Magnetic Moment of a Magnetized Fermi Gas

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In this paper we have calculated the exact expression for the magnetic moment of a Fermi gas in a magnetic field (magnetized Fermi gas). The susceptibility obtained from our result becomes the classical Curie-Langevin law in the weak-field limit for a nondegenerate and nonrelativistic gas. The induced magnetic moment $\mathfrak{M}$ was found to be proportional to the square of the electron density, but at higher densities $\mathfrak{M}$ passes through a maximum and eventually becomes negative; hence a magnetized gas is first paramagnetic but later becomes diamagnetic. We have also shown that spontaneous magnetization cannot take place in a noninteracting electron gas.

I. INTRODUCTION

IN two previous papers,[1,2] we derived a general expression for the equation of state for a system of noninteracting Fermi particles in a magnetic field (a magnetized Fermi gas), and studied the thermodynamic properties of such a gas. We found that in the limit of large quantum numbers, a magnetized Fermi gas reduces to an ordinary Fermi gas, whose properties have been described elsewhere.[3,4] In the case of small quantum numbers, the properties of a magnetized gas are quite different from a Fermi gas. The lowest magnetized state is characterized by the alignment of all electron spins with the field, and in this limit a magnetized gas behaves exactly as a one-dimensional gas, which has been studied extensively elsewhere as a theoretical problem.[5] As the density or temperature increases, higher magnetized states are excited. The gas behaves more like an ordinary Fermi gas and eventually approaches it in the classical limit. In previous papers,[1,2] criteria were given for a gas to be considered as a magnetized gas, and will not be discussed here.

In this paper we are interested in the magnetic properties of a magnetized Fermi gas.[6] The magnetic properties of a system are intimately related to the macroscopic magnetic moment of the system, which we will calculate in the sections that follow. We are also interested in an important question: Can magnetization arise spontaneously in an electron gas? Answers to this question will be found through a study of the total magnetic moment of the system.

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[1] V. Canuto and H. Y. Chiu, second preceding paper, Phys. Rev. 172, 1210 (1968), hereafter referred to as Paper I.

[2] V. Canuto and H. Y. Chiu, first preceding paper, Phys. Rev. 172, 1220 (1968), hereafter referred to as Paper II.

[3] E. Schatzman, *White Dwarfs* (North-Holland Publishing Co., Amsterdam, 1958), Chap. 4.

[4] R. Kubo, *Statistical Mechanics* (North-Holland Publishing Co., Amsterdam, 1965).

[5] E. H. Lieb and D. C. Mattis, *Mathematical Physics of One Dimension* (Academic Press Inc., New York, 1966).

[6] The magnetic properties of the electron in solids and in rarefied gases have been so extensively studied that a comprehensive list is impractical. See, for example, D. C. Mattis, *The Theory of Magnetism* (Harper and Row, New York, 1965); R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, New York, 1955).

II. GRAND PARTITION FUNCTION

The grand partition function $\mathfrak{Z}$ is defined by[7]

$$\ln\mathfrak{Z}=\ln \operatorname{Tr} \exp[\hat{\mathfrak{H}}-\tilde{\beta}(-\tilde{\mu}\hat{N})], \quad (1)$$

where $\tilde{\beta}=(kT)^{-1}$, $\hat{\mathfrak{H}}$ is the Hamiltonian operator for the system, $\tilde{\mu}$ is the chemical potential plus the rest energy in cgs units, and $\hat{N}$ is the particle occupation-number operator. From Eq. (1) we have derived[2] the following expression for $\mathfrak{Z}$:

$$\ln\mathfrak{Z}=\sum_{n,p_z}\omega_n\{\ln[1+\exp(-\tilde{\beta}(E_{1n}-\tilde{\mu}))] +\ln[1+\exp(-\tilde{\beta}(E_{2n}-\tilde{\mu}))]\}, \quad (2)$$

where ω_n is the statistical weight[7] for states of quantum number n for a system with volume Ω:

$$\omega_n=\Omega^{2/3}eH/2\pi\hbar c \quad (3)$$

and

$$E_{nr}=\pm mc^2\left[1+\left(\frac{p_z}{mc}\right)^2+2\frac{H}{H_c}(n+r-1)\right]^{1/2}, \quad (4)$$

$$r=1,\,2,\quad n=0,\,1,\,2,\,\cdots,\,\infty\,.$$

p_z is the z component of the momentum of the electron, H is the magnetic field which is in the z direction, E_{nr} is the quantized energy of the electron in the magnetic field, and

$$H_c=m^2c^3/e\hbar=4.414\times10^{13}\text{ G}. \quad (5)$$

The symbols e, $\hbar$, m, c, and k all have their usual meaning.

