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# Self-Consistent Homogeneous Isotropic Cosmological Models. I. Generalities

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Self-consistent models of uniform universes are provided by coupling Einstein's equations to the one-particle Liouville equation. Correlations between the "particles" of the cosmic gas are thus neglected. As a consequence, an equation of state is not needed in the theory but is, rather, provided from these statistical considerations. It is shown that an expanding self-consistent uniform universe behaves asymptotically as the relativistic polytrope $p\sim\mu^{5/3}$, and finally, as an expanding Friedmann universe. Near the possible singularity $R=0$ ($R$ = scale factor), self-consistent models are hot models. Friedmann models are shown to be a particular case of self-consistent models.

## 1. INTRODUCTION

In a preceding paper we studied some simple properties of the self-gravitating relativistic gas.[1] Essentially, we used a Vlasov approximation. In other words, the relativistic one-particle Liouville equation

$$u^\mu\partial_\mu\mathfrak{N}(x^\nu,u^\nu)-\Gamma_{\alpha\beta}{}^\mu(x^\nu)u^\alpha u^\beta\frac{\partial}{\partial u^\mu}\mathfrak{N}(x^\nu,u^\nu)=0 \quad (1.1)$$

was coupled to Einstein's equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}-\lambda g_{\mu\nu}=\chi T_{\mu\nu} \quad (1.2)$$

through the definition of the momentum-energy tensor

$$T^{\mu\nu}(x^\rho)=m(\surd g)\int\frac{d_3u}{u^0}\mathfrak{N}(x^\rho,u^\rho)u^\mu u^\nu. \quad (1.3)$$

In the following the metric tensor $g^{\mu\nu}$ is of signature $(+---)$ and $g\equiv|\det(g^{\mu\nu})|$. We also use $c=1$, where $c$ is the velocity of light in a vacuum. In Eq. (1.2), $R_{\mu\nu}$ is the Ricci tensor, $R$ is the scalar curvature, and $\lambda$ is the cosmological constant. $\chi$ denotes the gravitational constant. The $\Gamma_{\alpha\beta}{}^\mu$'s are the well-known Christoffel symbols of the second kind. In Eqs. (1.1) and (1.3), $\mathfrak{N}(x^\rho,u^\rho)$ is the invariant distribution function describing the gas. In Eq. (1.3), $m$ is the mass of a typical particle and the integral is extended to the hyperboloid

$$g_{\mu\nu}(x^\rho)u^\mu u^\nu=1\,,\quad u^0>0 \quad (1.4)$$

except when dealing with particles of vanishing mass. In this latter case the integral is extended to the light cone

$$g_{\mu\nu}(x^\rho)u^\mu u^\nu=0\,,\quad u^0\geqslant 0\,. \quad (1.4')$$

Finally, we will use the Einstein summation convention, Greek indices running from 0 to 3 and Latin ones from 1 to 3.

Since no explicit solution of Einstein's equations is known in terms of an arbitrary momentum-energy tensor $T_{\mu\nu}$, it was of course impossible to get a Vlasov equation as is commonly done in the case of electromagnetic interactions[2] (i.e., the electromagnetic field is expressed as a functional of the distribution function and next eliminated in the one-particle Liouville equation). Accordingly, we obtained a *linearized* kinetic equation by considering only small deviations of given "background quantities" (i.e., $g_{\mu\nu}$ and $\mathfrak{N}$). It then follows that our previous paper can be mainly applied to problems of stability. We also stressed that the only known relativistic self-gravitating system where *collective* effects are dominant is constituted by the universe as a whole.[3]

In this paper we deal with homogeneous, isotropic cosmological models. It is indeed well known that, in this particular case, Einstein's equations reduce to two differential equations[4] for the pressure $p$, the mass

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[1] Ph. Droz-Vincent and R. Hakim, Ann. Inst. H. Poincaré (to be published).

[2] See, e.g., S. Gartenhaus, *Elements of Plasma Kinetic Theory* (Holt, Rinehart and Winston, Inc., New York, 1964).

[3] This is not completely true since the *first stages* of the gravitational collapse of a massive star is another such example. However, as the star collapses, the neglect of correlations is less and less valid.

[4] See, e.g., H. Bondi, *Cosmology* (Cambridge University Press, Cambridge, 1961), 2nd ed.; G. C. McVittie, *General Relativity and Cosmology* (Chapman and Hall Ltd., London, 1965); R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1958).

density $\mu$, and a scale factor $R(t)$, sometimes called the "radius of the universe." The problem is fully determined when an equation of state connecting $p$ and $\mu$ is known. Here we reverse the procedure used previously[1]: Instead of eliminating the field (or rather the perturbation of the gravitational field), we rather eliminate the distribution function involved in $T_{\mu\nu}$; then $T_{\mu\nu}$ becomes a functional of $g_{\mu\nu}$ rather than of $\mathfrak{N}$. This is possible only because, for a uniform model universe, the one-particle Liouville equation is easily solved. In other words, $p$ and $\mu$ are expressed as functions of the scale factor $R(t)$ and an equation of state is not needed. Furthermore, $p$ and $\mu$ depend on an initial distribution function which characterizes the state of the universe at time $t=0$. Doing so, we obtain more general cosmological models than the usual ones. When they represent an expanding universe, all these models tend asymptotically towards expanding Friedmann models,[4] i.e., asymptotically describe dust-filled universes ($p\sim 0$).

At this point we have to emphasize strongly that, in spite of the fact that no equation of state is needed, there is still an arbitrariness in our models: the initial distribution function. At first sight, it could appear that this latter arbitrariness balances one of the equations of state to be chosen in the usual models. However, this is not so since the self-consistent models *restrict* the class of admissible equations of state, although we are not yet able to specify them more precisely.

