{\sin\theta}\right)=0. \tag{34}$$

Equations (31) and (34) represent the two basic boundary conditions, expanded as power series in ϵ.

The critical step now consists of requiring that the coefficients of each power of ϵ in Eqs. (31) and (34) vanish individually. This in effect replaces the two necessary boundary conditions by an infinite set of "sufficient" boundary conditions. The consistency of this procedure is easily demonstrated. Thus, if the coefficients a_n and b_n are determined by this method, the resulting fields will *a fortiori* satisfy the boundary conditions at the surface of the conductor. Moreover, the choice $z_n(\rho)=h_n{}^{(1)}(\rho)$ in the expression for the scattered field guarantees that the boundary condition at infinity is likewise satisfied. Finally, regardless of what values we finally obtain for the scattering coefficients a_n and b_n, Eq. (8) shows that the resulting field is a solution of the vector Helmholtz equation. We may thus appeal to the applicable general uniqueness theorem to demonstrate that the solution obtained in this manner represents indeed the unique solution to the problem.

Accordingly, we now set the coefficient of ϵ^l (l arbitrary) in each of Eqs. (31) and (34) equal to zero. This yields

$$\sum_{n=1}^{\infty}\nu_n\left\{\frac{\alpha_n^l\rho_0^lf^l}{l!}\frac{P_n^1}{\sin\theta}-i\frac{\alpha_n^{l+1}}{l!}\rho_0^lf^l\frac{dP_n^1}{d\theta}+\sum_{q=0}^{l}\frac{\beta_n^q}{q!}\rho_0^qf^qa_n^{l-q}\frac{P_n^1}{\sin\theta}-i\sum_{q=0}^{l}\frac{\beta_n^{q+1}}{q!}\rho_0^qf^qb_n^{l-q}\frac{dP_n^1}{d\theta}\right.$$
$$\left.-in(n+1)\rho_0f'\left[\frac{\gamma_n^{l-1}}{(l-1)!}\rho_0^{l-1}f^{l-1}P_n^1+\sum_{q=0}^{l-1}\frac{\delta_n^q}{q!}\rho_0^qf^qb_n^{l-1-q}P_n^1\right]\right\}=0, \qquad (35)$$

$$\sum_{n=1}^{\infty}\nu_n\left(-\frac{\alpha_n^l}{l!}\rho_0^lf^l\frac{dP_n^1}{d\theta}+i\frac{\alpha_n^{l+1}}{l!}\rho_0^lf^l\frac{P_n^1}{\sin\theta}-\sum_{q=0}^{l}\frac{\beta_n^q}{q!}\rho_0^qf^qa_n^{l-q}\frac{dP_n^1}{d\theta}+i\sum_{q=0}^{l}\frac{\beta_n^{q+1}}{q!}\rho_0^qf^qb_n^{l-q}\frac{P_n^1}{\sin\theta}\right)=0. \qquad (36)$$

Our task is now to extract analytic expressions for the perturbation scattering coefficients a_n^l, b_n^l from the set of Eqs. (35) and (36). Toward this end, we shall first isolate the highest-order perturbation coefficients occurring in Eqs. (35) and (36). These correspond to the terms $q=0$ in the summation over the index q. (Note that the remaining terms in the q summation involve perturbation coefficients only up to order $l-1$.) Thus, we may rewrite Eqs. (35) and (36) in the form

$$\sum_{n=1}^{\infty}\nu_n\left(\beta_n^0a_n^l\frac{P_n^1}{\sin\theta}-i\beta_n^1b_n^l\frac{dP_n^1}{d\theta}\right)+\sum_{n=1}^{\infty}\nu_n\left\{\frac{\alpha_n^l}{l!}\rho_0^lf^l\frac{P_n^1}{\sin\theta}-i\frac{\alpha_n^{l+1}}{l!}\rho_0^lf^l\frac{dP_n^1}{d\theta}\right.$$
$$+\sum_{q=1}^{l}\frac{\beta_n^q}{q!}\rho_0^qf^qa_n^{l-q}\frac{P_n^1}{\sin\theta}-i\sum_{q=1}^{l}\frac{\beta_n^{q+1}}{q!}\rho_0^qf^qb_n^{l-q}\frac{dP_n^1}{d\theta}$$
$$\left.-in(n+1)\rho_0f'\left[\frac{\gamma_n^{l-1}}{(l-1)!}\rho_0^{l-1}f^{l-1}P_n^1+\sum_{q=0}^{l-1}\frac{\delta_n^q}{q!}\rho_0^qf^qb_n^{l-1-q}P_n^1\right]\right\}=0, \qquad (37)$$

$$\sum_{n=1}^{\infty}\nu_n\left(-\beta_n^0a_n^l\frac{dP_n^1}{d\theta}+i\beta_n^1b_n^l\frac{P_n^1}{\sin\theta}\right)+\sum_{n=1}^{\infty}\nu_n\left(-\frac{\alpha_n^l}{l!}\rho_0^lf^l\frac{dP_n^1}{d\theta}+i\frac{\alpha_n^{l+1}}{l!}\rho_0^lf^l\frac{P_n^1}{\sin\theta}\right.$$
$$\left.-\sum_{q=1}^{l}\frac{\beta_n^q}{q!}\rho_0^qf^qa_n^{l-q}\frac{dP_n^1}{d\theta}+i\sum_{q=1}^{l}\frac{\beta_n^{q+1}}{q!}\rho_0^qf^qb_n^{l-q}\frac{P_n^1}{\sin\theta}\right)=0. \qquad (38)$$

We observe that the highest-order coefficients a_n^l and b_n^l occur only in the first sum of each of the Eqs. (37) and (38); all remaining perturbation coefficients are of lower order in the perturbation. Unfortunately, the coefficients a_n^l and b_n^l still occur under a summation sign and cannot be extracted immediately, inasmuch as the series as they stand do not have the form of orthogonal series. However, this may be achieved by making use of more complicated orthogonalities. In particular, we may employ the following easily verified integral properties of the associated Legendre functions:

$$\int_0^\pi\left(\frac{P_n^1P_k^1}{\sin^2\theta}+\frac{dP_n^1}{d\theta}\frac{dP_k^1}{d\theta}\right)\sin\theta d\theta=\frac{2n^2(n+1)^2}{2n+1}\delta_{nk}, \qquad (39)$$

$$\int_0^\pi\left(P_n^1\frac{dP_k^1}{d\theta}+P_k^1\frac{dP_n^1}{d\theta}\right)d\theta=0, \qquad (40)$$

where δ_{nk} represents the usual Kronecker δ.

We observe that by taking suitable linear combinations of Eqs. (37) and (38), we may make use of the orthogonality properties (39) and (40) in order to extract the desired coefficients a_n^l and b_n^l.