Carrying out the sum over p_z in the usual manner, we have the following expression for $\ln\mathfrak{Z}$ (see Paper II for

[7] K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963), Chap. 11, pp. 237–243.

details):

$$\frac{1}{\Omega}\ln\mathfrak{z}=\frac{eH}{2\pi\hbar}\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}dx\{\ln[1+\exp(-\tilde{\beta}(E_{1n}-\tilde{\mu}))]+\ln[1+\exp(-\tilde{\beta}(E_{2n}-\tilde{\mu}))]\},$$

which can be written as

$$\frac{1}{\Omega}\ln\mathfrak{z}=\frac{eH}{\pi\hbar}\Big(\frac{1}{2}\int_{-\infty}^{+\infty}dx\,\ln\{1+\exp[-\beta(E(x,0,H)-\mu)]\}+\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}dx\,\ln\{1+\exp[-\beta(E(x,n,H)-\mu)]\}\Big),\quad(6)$$

where

$$\beta=\tilde{\beta}mc^2,\quad \mu=\tilde{\mu}/mc^2,\quad x=p_z/mc,\qquad E(x,n,H)=(1+x^2+2nH/H_c)^{1/2}.\quad(6')$$

III. MAGNETIC MOMENT

The magnetic moment operator $\hat{\mathfrak{M}}$ is defined by[8]

$$\hat{\mathfrak{M}}=-\partial\hat{\mathcal{H}}/\partial H,\quad(7)$$

where $\hat{\mathcal{H}}$ is the Hamiltonian for a Dirac particle (see Paper I). The statistical average of $\hat{\mathfrak{M}}$ then gives the total magnetic moment of the system, $\mathfrak{M}$:

$$\mathfrak{M}=\mathrm{Tr}(\rho)\hat{\mathfrak{M}}/\mathrm{Tr}\rho,\quad(8)$$

where ρ is the density matrix, in the grand canonical ensemble

$$\rho=\exp[-\tilde{\beta}(\hat{\mathcal{H}}-\tilde{\mu}\hat{N})].\quad(9)$$

We now have

$$\mathfrak{M}=\frac{\mathrm{Tr}\{-(\partial\hat{\mathcal{H}}/\partial H)\exp[-\tilde{\beta}(\hat{\mathcal{H}}-\tilde{\mu}\hat{N})]\}}{\mathrm{Tr}\{\exp[-\tilde{\beta}(\hat{\mathcal{H}}-\tilde{\mu}\hat{N})]\}}.\quad(10)$$

If we keep the chemical potential $\tilde{\mu}$ constant with respect to the magnetic field,[7] then Eq. (10) becomes

$$\mathfrak{M}=\frac{1}{\tilde{\beta}}\frac{\partial}{\partial H}\ln\mathrm{Tr}\exp[-\tilde{\beta}(\hat{\mathcal{H}}-\tilde{\mu}\hat{N})],\qquad \mathfrak{M}=\frac{1}{\tilde{\beta}}\frac{\partial}{\partial H}\ln\mathfrak{z},\quad(11)$$

where $\mathfrak{z}$ is the grand partition function defined in Eq. (1) and explicitly given in Eq. (6). Equation (11) is a generalization to the nonzero-temperature case of the Pauli-Feynman theorem,[8,9] which states that

$$\langle n|\hat{\mathfrak{M}}|n\rangle=-\frac{\partial}{\partial H}\langle n|\hat{\mathcal{H}}|n\rangle\quad(12)$$

when $|\eta\rangle$ is the eigenstate of $\hat{\mathcal{H}}$.

Using Eq. (6), we find that, explicitly, $\mathfrak{M}$ is given by

$$\mathfrak{M}=\frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\Big\{\tfrac{1}{2}\phi\int_0^\infty\ln\Big[1+\exp\Big(-\frac{E(x,0,H)-\mu}{\phi}\Big)\Big]dx+\phi\sum_{n=1}^{\infty}\int_0^\infty\ln\Big[1+\exp\Big(-\frac{E(x,n,H)-\mu}{\phi}\Big)\Big]dx-\frac{H}{H_c}\sum_{n=1}^{\infty}n\int_0^\infty E^{-1}(x,n,H)\times\Big[1+\exp\Big(-\frac{E(x,n,H)-\mu}{\phi}\Big)\Big]dx,\quad(13)$$

where $\phi=kT/mc^2$, $\mu_B=e\hbar/2mc$ is the Bohr magneton, and $\lambda_c=\hbar/mc$ is the Compton wavelength of the electron.