In Paper II a more detailed analysis of self-consistent models will be studied and some refinements given. Section 2 is devoted to the basic assumptions and definitions used throughout this paper. In Sec. 3 the main general features of consistent models are studied. In Sec. 4 we discuss our results.

## 2. BASIC ASSUMPTIONS AND EQUATIONS

In order to show the manner in which the present formalism differs from the traditional one for uniform cosmological models, we have to be very careful while explicating our basic postulates. Accordingly, we first recall the usual assumptions, their implications here, and the hypotheses added.

(1) The universe may be considered as a relativistic gas whose molecules are galaxies (or clusters of galaxies). Therefore we shall describe this gas in a statistical way using methods developed elsewhere.[5,6]

(2) Correlations between "molecules" of the cosmic gas are neglected; the cosmic gas is sufficiently diluted. Hence only collective motions are considered. This amounts to describing the universe with a kinetic equation in the Vlasov approximation; i.e., the cosmic gas is described by Eq. (1.1) coupled to Eqs. (1.2) and (1.3). Of course, near a possible singularity this is not very accurate.

(3) In the space-time manifold there exists a congruence of timelike, rotational, and shear-free geodesics denoted by $\bar{u}^\mu$ which corresponds to the average motions of the galaxies (i.e., when one neglects their random motions). This is Weyl's hypothesis. More precisely, besides $\bar{u}^\mu\bar{u}_\mu=1$, $\bar{u}^\mu$ satisfies

$$d\bar{u}^\mu/ds+\Gamma_{\alpha\beta}{}^\mu\bar{u}^\alpha\bar{u}^\beta=0\,, \tag{2.1}$$

$$\omega_{\mu\nu}=(\nabla_\rho\bar{u}_\sigma-\nabla_\sigma\bar{u}_\rho)\Delta_\mu{}^\rho\Delta_\nu{}^\sigma=0\,, \tag{2.2}$$

$$\sigma_{\mu\nu}=(\nabla_\rho\bar{u}_\sigma+\nabla_\sigma\bar{u}_\rho)\Delta_\mu{}^\rho\Delta_\nu{}^\sigma-\tfrac{1}{2}\theta\Delta_{\mu\nu}=0\,, \tag{2.3}$$

where $\Delta_{\mu\nu}$ is the projector in the space orthogonal to $\bar{u}^\mu$,

$$\Delta_{\mu\nu}=g_{\mu\nu}-\bar{u}_\mu\bar{u}_\nu\,, \tag{2.4}$$

and where $\theta$ is the scalar of expansion,

$$\theta=\nabla_\mu\bar{u}^\mu. \tag{2.5}$$

These postulates and assumptions immediately lead to the existence of a cosmic time, say, $t$, such that the spacelike hypersurfaces $t=\text{const}$ are orthogonal to the congruence defined by $\bar{u}^\mu$. Possibly the "congruence" intersects at a given point in the far past.

(4) The gas representing the universe is a perfect fluid whose stream lines are defined by $\bar{u}^\mu$. Equivalently, the distribution function $\mathfrak{N}(x^\nu,u^\nu)$ depends only on one 4-vector (besides $u^\nu$) which we have already chosen to be $\bar{u}^\mu$. Consequently, it follows that the momentum-energy tensor $T^{\mu\nu}$ is a linear combination of the only two tensors at our disposal: $g^{\mu\nu}$ and $\bar{u}^\mu\bar{u}^\nu$. Hence it is of the usual form

$$T^{\mu\nu}=(\mu+p)\bar{u}^\mu\bar{u}^\nu-pg^{\mu\nu}. \tag{2.6}$$

This means that our gas is free from transport phenomena, such as heat conduction or viscosity.

(5) The universe is isotropic and spatially homogeneous. The spatial homogeneity implies that, in comoving coordinates, the two scalars $\mu$ and $p$ do not depend on the spatial coordinates $x^i$ although they generally depend on $x^0\equiv t$. Note that the isotropy property is partially included in the form of $T^{\mu\nu}$ given by Eq. (2.6) since the pressure is isotropic and $\bar{u}^\mu$ is shear-free. Furthermore, isotropy requires that $\mathfrak{N}(x^\nu,u^\nu)$ depend on $u^\nu$ only through the combination $\bar{u}^\mu u_\mu$, i.e., in comoving coordinates that $\mathfrak{N}$ depend on $u^iu_i$ (or $u^0$).

Postulates (3), (4), and (5) joined to Einstein's equations (1.2) lead, as is well known,[4,7] to the so-called Robertson-Walker line element[8]:

$$\begin{aligned}ds^2&=g_{\mu\nu}dx^\mu dx^\nu\\&=dt^2-R^2(t)K^2(r)[(dx^1)^2+(dx^2)^2+(dx^3)^2]\,,\end{aligned} \tag{2.7}$$

written in comoving coordinates. In Eq. (2.7), $R(t)$ is

[5] See the general bibliography given in R. Hakim, Ann. Inst. H. Poincaré 3, 225 (1967).

[6] See also R. Hakim, J. Math. Phys. 8, 1315 (1967); 8, 1379 (1967).

[7] J. L. Anderson, *Principles of Relativity Physics* (Academic Press Inc., New York, 1967), Chap. 14.

[8] H. P. Robertson, Astrophys. J. 82, 284 (1935); 83, 257 (1936); A. G. Walker, Proc. London Math. Soc. (2), 42, 90 (1936).

the yet unknown scale factor, $r=[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$, and $K(r)$ is given by

$$K(r)=(1+\tfrac{1}{4}kr^2)^{-1}, \tag{2.8}$$

where the *spatial* scalar curvature $k$ can take the values 0, ±1.