Thus, in order to obtain a_n^l, we multiply Eq. (37) by P_k^1, Eq. (38) by $-(dP_k^1/d\theta)\sin\theta$, and add the two resulting equations. Introducing the abbreviations

$$\xi_{kn}=\frac{P_n^1P_k^1}{\sin^2\theta}+\frac{dP_n^1}{d\theta}\frac{dP_k^1}{d\theta}, \qquad (41)$$

$$\eta_{kn}=P_n^1\frac{dP_k^1}{d\theta}+P_k^1\frac{dP_n^1}{d\theta}, \qquad (42)$$

the result is

$$\sum_{n=1}^{\infty} \nu_n(\beta_n{}^0 a_n{}^l \xi_{kn} \sin\theta - i\beta_n{}^1 b_n{}^l \eta_{kn})$$

$$+\sum_{n=1}^{\infty} \nu_n \bigg\{ \frac{\alpha_n{}^l}{l!}\rho_0{}^l f^l \xi_{kn} \sin\theta - i\frac{\alpha_n{}^{l+1}}{l!}\rho_0{}^l f^l \eta_{kn} + \sum_{q=1}^{l} \frac{\beta_n{}^q}{q!}\rho_0{}^q f^q a_n{}^{l-q} \xi_{kn} \sin\theta - i\sum_{q=1}^{l} \frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q f^q b_n{}^{l-q}\eta_{kn}$$

$$-in(n+1)\rho_0 f' \bigg[\frac{\gamma_n{}^{l-1}}{(l-1)!}\rho_0{}^{l-1} f^{l-1} + \sum_{q=0}^{l-1} \frac{\delta_n{}^q}{q!}\rho_0{}^q f^q b_n{}^{l-1-q} \bigg] P_n{}^1 P_k{}^1 \bigg\} = 0. \quad (43)$$

If we now integrate this equation with respect to θ between the limits 0 and π, we find that because of the orthogonality properties (39) and (40), the first infinite sum contains only a single nonvanishing term, namely, $\nu_k\beta_k{}^0 a_k{}^l 2k^2 \times(k+1)^2/(2k+1)$. We have thus succeeded in isolating the lth-order perturbation coefficient $a_k{}^l$, which may be written as

$$a_k{}^l = -\frac{(2k+1)}{2k^2(k+1)^2\beta_k{}^0\nu_k}\int_0^\pi R_{kl}d\theta, \quad (44)$$

where R_{kl} represents the second sum (over n) of Eq. (43). It is important to note that R_{kl} contains only known constants and perturbation coefficients of *lower* order.

In order to isolate $b_n{}^l$, we proceed in an analogous manner. Thus, if we multiply Eq. (37) by $-(dP_k{}^1/d\theta)\sin\theta$, Eq. (38) by $P_k{}^1$, and add the two resulting equations, we obtain

$$\sum_{n=1}^{\infty} \nu_n(-\beta_n{}^0 a_n{}^l \eta_{kn} + i\beta_n{}^1 b_n{}^l \xi_{kn} \sin\theta)$$

$$+\sum_{n=1}^{\infty} \nu_n \bigg\{ -\frac{\alpha_n{}^l}{l!}\rho_0{}^l f^l \eta_{kn} + i\frac{\alpha_n{}^{l+1}}{l!}\rho_0{}^l f^l \xi_{kn} \sin\theta - \sum_{q=1}^{l} \frac{\beta_n{}^q}{q!}\rho_0{}^q f^q a_n{}^{l-q} \eta_{kn} + i\sum_{q=1}^{l} \frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q f^q b_n{}^{l-q}\xi_{kn}\sin\theta$$

$$+in(n+1)\rho_0 f' \bigg[\frac{\gamma_n{}^{l-1}}{(l-1)!}\rho_0{}^{l-1} f^{l-1} + \sum_{q=0}^{l-1} \frac{\delta_n{}^q}{q!}\rho_0{}^q f^q b_n{}^{l-1-q} \bigg] P_n{}^1 \frac{dP_k{}^1}{d\theta}\sin\theta \bigg\} = 0. \quad (45)$$

If we again integrate with respect to θ between the limits 0 and π, and make use of the orthogonality properties (39) and (40), we obtain

$$b_k{}^l = i\frac{(2k+1)}{2k^2(k+1)^2\nu_k\beta_k{}^1}\int_0^\pi S_{kl}d\theta, \quad (46)$$

where S_{kl} represents the second sum (over n) of Eq. (45). Again, S_{kl} contains perturbation coefficients only of orders lower than l.

Finally, for reasons of maximum clarity, we shall rewrite the analytic solutions obtained here for the scattering coefficients in explicit form. Introducing the integral notations

$$\xi_{kn}{}^l = \int_0^\pi f^l \xi_{kn} \sin\theta d\theta, \quad (47)$$

$$\eta_{kn}{}^l = \int_0^\pi f^l \eta_{kn} d\theta, \quad (48)$$

$$\sigma_{kn}{}^l = \int_0^\pi f' f^l P_n{}^1 P_k{}^1 d\theta, \quad (49)$$

$$\mu_{kn}{}^l = \int_0^\pi f' f^l P_n{}^1 \frac{dP_k{}^1}{d\theta}\sin\theta d\theta, \quad (50)$$

the solutions (44) and (46) for $a_k{}^l$ and $b_k{}^l$ may be written in the form

$$a_k{}^l=-[(2k+1)/2k^2(k+1)^2\beta_k{}^0\nu_k]\bar{R}_{kl}, \tag{51}$$

$$b_k{}^l=[i(2k+1)/2k^2(k+1)^2\beta_k{}^1\nu_k]\bar{S}_{kl}, \tag{52}$$

where

$$\bar{R}_{kl}=\int_0^\pi R_{kl}d\theta=\sum_{n=1}^{\infty}\nu_n\left\{\frac{\alpha_n{}^l}{l!}\rho_0{}^l\xi_{kn}{}^l-i\frac{\alpha_n{}^{l+1}}{l!}\rho_0{}^l\eta_{kn}{}^l+\sum_{q=1}^{l}\frac{\beta_n{}^q}{q!}\rho_0{}^q a_n{}^{l-q}\xi_{kn}{}^q-i\sum_{q=1}^{l}\frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q b_n{}^{l-q}\eta_{kn}{}^q\right.$$
$$\left.-in(n+1)\rho_0\left[\frac{\gamma_n{}^{l-1}}{(l-1)!}\rho_0{}^{l-1}\sigma_{kn}{}^{l-1}+\sum_{q=0}^{l-1}\frac{\delta_n{}^q}{q!}\rho_0{}^q b_n{}^{l-1-q}\sigma_{kn}{}^q\right]\right\}, \tag{53}$$

$$\bar{S}_{kl}=\int_0^\pi S_{kl}d\theta=\sum_{n=1}^{\infty}\nu_n\left\{-\frac{\alpha_n{}^l}{l!}\rho_0{}^l\eta_{kn}{}^l+\frac{i\alpha_n{}^{l+1}}{l!}\rho_0{}^l\xi_{kn}{}^l-\sum_{q=1}^{l}\frac{\beta_n{}^q}{q!}\rho_0{}^q a_n{}^{l-q}\eta_{kn}{}^q+i\sum_{q=1}^{l}\frac{\beta_n{}^{q+1}}{q!}\rho_0{}^q\xi_{kn}{}^q b_n{}^{l-q}\right.$$
$$\left.+in(n+1)\rho_0\left[\frac{\gamma_n{}^{l-1}}{(l-1)!}\rho_0{}^{l-1}\mu_{kn}{}^{l-1}+\sum_{q=0}^{l-1}\frac{\delta_n{}^q}{q!}\rho_0{}^q b_n{}^{l-1-q}\mu_{kn}{}^q\right]\right\}. \tag{54}$$

Equations (51) and (52), together with (53) and (54), represent our final analytic solution for the perturbation scattering coefficients. As noted earlier, the right-hand sides of Eqs. (51) and (52) involve only known constants and perturbation scattering coefficients of lower order. Accordingly, inasmuch as the zero-order ($l=0$) coefficients corresponding to the case of a perfect sphere are known, these equations may be used to successively compute the perturbation scattering coefficients up to any arbitrary order. The nature of the analytic expressions is such that this may be accomplished in a completely systematic manner which is well suited to machine calculation. The over-all scattering coefficients are then obtained by means of Eqs. (19) and (20). All scattering quantities of interest may be calculated in terms of these coefficients; in particular, the actual scattered field is given by Eq. (8).