Equation (13) is the most general expression of the magnetic moment of a noninteracting Fermi gas in an external magnetic field. One can use the exact wave function (discussed in Paper II) to evaluate (12) and to show that, after taking statistical averages, Eq. (13) be obtained from Eq. (8) using the Hamiltonian of the Dirac equation as described in Paper I.

We shall now study the classical properties of $\mathfrak{M}$ as given in Eq. (13).

IV. CLASSICAL LIMIT—CURIE-LANGEVIN LAW

In Paper I, the classical limit was defined as the limit in which the sum over n can be replaced by an integration over n. We shall now show that in the classical limit the nonrelativistic and nondegenerate expression of Eq. (13) gives the Curie-Langevin law for the magnetic susceptibility.

The nondegenerate and nonrelativistic case is given by the following condition:

$$\exp\Big(\frac{E(x,n,H)-\mu}{\phi}\Big)\gg1,\quad E(x,n,H)\rightarrow1+\tfrac{1}{2}x^2+n\frac{H}{H_c},\quad(14)$$

$$\ln\Big[1+\exp\Big(-\frac{E(x,n,H)-\mu}{\phi}\Big)\Big]=\exp\Big(-\frac{E(x,n,H)-\mu}{\phi}\Big).\quad(15)$$

Equation (13) becomes

$$\mathfrak{M}=\frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\Big[\frac{1}{2}\phi\exp\Big(\frac{\mu-1}{\phi}\Big)+\phi\sum_{n=1}^{\infty}\exp\Big(\frac{\mu-1}{\phi}\Big)\times\exp\Big(-\frac{nH}{\phi H_c}\Big)-\frac{H}{H_c}\sum_{n=1}^{\infty}n\exp\Big(\frac{\mu-1}{\phi}\Big)\times\exp\Big(-\frac{nH}{\phi H_c}\Big)\Big]I_0,\quad(16)$$

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[8] C. Kittel, *Elementary Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1958).

[9] W. Pauli, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958); see also Refs. 7 and 8.

where

$$I_0=\int_0^\infty \exp\left(-\frac{x^2}{2\phi}\right)dx.$$

Let

$$\lambda=\exp[(\mu-1)/\phi] \tag{17}$$

and, after rearranging terms in the sum (which is allowed because these series are easily seen to be absolutely convergent), we have

$$\mathfrak{M}=\frac{1}{\pi^2}\frac{\mu_B}{\lambda_c^3}\left\{\lambda\phi\left[1+\exp\left(-\frac{H}{\phi H_c}\right)\right]\sum_{n=0}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right) -2\lambda\frac{H}{H_c}\sum_{n=1}^{\infty} n\exp\left(-\frac{nH}{\phi H_c}\right)\right\}I_0. \tag{18}$$

The following results are apparent:

$$I_0=\int_0^\infty \exp\left(-\frac{x^2}{2\phi}\right)dx=(2\pi\phi)^{1/2}, \tag{19}$$

$$\sum_{n=0}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right)=\left[1+\exp\left(-\frac{H}{\phi H_c}\right)\right]^{-1}, \tag{20}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n\exp\left(-\frac{nH}{\phi H_c}\right)&=-\phi H_c\frac{\partial}{\partial H}\sum_{n=1}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right)=-\tfrac{1}{2}\phi H_c\frac{\partial}{\partial H}\left[1+2\sum_{n=1}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right)\right]\\ &=-\tfrac{1}{2}\phi H_c\frac{\partial}{\partial H}\left[\sum_{n=0}^{\infty}\exp\left(-\frac{(n+1)H}{\phi H_c}\right)+\sum_{n=0}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right)\right]=-\tfrac{1}{2}\phi H_c\frac{\partial}{\partial H}\left\{\left[1+\exp\left(-\frac{H}{\phi H_c}\right)\right]\sum_{n=0}^{\infty}\exp\left(-\frac{nH}{\phi H_c}\right)\right\}\\ &=-\tfrac{1}{2}\phi H_c\frac{\partial}{\partial H}\frac{1+\exp(-H/\phi H_c)}{1-\exp(-H/\phi H_c)}=-\tfrac{1}{2}\phi H_c\frac{\partial}{\partial H}\coth\left(\frac{\mu_B H}{kT}\right),\end{aligned} \tag{21}$$

where

$$\mu_B H/kT=\tfrac{1}{2}H/\phi H_c. \tag{22}$$

We therefore find

$$\begin{aligned}\mathfrak{M}&=\frac{(2\pi\phi)^{1/2}\lambda\mu_B}{\pi^2\lambda_c^3}\left(\coth\eta+\eta\frac{\partial}{\partial\eta}\coth\eta\right),\\ \mathfrak{M}&=\frac{(2\pi\phi)^{1/2}\lambda\mu_B}{\pi^2\lambda_c^3}\frac{\partial}{\partial\eta}(\eta\coth\eta),\end{aligned} \tag{23}$$

where $\eta=\mu_B H/kT$.