### Simple Properties of the Distribution Function

In order to express the homogeneity of the model in the distribution function, we first calculate $p$ and $\mu$ as functional of $\mathfrak{N}(x^\nu,u^\nu)$. Using Eq. (2.6), we find

$$\mu=T^{\alpha\beta}\bar{u}_\alpha\bar{u}_\beta, \tag{2.9}$$

$$\mu-3p=T^\alpha{}_\alpha, \tag{2.10}$$

from which we obtain

$$p=\tfrac{1}{3}(T^{\alpha\beta}\bar{u}_\alpha\bar{u}_\beta-T^\alpha{}_\alpha). \tag{2.11}$$

Next, inserting the definition (1.3) of $T^{\alpha\beta}$ into Eqs. (2.9) and (2.11), we get

$$\mu=m(\sqrt{g})\int\frac{d_3u}{u^0}\mathfrak{N}(x^\nu,u^\nu)(\bar{u}^\alpha u_\alpha)^2 \tag{2.12}$$

and

$$p=\tfrac{1}{3}m(\sqrt{g})\int\frac{d_3u}{u^0}\mathfrak{N}(x^\nu,u^\nu)[(\bar{u}^\alpha u_\alpha)^2-1]. \tag{2.13}$$

At this stage we want to emphasize strongly that the mass density $\mu$ is *different* from the *numerical* density $\rho$ defined through the numerical current[9]

$$j^\mu(x^\nu)\equiv\rho(x^\nu)\bar{u}^\mu(x^\nu)=(\sqrt{g})\int\frac{d_3u}{u^0}u^\mu\mathfrak{N}(x^\nu,u^\nu). \tag{2.14}$$

In comoving coordinates $\mathfrak{N}(x^\nu,u^\nu)=\mathfrak{N}(x^\nu,u^0)$ because of the assumed isotropy. In order to express the spatial homogeneity of the system in the distribution function, we use Eq. (2.12). It is indeed not obvious that $\mathfrak{N}(x^\nu,u^0)=\mathfrak{N}(t,u^0)$. The spatial homogeneity of the system is expressed by $\mu=\mu(t)$ in comoving coordinates and a glance at Eq. (2.12) does not *a priori* imply $\partial_i\mathfrak{N}=0$ since (a) a factor $\sqrt{g}$ (depending on $x^\nu$) is involved in the right-hand side of Eq. (2.12) and (b) the integration is extended to a subfiber [the one defined by Eq. (1.4)] of the tangent fiber bundle of the space-time manifold.[10] In other words, the domain of integration is $x^\nu$-dependent and hence we have to be careful in expressing homogeneity. However, this apparent difficulty can easily be removed by using a suitable coordinate system in the subfiber (1.4). Since $\mathfrak{N}$ depends on $u^0$, we use this variable to express Eq. (2.12). Taking the Robertson-Walker line element into account, we have

$$u^2\equiv\sum_{i=1}^{i=3}(u^i)^2=[(u^0)^2-1]R^{-2}(t)K^{-2}(r) \tag{2.15}$$

and

$$d_3u=u^2\sin\theta\,dud\theta d\varphi \tag{2.16}$$

(in polar coordinates in $u$ space), so that finally we obtain

$$\mu(t)=\int_1^\infty 4\pi m du^0(u^0)^2[(u^0)^2-1]^{1/2}\mathfrak{N}(t,x^i;u^0) \tag{2.17}$$

and in the same way,

$$p=\tfrac{4}{3}\pi m\int_1^\infty du^0[(u^0)^2-1]^{3/2}\mathfrak{N}(t,x^i;u^0). \tag{2.18}$$

Then Eq. (2.17) immediately implies that $\mathfrak{N}(t,x^i;u^0)\equiv\mathfrak{N}(t,u^0)$, as we expected. It follows that $p=p(t)$ and conversely.

Isotropy and homogeneity also simplify considerably the one-particle Liouville equation (1.1). Indeed, the only surviving terms are such that (1.1) reads

$$u^0(\partial/\partial t)\mathfrak{N}(t,u^0)-\Gamma_{\alpha\beta}{}^0u^\alpha u^\beta(\partial/\partial u^0)\mathfrak{N}(t,u^0)=0, \tag{2.19}$$

and since the only nonvanishing $\Gamma_{\alpha\beta}{}^0$'s are

$$\Gamma_{ii}{}^0=-(\partial/\partial t)g_{ii}=R(t)\dot{R}(t)K^2(r), \tag{2.20}$$
$$g_{11}=g_{22}=g_{33}=R^2(t)K^2(r),$$

Eq. (2.19) is finally written as

$$u^0(\partial/\partial t)\mathfrak{N}(t,u^0)-\dot{R}(t)R^{-1}(t)\times[(u^0)^2-1](\partial/\partial u^0)\mathfrak{N}(t,u^0)=0. \tag{2.21}$$

A dot on $R(t)$ in the above equations denotes a time derivative.

It is interesting to notice that, in this statistical context, the isotropy of the distribution function implies its spatial homogeneity. Indeed, isotropy implies Eq. (2.19) *plus* a term of the form $u^i\partial_i\mathfrak{N}$ and next it is easy to realize by examining the resulting Liouville equation that $\mathfrak{N}(t,x^i;u^0)=\mathfrak{N}(t;u^0)$. This property is obviously a consequence of the form of the Christoffel symbols, and hence of the Robertson-Walker line element which we used in the one-particle Liouville equation. On the other hand, this line element is derived on the basis of isotropy (spatial homogeneity is then a consequence of isotropy by virtue of Schur's theorem).

## 3. SELF-CONSISTENT COSMOLOGICAL MODELS

It is well known that the Robertson-Walker line element (2.7) implies that Einstein's equations greatly simplify and finally yield[4,7]

$$\chi\mu(t)=3R^{-2}(t)[k+R^2(t)]-\lambda, \tag{3.1}$$

$$\chi p(t)=-2\ddot{R}(t)R^{-1}(t)-\dot{R}^2(t)R^{-2}(t)-kR^{-2}(t)+\lambda, \tag{3.2}$$

[9] See, e. g., J. L. Synge, *The Relativistic Gas* (North-Holland Publishing Co., Amsterdam, 1957).