We have thus accomplished what we set out to do, i.e., we have found an explicit analytical solution to the scattering problem under consideration, which is capable of yielding numerical results to any desired degree of accuracy. Although this solution was arrived at by means of a perturbation technique, this technique was used only as an analytic tool and it should be emphasized that the solution obtained is *exact*, inasmuch as no approximations of any kind, either mathematical or physical, were introduced. Moreover, the solution is general, i.e., it is equally valid in the near and far zones, as well as over the entire frequency range.

Finally, we should like to point out that although the final expressions (53) and (54) appear complicated, inasmuch as they involve infinite series, for most "reasonable" shapes these series will terminate after a moderate number of terms, due to orthogonality properties inherent in the associated Legendre functions.

In order to illustrate the analytic nature and basic simplicity of the general solution obtained above, we shall next consider some special cases in more detail.

IV. SPECIAL CASES OF GENERAL SOLUTION

A. Verification of the General Perturbation Solution and its Convergence for the Case of a Sphere

It would be interesting to verify the validity of the analytic perturbation solution by checking it against a known analytic solution for some special irregularly shaped object. This is impossible, however, for the simple reason that no exact analytic solution is known, other than that for the case of the sphere.

Accordingly, we shall here obtain the perturbation solution for a sphere of radius $r_s=a(1+\epsilon)$, considered as a perturbation of a sphere of radius a, and compare it with the known exact result. This comparison is not trivial, inasmuch as the general perturbation solution and the known exact solution have entirely different analytic forms.

In terms of our formalism, the sphere of radius $r_s=a(1+\epsilon)$ is characterized by $f(\theta)=1$, $f'(\theta)=0$. With this function $f(\theta)$, the integrals (47)–(50) become

$$\xi_{kn}{}^l=\int_0^\pi \xi_{kn}\sin\theta d\theta=\frac{2n^2(n+1)^2}{(2n+1)}\delta_{nk}, \tag{55}$$

$$\eta_{kn}{}^l=\int_0^\pi \eta_{kn}d\theta=0, \tag{56}$$

$$\sigma_{kn}{}^l=\mu_{kn}{}^l=0. \tag{57}$$

The second equalities of Eqs. (55) and (56) are true by virtue of Eqs. (39) and (40).

Substituting these values into Eqs. (53) and (54) for $\bar{R}_{kl}$ and $\bar{S}_{kl}$, we immediately obtain

$$\bar{R}_{kl}=\nu_k\frac{2k^2(k+1)^2}{(2k+1)}\left(\frac{\alpha_k{}^l}{l!}\rho_0{}^l+\sum_{q=1}^{l}\frac{\beta_k{}^q}{q!}\rho_0{}^q a_k{}^{l-q}\right), \tag{58}$$

$$\bar{S}_{kl}=i\nu_k\frac{2k^2(k+1)^2}{(2k+1)}\left(\frac{\alpha_k{}^{l+1}}{l!}\rho_0{}^l+\sum_{q=1}^{l}\frac{\beta_k{}^{q+1}}{q!}\rho_0{}^q b_k{}^{l-q}\right). \tag{59}$$

When these expressions are substituted into Eqs. (51) and (52), we obtain the following expressions for the perturbation scattering coefficients:

$$a_k{}^l = -\frac{1}{\beta_k{}^0}\left(\frac{\alpha_k{}^l\rho_0{}^l}{l!} + \sum_{q=1}^{l}\frac{\beta_k{}^q\rho_0{}^q a_k{}^{l-q}}{q!}\right), \qquad (60)$$

$$b_k{}^l = -\frac{1}{\beta_k{}^1}\left(\frac{\alpha_k{}^{l+1}\rho_0{}^l}{l!} + \sum_{q=1}^{l}\frac{\beta_k{}^{q+1}}{q!}\rho_0{}^q b_k{}^{l-q}\right). \qquad (61)$$

It is immediately verified that these expressions reproduce the correct zero-order results for an unperturbed sphere, corresponding to the case $l=0$. For the "perturbed" sphere of radius $a(1+\epsilon)$, the solution for the scattering coefficients given by our method is provided by Eqs. (19) and (20), with $a_k{}^l$ and $b_k{}^l$ given by Eqs. (60) and (61). On the other hand, the known exact solutions for the same scattering coefficients are provided by Eqs. (11) and (12), which for this case take the form

$$a_k = -\frac{j_n[\rho_0(1+\epsilon)]}{h_n{}^{(1)}[\rho_0(1+\epsilon)]}, \qquad (62)$$

$$b_k = -\left\{\frac{d}{d\rho}[\rho j_n(\rho)]\right\}_{\rho=\rho_0(1+\epsilon)} \Big/ \left\{\frac{d}{d\rho}[\rho h_n{}^{(1)}(\rho)]\right\}_{\rho=\rho_0(1+\epsilon)}. \qquad (63)$$

The question before us is whether or not the solutions (62) and (63) are identical with the solutions given by Eqs. (19) and (20), together with (60) and (61). In order to answer this question for the coefficient a_k, for example, we must expand expression (62) in a power series in ϵ, and compare the result with the perturbation expansion (19) and (60). In order to facilitate the comparison, we multiply both the numerator and denominator of expression (62) by $\rho_0(1+\epsilon)$ and expand the resulting expressions in the form of Taylor series about ρ_0. In this manner, we obtain

$$a_k = -\sum_{p=0}^{\infty}\frac{\epsilon^p\alpha_n{}^p\rho_0{}^p}{p!} \Big/ \sum_{p=0}^{\infty}\frac{\epsilon^p\beta_n{}^p\rho_0{}^p}{p!}. \qquad (64)$$

We observe that the analytic form of (64) is quite different from that of our perturbation solution, given by Eqs. (19) and (60). The former is in the form of a quotient of two infinite power series in ϵ, each of whose coefficients are known explicitly. On the other hand, the perturbation solution is in the form of a single power series in ϵ, where each expansion coefficient $a_k{}^l$ is given by a sort of "recursion" relation involving all lower coefficients $a_k{}^r$ with $r<l$.

The equivalence of the two solutions to low order of ϵ may easily be verified by means of direct expansion. In order to demonstrate that the two expressions are indeed equivalent to all orders in ϵ, we must make use of the following theorem concerning the quotient of two power series:

If we write

$$\sum_{n=0}^{\infty} a_n x^n \Big/ \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n, \qquad (65)$$

then the coefficients c_n may be written in the form

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{p=1}^{n} c_{n-p}b_p\right). \qquad (66)$$

This theorem is easily proved. If we now apply this theorem to expression (64) by making the appropriate identifications, the analytical equivalence of the solution (64) for a_k with the perturbation solution (19) and (60) is easily established. In an exactly analogous manner we can also demonstrate the equivalence of the corresponding solutions for b_k.