Expanding $\eta\coth\eta$ and retaining terms of the lowest order, we find

$$\frac{\partial}{\partial\eta}(\eta\coth\eta)\to\tfrac{2}{3}\eta,\quad \eta\to 0. \tag{24}$$

The magnetic susceptibility χ is defined by[4]

$$\chi=\lim_{H\to 0}(\mathfrak{M}/H). \tag{25}$$

From Eqs. (23)–(25) we obtain

$$\chi=\frac{2}{3}\frac{\mu_B^2}{kT}\lambda\left[\frac{1}{4\pi^3}\left(\frac{2\pi kT}{mc^2}\right)^{3/2}\frac{1}{\lambda_c^3}\right]. \tag{26}$$

We shall now eliminate λ. The definition of the particle density $\mathfrak{N}$ is

$$\mathfrak{N}=\frac{1}{\Omega}\lambda\frac{\partial}{\partial\lambda}\ln\mathfrak{Z}. \tag{27}$$

A simplified expression for $\ln\mathfrak{Z}$ can be obtained from Eqs. (6), (14), (15), and (19)–(21). We have

$$\ln\mathfrak{Z}=\frac{\Omega}{4\pi^3}\frac{\lambda}{\lambda_c^3}\left(\frac{2\pi kT}{mc^2}\right)^{3/2}. \tag{28}$$

Hence

$$\mathfrak{N}=\frac{1}{\Omega}\ln\mathfrak{Z}=\frac{1}{4\pi^3}\frac{\lambda}{\lambda_c^3}\left(\frac{2\pi kT}{mc^2}\right)^{3/2}. \tag{29}$$

Substituting Eq. (29) into Eq. (26), we obtain

$$\chi=\tfrac{2}{3}(\mu_B^2/kT)\mathfrak{N}\propto 1/T, \tag{30}$$

which is the Curie-Langevin law for a classical gas at high temperature and low density.[4,7]

This classical result includes both the diamagnetic part (due to particle motions in a magnetic field) and the paramagnetic part (due to the spin or the intrinsic magnetic moment of the particles). One can show that the paramagnetic contribution to χ is $\mu_B\mathfrak{N}/kT$ and the diamagnetic contribution is $-\frac{1}{3}$ of the paramagnetic contribution, so that the resultant is that given in Eq. (30).

It is difficult, however, to separate the general expression (13) into the corresponding diamagnetic and paramagnetic parts. Nevertheless, we can still decompose[10] (Gordon decomposition) the four-current j_μ

$$j_\mu=ie\bar{\psi}\gamma_\mu\psi=j_\mu^{(1)}+j_\mu^{(2)}$$

[10] J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1967), pp. 107–110.

such that

$$j_\mu^{(1)}A_\mu=(ie\hbar/2m)[\partial_\mu\bar\psi\psi-\psi\partial_\mu\bar\psi]A_\mu-(e^2/mc)A_\mu^2\bar\psi\psi$$

becomes the Landau Hamiltonian giving rise to diamagnetism in the nonrelativistic limit.

The other part,

$$j_\mu^{(2)}A_\mu=-\frac{e\hbar}{2mc}\left[\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}(\bar\psi\sigma_{\mu\nu}\psi)+\frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}(\bar\psi\sigma_{\mu\nu}\psi)\right],$$

$$\sigma_{\mu\nu}=-i\gamma_\mu\gamma_\nu\quad(\mu\neq\nu),$$

becomes the Pauli Hamiltonian giving rise to spin paramagnetism in the nonrelativistic limit.