[10] We are indebted to Dr. Ph. Droz-Vincent for this remark.

while the conservation of momentum energy $\nabla_\mu T^{\mu\nu}=0$ provides

$$\dot{\mu}/(\mu+p)=-3\dot{R}(t)/R(t). \tag{3.3}$$

Note that only two of the three equations (3.1)–(3.3) are independent.

Self-consistent models will now be obtained in the following manner. First, we solve the Liouville equation (2.21). Thus we obtain the distribution function $\mathfrak{N}(t,u^0)$ as a *functional* of $R(t)$. Next, we calculate $\mu(t)$ and $p(t)$ with the help of Eqs. (2.17) and (2.18). As a consequence, $\mu(t)$ and $p(t)$ are also functionals of $R(t)$, which we introduce in one of the two equations (3.1) or (3.2). [Equation (3.3) would reduce to an identity.] Finally, we obtain an equation involving only $R(t)$ *and* an initial distribution function.

### Solution of One-Particle Liouville Equation

Instead of the variable $u^0$ we rather use the new variable

$$v^2=[(u^0)^2-1] \tag{3.4}$$

and the new time variable

$$\tau=\ln R(t). \tag{3.5}$$

In the following we shall choose $R(t)$ such that $R(0)=1$; hence $\tau=0$ when $t=0$. Furthermore, we will denote by $t_0$ a time such that $R(t_0)=0$ (when it exists).

Let us also point out that $v$ is neither the modulus of the spatial components of the 4-velocity (this would be the case only in Minkowski space-time *and* using Lorentzian coordinates) nor is it a usual velocity. We can only say that $v$ is a velocitylike variable.

With the change of variables (3.4) and (3.5), Eq. (2.21) reads

$$(\partial/\partial\tau)\mathfrak{N}(\tau,v)-v(\partial/\partial v)\mathfrak{N}(\tau,v)=0, \tag{3.6}$$

whose most general solution is of the form

$$\mathfrak{N}(\tau,v)=\mathfrak{N}_0(v\exp\tau), \tag{3.7}$$

where $\mathfrak{N}_0$, the initial distribution function, is of course an *arbitrary* function. Equivalently, we have

$$\mathfrak{N}(t,v)=\mathfrak{N}_0[vR(t)], \tag{3.8}$$

which shows that $vR(t)$ is a first integral of the motion for homogeneous, isotropic universes.[11]

It should be noted that, in spite of the appearance of $u^0$ in the above equations (or of $v$, which is a function of $u^0$), all the preceding results are covariant since actually we use the invariant $\bar{u}^\mu u_\mu$ which reduces to $u^0$ in comoving coordinates.

Now using the variable $v$ instead of $u^0$ and Eq. (3.8), the expressions for the mass density and pressure read

$$\mu(t)=4\pi m\int_0^\infty dv(v^2+1)^{1/2}v^2\mathfrak{N}_0(vR(t)), \tag{3.9}$$

$$p(t)=\tfrac{4}{3}\pi m\int_0^\infty dv(v^2+1)^{-1/2}v^4\mathfrak{N}_0(vR(t)), \tag{3.10}$$

or, equivalently,

$$\mu(t)=4\pi mR^{-4}(t)\int_0^\infty dv[v^2+R^2(t)]^{1/2}v^2\mathfrak{N}_0(v) \tag{3.11}$$

and

$$p(t)=\tfrac{4}{3}\pi mR^{-4}(t)\int_0^\infty dv[v^2+R^2(t)]^{-1/2}v^4\mathfrak{N}_0(v). \tag{3.12}$$

In the same way the *numerical* density $\rho(t)$ is easily computed from Eq. (2.14). It turns out that

$$\rho(t)=4\pi R^{-3}(t)\int_0^\infty dv\, v^2\mathfrak{N}_0(v) \tag{3.13}$$

$$=R^{-3}(t)\rho(0). \tag{3.14}$$

### Self-Consistent Models

The equations for self-consistent models are now readily obtained by inserting expressions (3.11) and (3.12) into Eqs. (3.1)–(3.3). We find

$$4\pi m\chi R^{-4}(t)\int_0^\infty dv[v^2+R^2(t)]^{1/2}v^2\mathfrak{N}_0(v)\equiv\mu(R)\chi$$
$$=3R^{-2}(t)[k+\dot{R}^2(t)]-\lambda, \tag{3.15}$$

$$\tfrac{4}{3}\pi m\chi R^{-4}(t)\int_0^\infty dv[v^2+R^2(t)]^{-1/2}v^4\mathfrak{N}_0(v)\equiv p(R)\chi$$
$$=-2\ddot{R}(t)R^{-1}(t)-\dot{R}^2(t)R^{-2}(t)-kR^{-2}(t)+\lambda, \tag{3.16}$$

$$\frac{d}{dR}\left\{R^{-4}(t)\int_0^\infty dv[v^2+R^2(t)]^{1/2}v^2\mathfrak{N}_0(v)\right\}$$
$$=-3R^{-5}(t)\left\{\int_0^\infty dv\, v^2\mathfrak{N}_0(v)\right.$$
$$\left.\times\{[v^2+R^2(t)]^{1/2}-\tfrac{1}{3}v^2[v^2+R^2(t)]^{-1/2}\}\right\}. \tag{3.17}$$

It is interesting to note that in an expanding universe and for large $R(t)$'s [i.e., when $R(t)\gg R(0)=1$], $p(R)$ and $\mu(R)$ behave like

$$p(R)\sim\text{const}\times R^{-5}, \tag{3.18}$$

$$\mu(R)\sim(\text{another const})\times R^{-3}. \tag{3.19}$$

Therefore an expanding self-consistent universe behaves like a relativistic polytrope, i.e., its equation of states (asymptotically) is

$$p=\text{const}\times\mu^{5/3}. \tag{3.20}$$

[11] This result is of course not new [see, e.g., L. Landau and E. Lifshitz, *Classical Field Theory* (Addison-Wesley Publishing Co., Reading, Mass., 1962].