We have thus shown that for the special case consisting of the perturbation of a given sphere to a larger sphere, the solution obtained by our perturbation method actually agrees analytically with the *exact* result. While no other exact solutions are available for comparison purposes, we believe it is justified to assume that the analytical solution developed in the previous section represents the exact solution in all cases for which the series converges.

B. First-Order Solution

For reasons of increased insight into the nature of the perturbation solution and comparison with previous results, we shall here present the special case of the first-order solution explicitly. This is obtained simply by substituting $l=1$ into the expressions (51)–(54), which yields

$$a_k{}^1 = [(2k+1)/2k^2(k+1)^2\beta_k{}^0\nu_k]\bar{R}_{k1}, \qquad (67)$$

$$b_k{}^1 = [i(2k+1)/2k^2(k+1)^2\beta_k{}^1\nu_k]\bar{S}_{k1}, \qquad (68)$$

where

$$\bar{R}_{k1} = \rho_0\sum_{n=1}^{\infty}\nu_n[\alpha_n{}^1\xi_{kn}{}^1 - i\alpha_n{}^2\eta_{kn}{}^1 + \beta_n{}^1 a_n{}^0\xi_{kn}{}^1 - i\beta_n{}^2 b_n{}^0\eta_{kn}{}^1 - in(n+1)(\gamma_n{}^0+\delta_n{}^0 b_n{}^0)\sigma_{kn}{}^0], \qquad (69)$$

$$\bar{S}_{k1} = \rho_0\sum_{n=1}^{\infty}\nu_n[-\alpha_n{}^1\eta_{kn}{}^1 + i\alpha_n{}^2\xi_{kn}{}^1 - \beta_n{}^1 a_n{}^0\eta_{kn}{}^1 + i\beta_n{}^2\xi_{kn}{}^1 b_n{}^0 + in(n+1)(\gamma_n{}^0+\delta_n{}^0 b_n{}^0)\mu_{kn}{}^0]. \qquad (70)$$

Expressions (67) and (68), in conjunction with (69) and (70), give the *first-order* perturbation corrections to the zero-order scattering coefficients $a_k{}^0$, $b_k{}^0$, for any value of k. These results may be shown to be in agreement with those of Yeh,[13] once the latter are modified to apply to conductors and the analytical errors occurring therein are corrected (cf. Ref. 15).

C. Example: First-Order Scattering from an Oblate Spheroid

As a specific example of the application of the preceding general solution, we shall here calculate the first-order corrections to the scattering coefficients for the case of scattering from an oblate spheroid with semi-major axes of length $a(1+\epsilon)$ (along the x and y axes), and semi-minor axis of length a (along the z axis). The equation of this oblate spheriod in Cartesian coordinates is then given by

$$z^2+(1+\epsilon)^{-2}(x^2+y^2)=a^2. \tag{71}$$

In the present calculation we shall only be concerned with terms up to the first order in ϵ. To this order, Eq. (71) may be rewritten in spherical coordinates in the form

$$r=a(1+\epsilon\sin^2\theta). \tag{72}$$

In terms of our formalism, this shape is thus described by the perturbation function $f(\theta)=\sin^2\theta=1-x^2$.[19]

Our aim is now to evaluate the first-order expressions (69) and (70) for $\bar{R}_{k1}$ and $\bar{S}_{k1}$. We shall begin by considering the integrals $\xi_{kn}{}^1$, $\eta_{kn}{}^1$, $\sigma_{kn}{}^0$, and $\mu_{kn}{}^0$, with $f(\theta)=\sin^2\theta$. These integrals are identical with the integrals given in the Appendix of Ref. 13, which were there laboriously evaluated by numerical machine calculation. However, by making use of appropriate recursion relations, they may readily be evaluated analytically, which is accomplished in Appendix B. Their values are given respectively by Eqs. (B19), (B21), (B23), and (B27) of Appendix B. As may be seen from these expressions, the various integrals which enter into expressions (69) and (70) for $\bar{R}_{k1}$ and $\bar{S}_{k1}$ are nonvanishing for only a few values of n. Accordingly, the infinite-series expressions become finite.[20] If we now substitute the values (B19), (B21), (B23), and (B27) into expressions (69) and (70), these take the explicit form:

$$\begin{aligned}\bar{R}_{k1}/\rho_0=&\nu_{k-2}(\alpha_{k-2}{}^1+\beta_{k-2}{}^1a_{k-2}{}^0)\xi_{k,k-2}{}^1-i\nu_{k-1}[(\alpha_{k-1}{}^2+\beta_{k-1}{}^2b_{k-1}{}^0)\eta_{k,k-1}{}^1\\&+k(k-1)(\gamma_{k-1}{}^0+\delta_{k-1}{}^0b_{k-1}{}^0)\sigma_{k,k-1}{}^0]+\nu_k(\alpha_k{}^1+\beta_k{}^1a_k{}^0)\xi_{kk}{}^1-i\nu_{k+1}[(\alpha_{k+1}{}^2+\beta_{k+1}{}^2b_{k+1}{}^0)\eta_{k,k+1}{}^1\\&+(k+1)(k+2)(\gamma_{k+1}{}^0+\delta_{k+1}{}^0b_{k+1}{}^0)\sigma_{k,k+1}{}^0]+\nu_{k+2}(\alpha_{k+2}{}^1+\beta_{k+2}{}^1a_{k+2}{}^0)\xi_{k,k+2}{}^1,\end{aligned} \tag{73}$$

$$\begin{aligned}\bar{S}_{k1}/\rho_0=&i\nu_{k-2}[(\alpha_{k-2}{}^2+\beta_{k-2}{}^2b_{k-2}{}^0)\xi_{k,k-2}{}^1+(k-2)(k-1)(\gamma_{k-2}{}^0+\delta_{k-2}{}^0b_{k-2}{}^0)\mu_{k,k-2}{}^0]\\&-\nu_{k-1}(\alpha_{k-1}{}^1+\beta_{k-1}{}^1a_{k-1}{}^0)\eta_{k,k-1}{}^1+i\nu_k[(\alpha_k{}^2+\beta_k{}^2b_k{}^0)\xi_{kk}{}^1+k(k+1)(\gamma_k{}^0+\delta_k{}^0b_k{}^0)\mu_{kk}{}^0]\\&-\nu_{k+1}(\alpha_{k+1}{}^1+\beta_{k+1}{}^1a_{k+1}{}^0)\eta_{k,k+1}{}^1+i\nu_{k+2}[(\alpha_{k+2}{}^2+\beta_{k+2}{}^2b_{k+2}{}^0)\xi_{k,k+2}{}^1\\&+(k+2)(k+3)(\gamma_{k+2}{}^0+\delta_{k+2}{}^0b_{k+2}{}^0)\mu_{k,k+2}{}^0].\end{aligned} \tag{74}$$

The coefficients $\alpha_n{}^l$, $\beta_n{}^l$, $\gamma_n{}^l$, $\delta_n{}^l$; and $a_n{}^0$, $b_n{}^0$ are known constants which may be calculated by means of their respective defining equations. Accordingly, Eqs. (73) and (74), in conjunction with Eqs. (67) and (68), represent *explicit analytical* expressions for the first-order perturbation corrections for all of the scattering coefficients $a_k{}^0$, $b_k{}^0$; $k=1, 2, 3\cdots$. These expressions may then be used in appropriate formulas to calculate the over-all first-order perturbation corrections to any physical quantity of interest, such as various scattering cross sections, at any arbitrary frequency.