V. REDUCTIONS OF THE EXPRESSION FOR $\mathfrak{M}$

We can apply the same procedure as in Paper II to reduce $\mathfrak{M}$ into a sum over the same function over different arguments. The following transformation, which has been explained previously, is introduced:

$$v=x/a_n,\tag{31}$$

$$a_n=[1+2(H/H_c)n]^{1/2},\tag{32}$$

and the following functions are defined:

$$C_1(\phi,\mu)=\int_0^\infty(1+v^2)^{-1/2}\times\left[1+\exp\left(\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]^{-1}dv,\tag{33}$$

$$C_2(\phi,\mu)=\phi\int_0^\infty\ln\left[1+\exp\left(-\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]dv.\tag{34}$$

By partial integration, we find that

$$C_2(\phi,\mu)=\int_0^\infty\frac{v^2}{(1+v^2)^{1/2}}\times\left[1+\exp\left(\frac{(1+v^2)^{1/2}-\mu}{\phi}\right)\right]^{-1}dv.\tag{35}$$

Equation (13) can be written as

$$\mathfrak{M}=\frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\left\{\frac{1}{2}C_2(\phi,\mu)+\sum_{n=1}^{\infty}a_n^2C_2\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right)-\frac{H}{H_c}\sum_{n=1}^{\infty}nC_1\left(\frac{\phi}{a_n},\frac{\mu}{a_n}\right)\right\}.\tag{36}$$

Numerically,

$$\mathfrak{M}_0\equiv(2/\pi^2)\mu_B/\lambda_0^3=3.2637\times10^{10}\ \text{G}.\tag{37}$$

The properties of $C_1(\phi,\mu)$ and $C_2(\phi,\mu)$ have been extensively discussed in Paper II. In the degenerate limit $\phi\to0$, we have

$$C_1(\mu)\equiv\lim_{\phi\to0}C_1(\phi,\mu)=\ln[\mu+(\mu^2-1)^{1/2}],\tag{38}$$

$$C_2(\mu)\equiv\lim_{\phi\to0}C_2(\phi,\mu)=\tfrac{1}{2}\{\mu(\mu^2-1)^{1/2}-\ln[\mu+(\mu^2-1)^{1/2}]\},\tag{39}$$

for $\mu>1$, and

$$C_1(\mu)=C_2(\mu)=0,\quad\mu<1.\tag{40}$$

VI. DEGENERATE EXPRESSIONS FOR THE MAGNETIC MOMENT

As discussed in Paper II, the condition of degeneracy is expressed by

$$\mu-1>0,\quad\phi\to0.\tag{41}$$

From Eq. (40) it can be concluded that the sum over n in the expression for $\mathfrak{M}$ [Eq. (36)] terminates at s, where s satisfies the condition

$$a_s\leq\mu<a_{s+1}.\tag{42}$$

We therefore find

$$\mathfrak{M}=\frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\left[\frac{1}{2}C_2(\mu)+\sum_{n=1}^{s}a_n^2C_2\left(\frac{\mu}{a_n}\right)-\frac{H}{H_c}\sum_{n=1}^{s}nC_1\left(\frac{\mu}{a_n}\right)\right].\tag{43}$$

The first two terms are positive and the third term is negative. When $s=0$, the only nonvanishing term is the first term. As discussed in Paper II, this corresponds to the one-dimensional gas limit in which the spin of all electrons is aligned with the magnetic field. The resulting magnetic moment is therefore the magnetic moment of all electrons. As is well known in nonrelativistic quantum mechanics, the electron spin gives rise to paramagnetism, and the electron orbital motions give rise to diamagnetism. Hence, at the lowest-energy state there is only paramagnetism.

When quantum states other than the $n=0$ states are excited, the gas gradually becomes diamagnetic and $\mathfrak{M}$ then becomes negative. The nonrelativistic and nondegenerate limit of large quantum numbers n has been discussed in Sec. IV.

A. The Case $s=0$. Ground State of a Magnetized Gas

When the density is such that

$$1\leq\mu<a_1\equiv(1+2H/H_c)^{1/2},\tag{44}$$

the only nonvanishing term is the first term $\frac{1}{2}C_2(\mu)$. This corresponds to the one-dimensional gas limit, as discussed in Paper II. We shall obtain the magnetic moment $\mathfrak{M}$ as a function of particle density $\mathfrak{N}$ for this case.