Furthermore, if $R^{-5} \ll R^{-3}$, then $p(R) \ll \mu(R)$ and we recover the *expanding* Friedmann models[4] with

$$\mu(R) \simeq m\rho(R) = m\rho(0)R^{-3}, \qquad (3.21)$$

i.e., a cold dust-filled universe.

In the neighborhood of $t=t_0$, i.e., when $R \sim 0$, $\mu(R)$ and $p(R)$ behave like

$$p(R) \sim \text{const} \times R^{-4}, \qquad (3.22)$$

$$\mu(R) \sim 3 \text{ const} \times R^{-4}; \qquad (3.23)$$

it follows that self-consistent models behave like a relativistic perfect gas of incoherent radiation ($p=\frac{1}{3}\mu$). Equivalently, near an "initial singularity"[12] the self-consistent models behave like ultrarelativistic gases, i.e., like *hot models*. However, it is important to realize that our approximations fail long before the time $t_0$ is reached since, for instance, correlations are no longer negligible near $R \sim 0$.

Equations (3.11) and (3.12) for $p$ and $\mu$ exhibit a very simple dependence on the scale factor $R(t)$ and may be calculated in a limited number of cases only. However, the simple dependence on the scale factor permits the use of approximation methods such as the use of expansions in powers of $R^{-2}(t)$, etc.

A trivial case where $p(R)$ and $\mu(R)$ are easily calculated is provided by choosing

$$\mathfrak{N}_0(v) = \frac{\rho(0)}{4\pi v_0^2}\delta(v-v_0) \quad (v_0 = \text{const}). \qquad (3.24)$$

We then find

$$\mu(R) = m\rho(0)R^{-4}(v_0^2+R^2)^{1/2}, \qquad (3.25)$$

$$p(R) = \tfrac{1}{3}m\rho(0)v_0^2R^{-4}(v_0^2+R^2)^{-1/2}, \qquad (3.26)$$

and, as a consequence, the following equation of states:

$$1 = m\rho(0)(3p)^{1/2}\mu^{-1/2}/v_0(\mu-3p)^2, \qquad (3.27)$$

which is very complicated. The choice (3.24) for $\mathfrak{N}_0(v)$ means that the invariant

$$v \equiv [(\bar{u}^\mu u_\mu)^2 - 1]^{1/2} \qquad (3.28)$$

has a given value $v_0$. However, the main interest of Eq. (3.24) is that it allows us to get another—although very crude—approximation method by choosing a superposition of such distributions:

$$\mathfrak{N}_0(v) = \sum_{i=1}^{i=n} \frac{\rho_i}{4\pi v_i^2}\delta(v-v_i), \qquad (3.29)$$

and hence to get more and more "realistic" models. Finally, let us also note that when $v_0$ tends to zero in Eqs. (3.24)–(3.26), i.e., when $u^0=1$ in comoving coordinates (equivalently, when the initial state of the universe is a zero-temperature state), then we find anew the well-known Friedmann models including the oscillating models as well as the expanding ones.

Another case of initial distribution function has been considered by Bel, who chose[13]

$$\mathfrak{N}_0(u^\mu) \propto \exp(-\beta\Delta_{\mu\nu}u^\mu u^\nu) \sim \exp(-\beta v^2), \qquad (3.30)$$

with $\beta > 0$.

### Equilibrium as Initial State

Equilibrium states deserve a particular discussion because of their special importance. As is well known, the equilibrium distribution function is the Jüttner-Synge density[9]:

$$\mathfrak{N}(x^\nu, u^\nu) = [\alpha/4\pi K_2(\alpha)] \exp(-\alpha\bar{u}^\mu u_\mu), \qquad (3.31)$$

where $K_2(\alpha)$ is the Kelvin function of order 2 and $\alpha = mc^2/kT$. A question then arises. Does the equilibrium distribution (3.31) preserve its form in the course of time? The answer is obviously no, since

$$\mathfrak{N}(t,v) = [\alpha/4\pi K_2(\alpha)] \exp\{-\alpha[v^2R^2(t)+1]^{1/2}\}. \qquad (3.32)$$

It has indeed been shown by Chernikov[14] that (3.31) is a solution of the one-particle Liouville equation only when $\bar{u}^\mu$ is a Killing vector of the space-time manifold:

$$\nabla_\mu \bar{u}_\nu + \nabla_\nu \bar{u}_\mu = 0. \qquad (3.33)$$

However, this is not so; it is indeed well known that the uniform model universes have generally only one conformal Killing vector, say, $y^\mu$, i.e., such that

$$\nabla_\mu y_\nu + \nabla_\nu y_\mu = \Phi(x^\rho) g_{\mu\nu}. \qquad (3.34)$$

Here $y^\mu = R(t)\bar{u}^\mu$ and $\Phi(x^\rho) = 2\dot{R}$. Therefore $\bar{u}^\mu$ is *not* a Killing vector. However, if we consider a radiation-dominated universe in equilibrium at $t=t_0$, then it will remain in equilibrium in the course of time. In this case the Jüttner-Synge distribution function reads[9]

$$\mathfrak{N}(t,v) = (\alpha^3/8\pi) \exp[-\alpha v R(t)]. \qquad (3.35)$$

Since the mass of the photon vanishes, $u^0 = v$ and $\mathfrak{N}(t,v)$ may be rewritten as

$$\mathfrak{N}(t,u^0) \sim \exp[-\alpha R(t)\bar{u}^\mu u_\mu]. \qquad (3.36)$$

Chernikov has also shown[14] that the Jüttner-Synge distribution function for zero-rest-mass particles is a solution of the one-particle Liouville equation provided that $\bar{u}^\mu$ times a given factor is a conformal Killing vector. This is precisely the case here. Note that Eq. (3.35) or (3.36) shows that at time $t$ the system is in equilibrium at the *effective* temperature

$$T_{\text{eff}}(t) = T(0)R^{-1}(t), \qquad (3.37)$$

and hence in an expanding universe $T_{\text{eff}}$ is a decreasing

[12] It is not clear at all whether there exists such an initial singularity in the self-consistent models. However, a glance at Eqs. (3.1) and (3.2) indicates that this is probable.