For purposes of even more specific illustration, we shall now consider the so-called "Rayleigh region" in more detail. This region corresponds physically to the case where the wavelength of the incident wave is very large compared to any characteristic length of the scatterer. Analytically, this approximation is equivalent to keeping only the leading terms in the expansion of all expressions as power series in the parameter $\rho_0=ka$ ($\ll 1$). As is well known, only the lowest order ($n=1$) scattering coefficients a_1 and b_1 need to be considered in the Rayleigh limit. Accordingly, we need only calculate the leading terms of the first-order correction terms $a_1{}^1$, $b_1{}^1$.

These are obtained by substituting $k=1$ into the general expressions (73) and (74). Inasmuch as the summation of the finite series from which these expressions were derived commenced at $n=1$, we note that for $k=1$ the terms corresponding to $n=k-2$ and $n=k-1$ are absent. Accordingly, substituting $k=1$ into expressions (73) and (74), these reduce to

$$\begin{aligned}\bar{R}_{11}/\rho_0=&\nu_1(\alpha_1{}^1+\beta_1{}^1a_1{}^0)\xi_{11}{}^1-i\nu_2[(\alpha_2{}^2+\beta_2{}^2b_2{}^0)\eta_{12}{}^1\\&+6(\gamma_2{}^0+\delta_2{}^0b_2{}^0)\sigma_{12}{}^0]+\nu_3(\alpha_3{}^1+\beta_3{}^1a_3{}^0)\xi_{13}{}^1,\end{aligned} \tag{75}$$

$$\begin{aligned}\bar{S}_{11}/\rho_0=&i\nu_1[(\alpha_1{}^2+\beta_1{}^2b_1{}^0)\xi_{11}{}^1+2(\gamma_1{}^0+\delta_1{}^0b_1{}^0)\mu_{11}{}^0]\\&-\nu_2(\alpha_2{}^1+\beta_2{}^1a_2{}^0)\eta_{12}{}^1+i\nu_3[(\alpha_3{}^2+\beta_3{}^2b_3{}^0)\xi_{13}{}^1\\&+12(\gamma_3{}^0+\delta_3{}^0b_3{}^0)\mu_{13}{}^0].\end{aligned} \tag{76}$$

We shall now proceed to evaluate these expressions numerically. The numerical values of the required integrals may be obtained by substituting $k=1$ into expressions (B19), (B21), (B23), and (B27) of Appendix B, which yields

$$\begin{aligned}&\xi_{11}{}^1=8/5,\quad \xi_{13}{}^1=-64/35,\quad \eta_{12}{}^1=-56/5,\\&\sigma_{12}{}^0=8/5,\quad \mu_{11}{}^0=8/15,\quad \mu_{13}{}^0=32/5.\end{aligned} \tag{77}$$

The remaining constants may be evaluated in the Rayleigh limit by making use of the expansions of the spherical Bessel functions $j_n(\rho)$, $h_n{}^{(1)}(\rho)$ as power series

[19] It should be emphasized that this is true only to first order in ϵ. The exact function $f(\theta)$ appropriate to the oblate spheroid described by Eq. (71) is more complex.

[20] This statement remains true even in higher orders, although the number of nonvanishing terms will increase.

in ρ. The leading terms of these expansions are

$$j_n(\rho)\sim\rho^n/(2n+1)!!, \quad (78)$$

$$h_n^{(1)}(\rho)\sim -i(2n-1)!!/\rho^{n+1}, \quad (79)$$

where the double factorial is defined by $(2n+1)!! \equiv 1\times3\times5\cdots(2n+1)$. Substituting the expansions (78) and (79) into the respective defining Eqs. (22), (27), (32), and (33) for α_n^p, β_n^p, γ_n^p, and δ_n^p, we obtain

$$\alpha_n^p=\left[\frac{d^p}{d\rho^p}\left(\frac{\rho^{n+1}}{(2n+1)!!}\right)\right]_{\rho=\rho_0}, \quad (80)$$

$$\beta_n^p=\left[\frac{d^p}{d\rho^p}\left(-\frac{i(2n-1)!!}{\rho^n}\right)\right]_{\rho=\rho_0}, \quad (81)$$

$$\gamma_n^p=\left[\frac{d^p}{d\rho^p}\left(\frac{\rho^{n-1}}{(2n+1)!!}\right)\right]_{\rho=\rho_0}, \quad (82)$$

$$\delta_n^p=\left[\frac{d^p}{d\rho^p}\left(-\frac{i(2n-1)!!}{\rho^{n+2}}\right)\right]_{\rho=\rho_0}. \quad (83)$$

Making use of these expressions, the coefficients required in the calculation of $\bar{R}_{11}$ and $\bar{S}_{11}$ are easily evaluated to be

$$\begin{aligned}
&\alpha_1^1=2\rho_0/3,\quad \alpha_1^2=\tfrac{2}{3};\quad \alpha_2^1=\rho_0^2/5,\quad \alpha_2^2=2\rho_0/5;\quad \alpha_3^1=4\rho_0^3/105,\quad \alpha_3^2=4\rho_0^2/35;\\
&\beta_1^0=-i/\rho_0,\quad \beta_1^1=i/\rho_0^2,\quad \beta_1^2=-2i/\rho_0^3;\quad \beta_2^1=6i/\rho_0^3,\quad \beta_2^2=-18i/\rho_0^4;\\
&\beta_3^1=45i/\rho_0^4,\quad \beta_3^2=-180i/\rho_0^5;\quad \gamma_1^0-\tfrac{1}{3},\quad \gamma_2^0=\rho_0/15,\quad \gamma_3^0=\rho_0^2/105;\\
&\delta_1^0=-i/\rho_0^3,\quad \delta_2^0=-3i/\rho_0^4,\quad \delta_3^0=-15i/\rho_0^2.
\end{aligned} \quad (84)$$

The leading terms of the required zero-order scattering coefficients a_n^0, b_n^0 are obtained by substituting expressions (78) and (79) into Eqs. (11) and (12). This yields

$$a_n^0=-\frac{i\rho_0^{2n+1}}{(2n-1)!!(2n+1)!!}, \quad (85)$$

$$b_n^0=\frac{i(n+1)\rho_0^{2n+1}}{n(2n-1)!!(2n+1)!!}. \quad (86)$$

From these expressions, the required numerical values are found to be

$$\begin{aligned}
&a_1^0=-i\rho_0^3/3,\quad a_2^0=-i\rho_0^5/45,\\
&a_3^0=-i\rho_0^7/(15)(105);\quad b_1^0=\tfrac{2}{3}i\rho_0^3,\quad b_2^0=i\rho_0^5/30,\\
&b_3^0=4i\rho_0^7/(45)(105).
\end{aligned} \quad (87)$$

Finally, the required numerical values for ν_n are obtained from Eq. (7).