The relation between the density $\mathfrak{N}$ and μ is [Eq. (81), Paper II], for the case given by (44),

$$\mathfrak{N}=\frac{1}{\pi^2}\frac{H}{H_c}\frac{1}{\lambda_c^3}\tfrac{1}{2}C_4(\mu)=\mathfrak{N}_0\frac{H}{H_c}C_4(\mu), \tag{45}$$

$$C_4(\mu)=(\mu^2-1)^{1/2}, \quad \mathfrak{N}_0=\frac{1}{2\pi^2}\frac{1}{\lambda_c^3}. \tag{46}$$

From Eqs. (46) and (39), we find the relation between $C_2(\mu)$ and $C_4(\mu)$:

$$\begin{aligned} C_2(\mu)&=\tfrac{1}{2}C_4(\mu)[C_4^2(\mu)+1]^{1/2} \\ &\qquad -\tfrac{1}{2}\ln\{C_4(\mu)+[C_4^2(\mu)+1]^{1/2}\} \\ &=\tfrac{1}{2}\xi(\xi^2+1)^{1/2}-\tfrac{1}{2}\ln[\xi+(\xi^2+1)^{1/2}], \end{aligned} \tag{47}$$

where

$$\xi=C_4(\mu)=\mathfrak{N}/(\mathfrak{N}_0 H/H_c). \tag{48}$$

Therefore,

$$\begin{aligned} \mathfrak{M}&=\frac{1}{2\pi^2}\frac{\mu_B}{\lambda_c^3}\xi(\xi^2+1)^{1/2}\left(1-\frac{\ln[\xi+(\xi^2+1)^{1/2}]}{\xi(\xi^2+1)^{1/2}}\right) \\ &=\frac{1}{2\pi^2}\frac{\mu_B}{\lambda_c^3}\frac{\mathfrak{N}}{\mathfrak{N}_0 H/H_c}\left[\left(\frac{\mathfrak{N}}{\mathfrak{N}_0 H/H_c}\right)^2+1\right]^{1/2} \\ &\qquad\times\mathfrak{F}_1\left(\frac{\mathfrak{N}}{\mathfrak{N}_0 H/H_c}\right), \end{aligned} \tag{49}$$

where

$$\mathfrak{F}_1(x)=1-\frac{\ln[x+(x^2+1)^{1/2}]}{x(x^2+1)^{1/2}}, \tag{50}$$

and

$$\mathfrak{F}_1(x)\to x^2/(1+x^2)^{1/2}, \quad x\ll 1 \tag{51}$$

$$\mathfrak{F}_1(x)\to 1, \quad x\gg 1. \tag{52}$$

From Eq. (49) it may be concluded that $\mathfrak{M}$ increases as $\mathfrak{N}^2$ at a given field strength. However, when $\mathfrak{N}$ is too large, higher magnetic states with $s>0$ will enter, causing $\mathfrak{M}$ to decrease. Hence, there is a combination of the field strength and the density such that $\mathfrak{M}$ is a maximum. This maximum value of $\mathfrak{M}$ can be estimated as follows:

The value of μ should be close to the upper limit given in Eq. (44), and for large values of H/H_c it is

$$\mu\approx(2H/H_c)^{1/2}. \tag{53}$$

This gives the following relation between $\mathfrak{N}$ and H/H_c for the maximum value of $\mathfrak{M}$:

$$\mathfrak{N}\approx\sqrt{2}\,\mathfrak{N}_0(H/H_c)^{3/2}. \tag{54}$$

Substituting Eq. (54) into Eq. (49), we obtain the maximum value of $\mathfrak{M}$ as a function of H for $\mathfrak{N}\gg\mathfrak{N}_0 \times H/H_c$:

$$\mathfrak{M}\simeq\frac{1}{\pi^2}\frac{\mu_B}{\lambda_c^3}\frac{H}{H_c}\approx 10^{-3}H. \tag{55}$$

B. The Case $s>0$

In the case $s>0$, the higher magnetic states are occupied and the gas gradually becomes diamagnetic. At a given field strength, $\mathfrak{M}$ increases with density until reaching a maximum given by Eq. (55), then decreases with density, eventually becoming negative. Figures 1 and 2 show the relation between $\mathfrak{M}$ and H/H_c, and $\mathfrak{M}$ and $\mathfrak{N}$. The undulating behavior of $\mathfrak{M}$ is due to the peculiar behavior of the density of states shown in Fig. 4 of Paper II.

VII. ABSENCE OF PERMANENT MAGNETISM IN A DENSE ELECTRON GAS

In solids, it has been discovered that the magnetic susceptibility χ is given by the Curie-Weiss Law[11]

$$\chi=C/(T-\theta), \tag{56}$$

where C is a constant, and θ is the transition temperature, below which there is ferromagnetism and above which there is paramagnetism. This may suggest that an electron gas will also possess a transition temperature so that it will become ferromagnetic at low temperatures. This is not the case in the absence of large-scale ordered phenomena, as will be shown.