[13] L. Bel, Astrophys. J. (to be published).

[14] N. A. Chernikov, Acta Phys. Polon. 26, 1069 (1964).

function of time, as is obvious from elementary physical considerations.

An apparently more "realistic" distribution function for radiation is provided by the "quantum" one:

$$\mathfrak{N}(t,v) \propto \{\exp[\alpha R(t)v]-1\}^{-1}, \qquad (3.38)$$

but, actually, this last distribution function does not change anything concerning the $R$ dependence of $p$ and $\mu$. Since in any case $T_\mu{}^\mu=0$ for radiation and because of the assumption that we deal with a perfect fluid, then $p=\frac{1}{3}\mu$. It then follows from Eq. (3.3) that

$$\mu=\mu_0 R^{-4} \qquad (3.39)$$

and

$$p=\tfrac{1}{3}\mu_0 R^{-4}. \qquad (3.40)$$

### Entropy of the Cosmic Gas

In relativistic kinetic theory[6,9] the total entropy is obtained through the 4-vector entropy density

$$S^\mu(x^\nu)=-(\sqrt{g})\int \frac{d_3u}{u^0}\mathfrak{N}(x^\nu,u^\nu)\ln\mathfrak{N}(x^\nu,u^\nu)u^\mu. \qquad (3.41)$$

In comoving coordinates, the only surviving component of $S^\mu$ is $S^0$, i.e., the entropy density. We have

$$S^0(t)=-4\pi\int_0^\infty dv\, v^2\mathfrak{N}(t,v)\ln\mathfrak{N}(t,v), \qquad (3.42)$$

and, finally,

$$S^0(t)=S^0(0)R^{-3}(t). \qquad (3.43)$$

The entropy of a given volume of cosmic gas is now

$$S(t)=VS^0(t)\sim R^3(t)S^0(0)R^{-3}(t) = S(0). \qquad (3.44)$$

Therefore, in this model the entropy of the universe is constant. This result was, of course, expected since the Vlasov approximation is reversible; irreversibility occurs when dealing with correlations between particles *or* when dealing with radiation emission.[6,15] It would be particularly interesting to derive the equations of motion of a test particle embedded in the cosmological space-time, taking gravitational radiation reaction into account. This would probably give rise to an intrinsic irreversibility of the universe; it is indeed the case for electromagnetic interactions.[6,15]

## 4. DISCUSSION

We now discuss briefly the above results.

(1) In the preceding sections the cosmic gas was assumed to be constituted of particles of only one mass, while in nature this is obviously not so. Therefore we have to generalize our equations slightly by considering the cosmic gas as constituted of different species of particles, i.e., by considering a several-fluids model. This generalization is rather trivial. We have indeed $n$ distribution functions $\mathfrak{N}_i$ ($i=1,\cdots,n$)—one per species—which satisfy $n$ one-particle Liouville equations of the type (1.1) coupled to Einstein's equations through the *total* momentum-energy tensor

$$T^{\mu\nu}=\sum_{i=1}^{i=n} T_i{}^{\mu\nu}. \qquad (4.1)$$

Each $T_i{}^{\mu\nu}$ is related to $\mathfrak{N}_i$ by an expression of the form (1.3). Furthermore, if we assume that there is only one vector field, say, $\bar{u}^\mu$, at our disposal, then the stream lines of the various subgases are alike and each $T_i{}^{\mu\nu}$ has the form (2.6). It then follows that

$$p=\sum_{i=1}^{i=n} p_i \qquad (4.2)$$

and

$$\mu=\sum_{i=1}^{i=n}\mu_i, \qquad (4.3)$$

as expected. The only (little) problem concerns homogeneity. However, if we assume—and this is in agreement with observation—that each kind of fluid is isotropic, then repeating the argument given at the end of Sec. 2, it follows that $\mathfrak{N}_i(x^\nu,u^\nu)\equiv\mathfrak{N}_i(t,u^0)$, in comoving coordinates.

(2) Furthermore, we neglected radiation. However, it is easy to see that radiation may be taken into account by simply adding a term of the form

$$\mu=\mu_0 R^{-4}(t) \qquad (4.4)$$

and/or

$$p=\tfrac{1}{3}\mu_0 R^{-4}(t) \qquad (4.5)$$

to $\mu$ and (or) $p$, respectively. Indeed, if we assume that the radiation fluid is a perfect fluid, the vanishing of the trace of its momentum-energy tensor provides the equation of state $p=\frac{1}{3}\mu$. As a consequence of Eq. (3.3), Eqs. (4.4) and (4.5) follow. Accordingly, there is no need for a kinetic theory of the radiation fluid. However, such a kinetic theory is really needed if one seriously wants to consider transport phenomena which occur near the initial singularity when our approximations fail.

(3) Near a possible initial singularity the consideration of only collective motions is no longer adequate and some refinements are necessary. First, we no longer have galaxies but only elementary particles, radiation, and neutrinos. Secondly, mutual transformations, nuclear reactions must be dealt with. Third, correlations must be properly taken into account.[16] We shall come back to these refinements in Paper II.

[15] R. Hakim and A. Mangeney, J. Math. Phys. 9, 116 (1968).