We are now ready to evaluate the first-order perturbation corrections a_1^1 and b_1^1 in the Rayleigh limit. Substitution of the numerical values (77), (84), and (87) into expressions (75) and (76) yields after some algebra,

$$\bar{R}_{11}=-(84/15)i\rho_0^2,\quad \bar{S}_{11}=-(32/5)\rho_0. \quad (88)$$

According to Eqs. (67) and (68), the first-order perturbation coefficients a_1^1 and b_1^1 are then given by

$$a_1^1=-(3/8\beta_1^0\nu_1)\bar{R}_{11},\quad b_1^1=(3i/8\beta_1^1\nu_1)\bar{S}_{11}, \quad (89)$$

which, together with (88), yields

$$a_1^1=(21/15)i\rho_0^3,\quad b_1^1=(8/5)i\rho_0^3. \quad (90)$$

Finally, the complete scattering coefficients which contribute in the Rayleigh limit are obtained by

$$a_1=a_1^0+\epsilon a_1^1,\quad b_1=b_1^0+\epsilon b_1^1 \quad (91)$$

and are numerically equal to

$$a_1=-\tfrac{1}{3}i\rho_0^3\left(1-\frac{21}{5}\epsilon\right),\quad b_1=\tfrac{2}{3}i\rho_0^3\left(1+\frac{12}{5}\epsilon\right). \quad (92)$$

The above results show that in this case the boundary-shape perturbation, represented by ϵ, contributes already in the lowest order of the expansion in $\rho_0=ka$. Thus, if the value of ϵ is non-negligible, it would be meaningless to attempt to calculate the scattering from an irregularly shaped conductor more accurately by taking into account terms of higher order in ka, i.e., to go beyond the Rayleigh limit, while neglecting the deviation of the conductor from a spherical shape. Yet this has frequently been the case in approximate calculations reported in the literature.

As an example of the first-order contribution of the shape perturbation to a cross section, we may consider the total scattering cross section which is defined by

$$\sigma_{\text{scat}}=\frac{4\pi}{k^2}\sum_{l=1}^{\infty}(l+\tfrac{1}{2})(|a_l|^2+|b_l|^2). \quad (93)$$

For our present purpose we need only keep the first term corresponding to $l=1$. If we then substitute expression (91) for a_1 and b_1, the result is

$$\sigma_{\text{scat}}/\sigma_g=(10/3)\rho_0^4(1+(4/25)\epsilon), \quad (94)$$

where the scattering cross section has been normalized to the geometric cross section seen by the incident wave, which for our geometry is given by $\sigma_g=\pi a^2(1+\epsilon)^2 \approx\pi a^2(1+2\epsilon)$.

In carrying out the above calculation, we have made no effort to choose an "optimum" radius for the unperturbed sphere. As will be discussed more fully in Appendix A, an optimum choice for the unperturbed sphere would serve to improve the convergence of the general perturbation series.

The chief purpose of the illustrative example calculated in this section has been to demonstrate that while the general series expressions obtained in Sec. III may at first sight appear forbiddingly complex, they are in fact basically simple to evaluate. Thus, for the case of an oblate spheroid, we were readily able to obtain explicit finite analytic expressions for the first-order perturbation corrections to all scattering coefficients a_k, b_k, $(k=1, 2, 3,\cdots)$, and to evaluate these numerically in the Rayleigh limit without having to resort to machine calculation.

V. DISCUSSION

The only electromagnetic scattering problems which have so far been solved exactly by analytical means have been the scattering from perfect spheres and infinite right circular cylinders. The present work greatly extends the class of bodies for which exact analytical solutions may be obtained. In particular, in the work reported here we have been able to obtain an exact analytic solution to the scattering problem for the case where the scatterer is a cylindrically symmetric perfect conductor, with the incident plane wave traveling along the axis of symmetry.

This was accomplished by developing an appropriate boundary-perturbation technique for treating the full boundary-value problem. It should be emphasized that the perturbation technique was used only as a tool in obtaining the final solution. The solution itself is exact, provided that the series converges, inasmuch as no approximations of any kind, either physical or mathematical, were introduced. It is thus equally valid in the near and far zones, as well as over all ranges of the physical parameters of interest.

The general solution is similar in nature to the well-known Mie series, although it is of course more complicated than the latter, inasmuch as each individual scattering coefficient a_n, b_n is itself expressed in terms of a perturbation series. However, these perturbation series differ from the more familiar perturbation series in the literature, in that in this case we are able to obtain an exact analytical expression for the contribution of every order in the perturbation.[21] This enables us to obtain numerical results in a completely routine and systematic manner to any desired degree of accuracy. The form of the final solution obtained is thus exceptionally well suited for numerical evaluation by computer.

The present work was restricted to the cylindrically symmetric case and to the scattering from perfect conductors. The next stage will be to generalize the method to the general problem of a completely arbitrarily shaped conductor, with an arbitrary angle of incidence of the incident wave. This includes such special cases of practical interest as the scattering from cylindrically symmetric shapes at arbitrary angles of incidence. Once the general solution has been obtained for conductors, it may subsequently be readily generalized for scatterers of arbitrary, though constant index of refraction (e.g., plasmas). Finally, it should also be possible to apply similar techniques for arbitrarily shaped plasmas with a varying refractive index, although in that case the fundamental spatial solutions would no longer be Bessel functions.

[21] In fact, for the case of a sphere we were able to demonstrate the analytic convergence of the perturbation series to the exact result.

APPENDIX A: CHOICE OF OPTIMUM UNPERTURBED SPHERE

The general method based on a perturbation technique which was developed in this paper is restricted to perfect conductors whose boundary surface in spherical coordinates can be described by the equation

$$r_s=a[1+\epsilon f(\theta)]; \quad |\epsilon f(\theta)|<1, \quad 0\leq\theta\leq\pi, \qquad \text{(A1)}$$

where a is the radius of the "unperturbed sphere". At first sight it would appear that Eq. (A1) is capable of describing only shapes which do not differ too drastically from the unperturbed sphere. However, this limitation is largely only apparent. We must keep in mind that for a particular irregularly shaped body there is no *a priori* given "unperturbed sphere." What is really given as part of any particular problem is the shape of the surface of the irregular conductor, which may in general be described by an equation of the form

$$r=g(\theta) \qquad \text{(A2)}$$

in spherical coordinates. This irregular conductor is now to be considered as the perturbation of some "unperturbed sphere." Clearly, we are free to choose the radius a of this sphere, as well as the location of its center at will. Because of these two degrees of freedom, it is possible to choose unperturbed spheres for an extremely large class of shapes in a manner such that these shapes, given originally by Eq. (A2), fit the requirements of Eq. (A1). For example, any convex body falls into this class. More generally, any shape which allows of a radial explicit representation can be described by Eq. (A1).