We can now answer the question of whether a dense electron gas can be self-magnetized. This question has been raised many times with respect to gravitational-collapse problems.

The relation between the resulting field B, the impressed field H, and the induced magnetic moment $\mathfrak{M}$ is

$$B=\mu_0(H+\mathfrak{M}), \tag{57}$$

where μ_0 is a constant which takes care of dimensions. Consider a small cavity in a magnetized body (Fig. 3). The field felt by a sample in the cavity is B; hence if

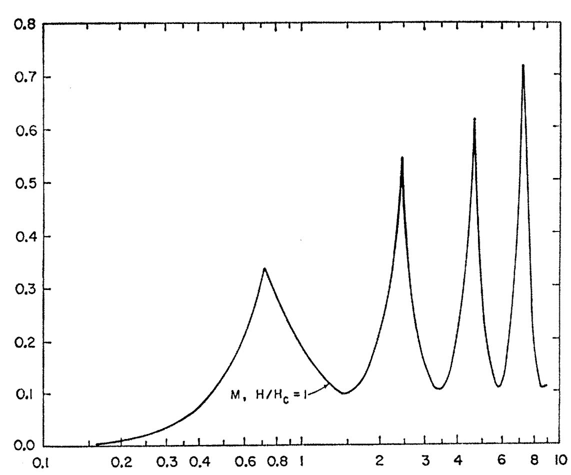

FIG. 1. The relation between $\mathfrak{M}$ and N/N_0 for $H/H_c=1$. $M=\mathfrak{M}/\mathfrak{M}_0$, where $\mathfrak{M}_0=(2/\pi^2)\mu_B/\lambda_c^3$ and $N_0=(1/\pi^2)(1/\lambda_c^3)$.

[11] A. H. Morrish, *Physical Principles of Magnetism* (John Wiley & Sons, Inc., New York, 1965).

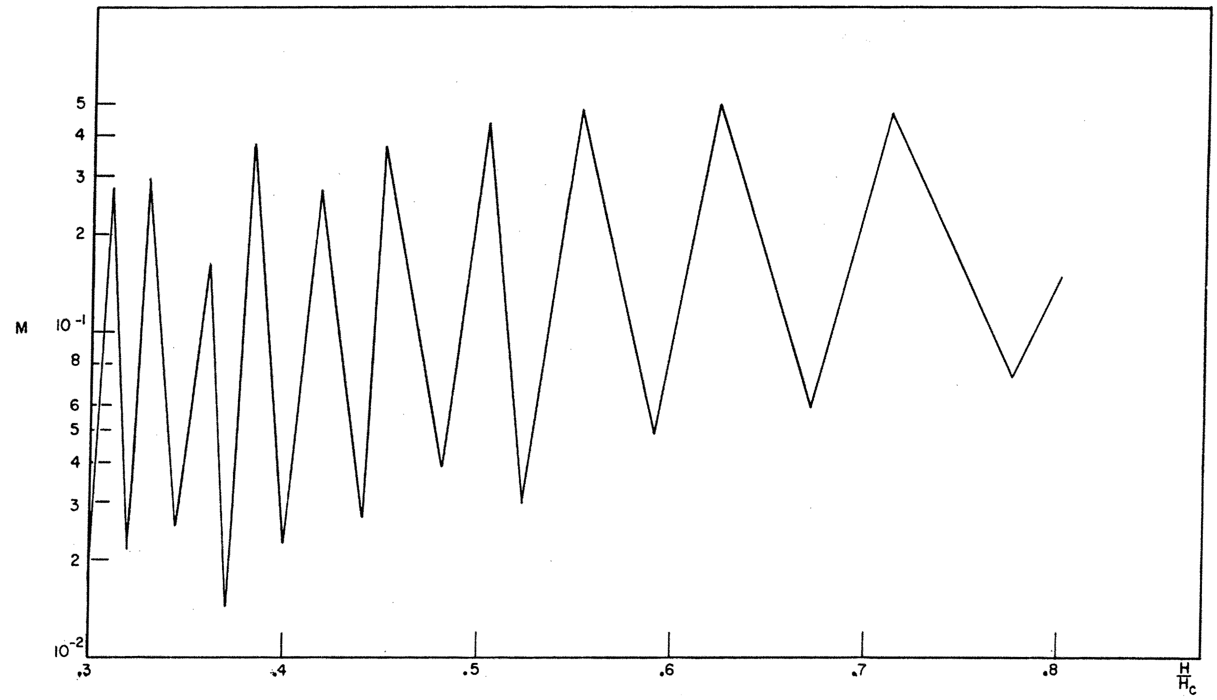

FIG. 2. The relation between $M=\mathfrak{M}/\mathfrak{M}_0$ and H/H_c.