[16] A simplified description of the initial fireball is being performed by H. Y. Chiu and A. Salmona on the basis of the Boltzmann equation (A. Salmona, private communication).

(4) In Sec. 3 we found that Friedmann models are particular cases of the self-consistent models. At first sight this might be surprising since Friedmann models are such that $\mu \sim R^{-3}(t)$, whereas self-consistent models are such that $\mu \sim R^{-4}(t)$ near the singularity. In fact, we must point out that this last evaluation was not derived rigorously but rather on the basis of implicit regularity hypotheses. Actually, the choice $\mathfrak{N}_0(v) \sim \delta(v)/v^2$, which leads to Friedmann models, amounts to choosing a rather "irregular" initial distribution. [Note that $\delta(\mathbf{v}) = \delta(v)\{2\pi^2 \sin\theta\, v^2\}^{-1}$.]

(5) The usual red-shift formula has been generalized by Bel[13] to the case of an arbitrary source-observer relative motion, and he gave an interesting application of the above ideas which is shown to be valid in more general cases than the one he considered.

With our notations, Bel's formula reads

$$1+z=\nu/\nu_0=a(t,v)/a_0(t_0,v_0)\,, \tag{4.6}$$

where

$$a(t,v)=R^{-1}(t)[(v^2+1)^{1/2}-v\cos\theta]\,. \tag{4.7}$$

In Eqs. (4.6) and (4.7), $\nu$ is the proper frequency, $\nu_0$ is the observed frequency, and $\theta$ is the angle between the spatial velocity of the source and the emission direction.

The quantity $a_0$ is defined in the same way as $a$ except that it refers to the observer's arrival time, the observer's $v_0$, and the angle between the observer's motion and the direction of the signal $\theta_0$.

Therefore, we shall be able to calculate the *average* red shift and its *dispersion* for a typical galaxy at time $t$. To do that we need a conditional distribution function ($x^\nu$ being fixed; $x_\nu$ is the space-time position of the typical galaxy). Elsewhere[15] we have shown that such a conditional distribution is provided by

$$\mathfrak{N}(x^\nu|u^\nu)=u^\mu\bar{u}_\mu\mathfrak{N}(x^\nu,u^\nu)/[j^\mu(x^\nu)j_\mu(x^\nu)]^{1/2}\,. \tag{4.8}$$

Here we have[17]

$$\mathfrak{N}(t|u^0)=[R^3(t)/\rho(0)]u^0\mathfrak{N}(t,u^0)\,, \tag{4.9}$$

and finally, we get

$$\mathfrak{N}(t|v)=\frac{R^3(t)}{\rho(0)}\mathfrak{N}(t,v)=\frac{R^3(t)}{\rho(0)}\mathfrak{N}_0[R(t)v]\,. \tag{4.10}$$

It follows that $\langle a(t,v)\rangle$ and $\langle a^2(t,v)\rangle$ are given by

$$\langle a(t,v)\rangle=\frac{R^{-2}(t)}{\rho(0)}\int d_3v\, v^2\mathfrak{N}_0(v) \times\{[v^2+R^2(t)]^{1/2}-v\cos\theta\} \tag{4.11}$$

and

$$\langle\{a(t,v)\}^2\rangle=\frac{R^{-4}(t)}{\rho(0)}\int d_3v\, v^2\mathfrak{N}_0(v) \times\{[v^2+R^2(t)]^{1/2}-v\cos\theta\}^2\,. \tag{4.12}$$

From Eqs. (4.11) and (4.12) we see that, in an expanding universe and for large $R$'s, we generally have

$$\langle a(t,v)\rangle \sim R^{-1}(t)\,, \tag{4.13}$$

$$\langle[a(t,v)]^2\rangle \sim R^{-2}(t)\,. \tag{4.14}$$

As a consequence, the dispersion[18] of $a(t,v)$ behaves as $R^{-2}(t)$ and hence there exists a dispersion in red shift (due to proper motion) which decreases in an expanding universe as a function of the *emission time*. It is interesting to note that the behavior (4.13) and (4.14) of $\langle a\rangle$ and $\langle a^2\rangle$ is the same as that given by Bel,[13] using a Gaussian distribution. Equations (4.11) and (4.12) may be simplified by using polar coordinates, the polar axis being chosen as the axis of the emission direction. We find

$$\langle a(t,v)\rangle=R^{-1}(t)[\mu(t)/m\rho(t)]\,. \tag{4.15}$$

It follows that the red-shift formula now is

$$1+\langle z\rangle=\frac{\langle a(t,v)\rangle}{a_0}=\frac{R(t_0)}{R(t)}\frac{\mu(t)}{m\rho(t)}\,, \tag{4.16}$$

where we have chosen $v_0=0$. Equation (4.16) shows that, in an expanding universe and when $R(t)\gg 1$ [i.e., when $\mu(t)\sim m\rho(t)$], we recover the usual red-shift formula.

In the same way we find

$$\langle a^2(t,v)\rangle=R^{-2}(t)+XR^{-4}(t)\,, \tag{4.17}$$

with

$$X=\frac{16\pi}{3}\rho^{-1}(0)\int_0^\infty dv\, v^4\mathfrak{N}_0(v)\,. \tag{4.18}$$

The red-shift dispersion is then

$$\sigma^2=a_0^{-2}(\langle a^2\rangle-\langle a\rangle^2) =\frac{R^2(t_0)}{R^2(t)}\left[1-\frac{\mu(t)}{m\rho(t)}\right]+X\frac{R^2(t)}{R^4(t)} \tag{4.19}$$

or

$$\sigma^2=(1+\langle z\rangle)^2\left[1-\frac{\mu(t)}{m\rho(t)}\right]+X\frac{(1+\langle z\rangle)^4}{R^2(t_0)}\left\{\frac{m\rho(t)}{\mu(t)}\right\}^4\,. \tag{4.20}$$

Similar considerations led Bel to the conclusion that "the dispersion of the red shift is an increasing function of the source-observer distance provided this distance is large enough."[15] This conclusion is still valid here for more general distribution functions than Gaussian densities. However, more accurate evaluations are necessary before convincing conclusions may be drawn as to the observed red-shift dispersion of quasars. In the same way, we can obtain the dispersion of other

[17] x does not appear because of homogeneity. Note also that all average values calculated with the help of this conditional probability depend implicitly on the hypersurface $t=$ const. In Ref. 15 we pointed out that the definition of such a conditional distribution function is generally ambiguous *except* in the case where we know only one 4-vector. This is the case here.