Let us assume for the moment that the center of the sphere is chosen to be identical with the origin of the spherical coordinate system to which Eq. (A2) (which describes the shape of the irregular conductor) refers. We are then still free to choose the radius a of the unperturbed sphere. It may be possible to make this choice in some optimal manner, in the sense of improving the convergence of the general series solution. Clearly, we desire to minimize in some average sense the difference between the irregular body and the sphere of which it is to be considered a perturbation. Although there is no unique prescription for choosing such an "optimum" unperturbed sphere, a reasonable criterion for the choice of a would be to a pick a value which minimizes the average square deviation between the sphere and the cross section of the cylindrically symmetric irregular conductor, i.e., to minimize

$$\Delta=\frac{1}{\pi}\int_0^\pi [g(\theta)-a]^2 d\theta \tag{A3}$$

by choosing the optimum value of a to be the solution of the equation

$$\partial\Delta/\partial a=0. \tag{A4}$$

The value of a which minimizes Δ is easily found to be

$$a=\frac{1}{\pi}\int_0^\pi g(\theta)d\theta, \tag{A5}$$

and we may accordingly pick this as the optimum choice for a.

In the above development we assumed that the center of the sphere is coincident with the center of the spherical coordinate system in terms of which Eq. (A2) is expressed. In most cases which can be envisaged in practice, the choice of location for the center of the unperturbed sphere will be obvious. If this should not coincide with the center of the spherical coordinate system of Eq. (A2), the simplest procedure would be to rewrite Eq. (A2) in terms of spherical coordinates centered at the center of the sphere, and then to use (A5) for the choice of optimum radius. If in any circumstances the location for the center of the unperturbed sphere should not be obvious, it may be left arbitrary and determined by varying Eq. (A3) with respect to both a and the location of the origin of the unperturbed sphere. As an illustration of the application of the foregoing, let us consider the prolate spheroid described by the first-order equation

$$r_s=a(1+\epsilon\sin^2\theta), \tag{A6}$$

which was used as an example in the calculation carried out in Sec. IV C. No attempt was made there to optimize the radius of the unperturbed sphere.

In order to avoid confusion in notation, let us denote the radius of the unperturbed sphere by c and the corresponding perturbation parameter by δ. Thus c replaces a, and δ replaces ϵ in all of the equations (A1)–(A5). In particular, the standard form (A1) for the oblate spheroid, considered as a perturbation of a sphere of radius c now takes the form

$$r_s=c[1+\delta f(\theta)]. \tag{A7}$$

The function $g(\theta)$ describing the actual oblate spheroid under consideration is given by Eq. (A6). The optimization criterion (A5) thus becomes

$$c=\frac{1}{\pi}\int_0^\pi a(1+\epsilon\sin^2\theta)d\theta=a(1+\tfrac{1}{2}\epsilon). \tag{A8}$$

The optimal radius c for the unperturbed sphere corresponding to the oblate spheroid described by Eq. (A6) is thus not a (as was used in Sec. IV C) but $c=a(1+\frac{1}{2}\epsilon)$.

If we now equate Eqs. (A7) and (A6), with the value of c given by Eq. (A8), we obtain

$$r_s=c[1+(\epsilon/(1+\tfrac{1}{2}\epsilon))(\sin^2\theta-\tfrac{1}{2})], \tag{A9}$$

which takes the place of the standard form (A6). Accordingly, the result of optimizing the unperturbed sphere for the case of the prolate spheroid described by Eq. (A6) is to change the perturbing function $f(\theta)$ from $f(\theta)=\sin^2\theta$ to $f(\theta)=\sin^2\theta-\frac{1}{2}$, while at the same time changing the perturbation parameter from ϵ to $\epsilon/(1+\frac{1}{2}\epsilon)$, and the unperturbed sphere radius from a to $a(1+\frac{1}{2}\epsilon)$. Inasmuch as in our example $\epsilon>0$, we will have $\delta<\epsilon$, and accordingly the convergence of the corresponding series solution is likely to be improved as a result of the optimization procedure.

APPENDIX B: EVALUATION OF FIRST-ORDER INTEGRALS

This Appendix is devoted to the analytical evaluation of the integrals needed to obtain the explicit expressions for the first-order perturbation coefficients a_1^1, b_1^1 for the case of an oblate spheroid. With $f(\theta)=\sin^2\theta$, these integrals are

$$\xi_{kn}{}^1=\int_0^\pi \xi_{kn}\sin^3\theta d\theta =\int_0^\pi \sin^3\theta\left[\frac{P_n{}^1P_k{}^1}{\sin^2\theta}+\frac{dP_n{}^1(x)}{d\theta}\frac{dP_k{}^1(x)}{d\theta}\right]d\theta, \tag{B1}$$

$$\eta_{kn}{}^1=\int_0^\pi \eta_{kn}\sin^2\theta d\theta =\int_0^\pi\left(P_n{}^1\frac{dP_k{}^1}{d\theta}+P_k{}^1\frac{dP_n{}^1}{d\theta}\right)\sin^2\theta d\theta, \tag{B2}$$

$$\sigma_{kn}{}^0=2\int_0^\pi \sin\theta\cos\theta P_n{}^1(x)P_k{}^1(x)d\theta, \tag{B3}$$

$$\mu_{kn}{}^0=2\int_0^\pi \sin^2\theta\cos\theta P_n{}^1\frac{dP_k{}^1}{d\theta}d\theta. \tag{B4}$$

These are the same integrals which were evaluated by means of numerical machine calculation in Ref. 13. However, as we shall see, through judicious use of appropriate recursion relations they may readily be evaluated analytically.

To begin with, we shall first calculate some preliminary auxiliary integrals. The first of these is

$$A_{kn}=\int_0^\pi P_k{}^1(x)P_n{}^1(x)\sin\theta d\theta=0, \quad k\neq n$$
$$=\frac{2k(k+1)}{(2k+1)}, \quad k=n \tag{B5}$$

which is merely the well-known orthogonality property of the associated Legendre functions.[22]

Next, let us consider the integral

$$B_{kn}=\int_0^\pi \cos\theta\, P_k{}^1P_n{}^1 \sin\theta d\theta=\int_{-1}^1 xP_k{}^1P_n{}^1dx. \quad \text{(B6)}$$

Making use of the recursion relation[22]

$$xP_n{}^1=[1/(2n+1)][nP_{n+1}{}^1+(n+1)P_{n-1}{}^1], \quad \text{(B7)}$$

this integral may be rewritten in the form

$$B_{kn}=\frac{1}{(2n+1)}\int_{-1}^1 [nP_{n+1}{}^1P_k{}^1+(n+1)P_{n-1}{}^1P_k{}^1]dx$$
$$=\frac{1}{(2n+1)}[nA_{k,n+1}+(n+1)A_{k,n-1}]. \quad \text{(B8)}$$

If we now apply the result (B5) to the second equality of expression (B8), we find

$$B_{kn}=\frac{2(k-1)k(k+1)}{(2k-1)(2k+1)}, \quad n=k-1$$
$$=\frac{2k(k+1)(k+2)}{(2k+1)(2k+3)}, \quad n=k+1$$
$$=0, \quad \text{otherwise.} \quad \text{(B9)}$$

Next, we shall consider the integral

$$C_{kn}=\int_0^\pi \sin^2\theta P_k{}^1\frac{dP_n{}^1(x)}{d\theta}d\theta$$
$$=\int_{-1}^1 (x^2-1)P_k{}^1\frac{dP_n{}^1}{dx}dx. \quad \text{(B10)}$$

Making use of the recursion relation[22]

$$(x^2-1)(dP_n{}^1/dx)=[1/(2n+1)]\times[n^2P_{n+1}{}^1-(n+1)^2P_{n-1}{}^1], \quad \text{(B11)}$$

this integral may be rewritten in the form

$$C_{kn}=[1/(2n+1)][n^2A_{k,n+1}-(n+1)^2A_{k,n-1}]. \quad \text{(B12)}$$

If we now again apply the result (B5) to expression (B12), we obtain

$$C_{kn}=\frac{2(k-1)^2k(k+1)}{(2k-1)(2k+1)}, \quad n=k-1$$
$$=-\frac{2k(k+1)(k+2)^2}{(2k-1)(2k+1)}, \quad n=k+1$$
$$=0, \quad \text{otherwise.} \quad \text{(B13)}$$

[22] See, for example, W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Co., New York, 1950).