[18] It is clear that, since the choice of $\mathfrak{N}_0(v)\sim\delta(v)$ leads to Friedmann models, these models cannot present a dispersion of red shift (absence of random motions).

observable quantities. This is studied in detail in Paper II.

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## APPENDIX A

The calculations of $\mu(R)$ and $p(R)$ can be performed only with a limited number of physically meaningful initial distributions. One of them is

$$\mathfrak{N}_0(v)=\rho(0)(\alpha^3/8\pi)\exp(-\alpha v)\,,\quad \alpha>0\,. \tag{A1}$$

Equation (A1) would correspond to pure radiation *if* the expressions which give $p$ and $\mu$ in terms of $\mathfrak{N}_0$ would be modified (for radiation $u^\mu$ is isotropic, $u^\mu u_\mu=0$, and the various formulas given previously would no longer be valid). However, if we consider that the initial state is extremely hot and in equilibrium, i.e., that the particles of the universe[19] are ultrarelativistic, then Eq. (A1) is a good approximation for one species of particles. Indeed, taking $v$ instead of $u^0$ amounts to neglecting the rest-mass energy with respect to thermal energy. Note that $\alpha=mc^2/kT$. This approximation will be valid as long as the cosmic gas may be considered as ultrarelativistic, i.e., as long as

$$\alpha R(t)\gg 1\,.$$

In other words, this approximation will be valid as long as the effective temperature

$$T_{\text{eff}}=TR^{-1}(t)\gg mc^2\,. \tag{A2}$$

Now, using Eqs. (3.9) and (3.10), we obtain

$$\mu(R)=\rho(0)(\alpha^3/8\pi)R^{-4}(t)(\partial^2/\partial\alpha^2)\mathfrak{L}[v^2+R^2(t)]^{1/2} \tag{A3}$$

and

$$p(R)=\rho(0)(\alpha^3/8\pi)R^{-4}\partial^4/\partial\alpha^4\mathfrak{L}[v^2+R^2(t)]^{-1/2}\,. \tag{A4}$$

In Eqs. (A3) and (A4), $\mathfrak{L}$ denotes the Laplace transform.

These Laplace transforms are found to be[20]

$$\mathfrak{L}[v^2+R^2(t)]^{-1/2}=\tfrac{1}{2}\pi\{\mathbf{H}_0[\alpha R(t)]-N_0[\alpha R(t)]\} \tag{A5}$$

and

$$\mathfrak{L}[v^2+R^2(t)]^{1/2} =[\pi R(t)/\alpha]\{\mathbf{H}_1[\alpha R(t)]-N_1[\alpha R(t)]\}\,, \tag{A6}$$

where $N_0$ and $N_1$ are Neumann functions of order 0 and 1 and where $\mathbf{H}_0$ and $\mathbf{H}_1$ are Struve functions of order 0 and 1.

## APPENDIX B

It may be of interest to obtain a true Vlasov equation for $\mathfrak{N}(t,v)$. This may be easily effected if we notice that we have to couple the following equations:

Liouville equation:

$$(\partial/\partial t)\mathfrak{N}(t,v)-\dot{R}(t)R^{-1}(t)v(\partial/\partial v)\mathfrak{N}(t,v)=0 \tag{B1}$$

and

$$\text{Einstein equation:}\quad \chi\mu(R)=3R^{-2}(t)(k+\dot{R}^2)-\lambda \tag{B2}$$

through Eq. (3.9) for $\mu(R)$. Finally, we obtain

$$\frac{\partial}{\partial t}\mathfrak{N}\mp\left\{4\pi m\chi\int_0^\infty dv\, v^2(v^2+1)^{1/2}\mathfrak{N}+\lambda -\tfrac{1}{3}k\left[\left(\frac{d}{dR}4\pi m\int_0^\infty dv\, v^2(v^2+1)^{1/2}\mathfrak{N}\right)^2 \times\left(4\pi m\int_0^\infty dv\, v^2[(v^2+1)^{1/2}-\tfrac{1}{3}v^2(v^2+1)^{-1/2}]\mathfrak{N}\right)^{-2}\right]\right\}^{1/2} \times v(\partial/\partial v)\mathfrak{N}=0\,, \tag{B3}$$

which is highly nonlinear, as expected. In Paper II we shall use a modified form of this equation to study problems of stability of the self-consistent models against an inhomogeneous perturbation.

## APPENDIX C

From the one-particle Liouville equation (3.6), hydrodynamical equations may easily be obtained.

(a) Multiplying Eq. (3.6) by $4\pi v^2$ and integrating over $v$, we find an equation for $\rho(t)$. Once integrated we obtain $\rho(t)=\rho(0)R^{-3}(t)$, as expected.

(b) Multiplying Eq. (3.6) by $4\pi v^2(v^2+1)^{1/2}m$ and integrating over $v$, we are finally led to Eq. (3.3). This is not surprising since in both cases we express the momentum-energy conservation law.

(c) Other equations may be obtained in a similar way, such as equations for $\partial\langle z\rangle/\partial t$ or $\partial(\sigma^2)/\partial t$, etc.

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[19] Here, in the initial state, the particles of the cosmic gas are not galaxies—which are not yet formed—but rather elementary particles.

[20] W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1954).