Finally, we consider the integral

$$D_{kn}=\int_0^\pi \sin^3\theta \frac{dP_n{}^1}{d\theta}\frac{dP_k{}^1}{d\theta}d\theta$$
$$=\int_{-1}^1 (1-x^2)^2\frac{dP_n{}^1}{dx}\frac{dP_k{}^1}{dx}dx. \quad \text{(B14)}$$

Applying the recursion relation (B11) twice, we may write

$$(x^2-1)^2(dP_n{}^1/dx)(dP_k{}^1/dx)=[1/(2n+1)(2k+1)]$$
$$\times[n^2k^2P_{n+1}{}^1P_{k+1}{}^1+(n+1)^2(k+1)^2P_{n-1}{}^1P_{k+1}{}^1$$
$$-k^2(n+1)^2P_{k+1}{}^1P_{n-1}{}^1-n^2(k+1)^2P_{n+1}{}^1P_{k-1}{}^1]. \quad \text{(B15)}$$

Accordingly, expression (B14) may be written in the form

$$D_{kn}=[1/(2n+1)(2k+1)]$$
$$\times[n^2k^2A_{k+1,n+1}+(n+1)^2(k+1)^2A_{k-1,n-1}-k^2(n+1)^2$$
$$\times A_{k+1,n-1}-n^2(k+1)^2A_{k-1,n+1}]. \quad \text{(B16)}$$

Applying the result (B5) to expression (B16) yields

$$D_{kn}=\frac{-2(k-2)^2(k-1)k(k+1)^2}{(2k-3)(2k-1)(2k+1)}, \quad n=k-2$$
$$=\frac{2k(k+1)}{(2k+1)^2}\left[\frac{k^3(k+2)}{(2k+3)}+\frac{(k-1)(k+1)^3}{(2k-1)}\right], \quad n=k$$
$$=\frac{-2k^2(k+1)(k+2)(k+3)^2}{(2k+1)(2k+3)(2k+5)}, \quad n=k+2$$
$$=0, \quad \text{otherwise.} \quad \text{(B17)}$$

We are now prepared to evaluate the required integrals (B1)–(B4). Thus, the integral (B1) may be written in the form

$$\xi_{kn}{}^1=\int_0^\pi P_n{}^1P_k{}^1 \sin\theta d\theta$$
$$+\int_0^\pi \sin^3\theta\frac{dP_n{}^1}{d\theta}\frac{dP_k{}^1}{d\theta}d\theta=A_{kn}+D_{kn}. \quad \text{(B18)}$$

If we then make use of the results (B5) and (B17), we obtain

$$\xi_{kn}{}^1=\frac{-2(k-2)^2(k-1)k(k+1)^2}{(2k-3)(2k-1)(2k+1)},\qquad n=k-2$$
$$=\frac{2k(k+1)}{(2k+1)}\left\{1+\frac{1}{2k+1}\left[\frac{k^3(k+2)}{2k+3}+\frac{(k-1)(k+1)^3}{2k-1}\right]\right\},\quad n=k$$
$$=\frac{-2k^2(k+1)(k+2)(k+3)^2}{(2k+1)(2k+3)(2k+5)},\qquad n=k+2$$
$$=0,\qquad \text{otherwise.}\qquad \text{(B19)}$$

In view of Eq. (B10), expression (B2) may be written in the form

$$\eta_{kn}{}^1=C_{kn}+C_{nk}.\qquad \text{(B20)}$$

Substitution of the expression (B13) immediately yields

$$\eta_{kn}{}^1=\frac{2k(k+1)(k-1)}{(2k-1)}\times\left[\frac{(k-1)}{(2k+1)}-\frac{(k+1)}{(2k-3)}\right],\quad n=k-1$$
$$=\frac{2k(k+1)(k+2)}{(2k+1)}\times\left[\frac{k}{2k+3}-\frac{(k+2)}{(2k-1)}\right],\qquad n=k+1$$
$$=0,\qquad \text{otherwise.}\qquad \text{(B21)}$$

We now turn to the integral $\sigma_{kn}{}^0$, given by Eq. (B3) as

$$\sigma_{kn}{}^0=2\int_0^\pi \sin\theta\,\cos\theta P_n{}^1(x)P_k{}^1(x)d\theta$$
$$=2\int_{-1}^1 xP_n{}^1(x)P_k{}^1dx.\qquad \text{(B22)}$$

In view of Eq. (B6), and the auxiliary result (B9), this integral is found to be

$$\sigma_{kn}{}^0=2B_{kn}=\frac{4(k-1)k(k+1)}{(2k-1)(2k+1)},\quad n=k-1$$
$$=\frac{4k(k+1)(k+2)}{(2k+1)(2k+3)},\quad n=k+1$$
$$=0,\qquad \text{otherwise.}\qquad \text{(B23)}$$

Finally, we consider the integral (B4), which may be rewritten in the form

$$\mu_{kn}{}^0=2\int_{-1}^1 (x^2-1)xP_n{}^1\frac{dP_k{}^1}{dx}dx.\qquad \text{(B24)}$$

Making use of the recursion relation (B11), the integrand may be written as follows:

$$(x^2-1)\frac{dP_k{}^1}{dx}xP_n{}^1=\frac{xP_n{}^1}{(2k+1)}\times[k^2P_{k+1}{}^1-(k+1)^2P_{k-1}{}^1].\qquad \text{(B25)}$$

In view of Eq. (B6), the integral (B24) thus becomes

$$\mu_{kn}{}^0=2/(2k+1)[k^2B_{k+1,n}-(k+1)^2B_{k-1,n}].\qquad \text{(B26)}$$

Substitution of the result (B9) into expression (B26) finally yields:

$$\mu_{kn}{}^0=\frac{-4(k-2)(k-1)k(k+1)^2}{(2k-3)(2k-1)(2k+1)},\qquad n=k-2$$
$$=\frac{4k(k+1)}{(2k+1)^2}\left[\frac{k^2(k+2)}{(2k+3)}-\frac{(k-1)(k+1)^2}{(2k-1)}\right],\quad n=k$$
$$=\frac{4k^2(k+1)(k+2)(k+3)}{(2k+1)(2k+3)(2k+5)},\qquad n=k+2$$
$$=0,\qquad \text{otherwise.}\qquad \text{(B27)}$$