the value of induced $\mathfrak{M}$ matches B, the body may become self-magnetized through the interaction among electrons, and permanent magnetism may result. In order that this may occur, it is necessary that the induced magnetic field be greater than the impressed field, that is,

$$\mathfrak{M}>H, \quad B=\mu_0\mathfrak{M}. \tag{58}$$

Equation (55) shows that $\mathfrak{M}\ll H$. *Hence, there is no possibility for permanent magnetism to exist in a dense electron gas.*

Higher-order electromagnetic corrections will change the value ot $\mathfrak{M}$ computed here, but the magnitude of these corrections will be 1/137 of the computed value of $\mathfrak{M}$, unless some collective phenomena take place. These collective phenomena may include the presence of an electric current or solid structure at high density and low temperature.

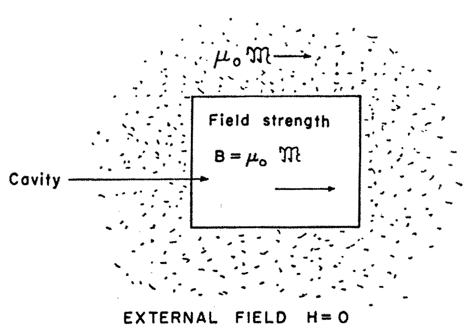

FIG. 3. The condition for permanent magnetism.

VIII. SUMMARY AND CONCLUSION

In this paper, we have computed the magnetic moment of a magnetized Fermi gas. A general expression for the magnetic moment was obtained. The properties of the magnetic moment in a degenerate magnetized gas were also studied. We obtained the following results:

(i) At a given field strength, the magnetization $\mathfrak{M}$ (magnetic moment per unit volume) increases more than linearly with particle density, and for high field strengths $\mathfrak{M}$ may even increase quadratically with the particle density $\mathfrak{N}$. However, if the density is too high, $\mathfrak{M}$ will decrease with the excitation of higher magnetic states. The maximum value of $\mathfrak{M}$ at a given field strength is roughly

$$\mathfrak{M}\simeq\frac{1}{\pi^2}\frac{\mu_B}{\lambda_c^3}\frac{H}{H_c}\sim\tfrac{1}{3}10^{-3}H\ll H. \tag{59}$$

(ii) Permanent magnetism will take place when the following condition isfulfilled:

$$H=\mu_0\mathfrak{M}, \tag{60}$$

where $\mu_0=1$ in Gaussian units (used in this paper). From Eq. (59), this condition cannot be fulfilled for an electron gas. Thus there is no permanent magnetism in a noninteracting electron gas.

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Self-Consistent Homogeneous Isotropic Cosmological Models. I. Generalities

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Self-consistent models of uniform universes are provided by coupling Einstein's equations to the one-particle Liouville equation. Correlations between the "particles" of the cosmic gas are thus neglected. As a consequence, an equation of state is not needed in the theory but is, rather, provided from these statistical considerations. It is shown that an expanding self-consistent uniform universe behaves asymptotically as the relativistic polytrope $p\sim\mu^{5/3}$, and finally, as an expanding Friedmann universe. Near the possible singularity $R=0$ (R=scale factor), self-consistent models are hot models. Friedmann models are shown to be a particular case of self-consistent models.

1. INTRODUCTION

IN a preceding paper we studied some simple properties of the self-gravitating relativistic gas.[1] Essentially, we used a Vlasov approximation. In other words, the relativistic one-particle Liouville equation

$$u^\mu\partial_\mu\mathfrak{N}(x^\nu,u^\nu)-\Gamma_{\alpha\beta}{}^\mu(x^\nu)u^\alpha u^\beta\frac{\partial}{\partial u^\mu}\mathfrak{N}(x^\nu,u^\nu)=0 \quad (1.1)$$

was coupled to Einstein's equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}-\lambda g_{\mu\nu}=\chi T_{\mu\nu} \quad (1.2)$$

through the definition of the momentum-energy tensor

$$T^{\mu\nu}(x^\rho)=m(\sqrt{g})\int\frac{d_3u}{u^0}\mathfrak{N}(x^\rho,u^\rho)u^\mu u^\nu. \quad (1.3)$$

In the following the metric tensor $g^{\mu\nu}$ is of signature $(+---)$ and $g\equiv|\det(g^{\mu\nu})|$. We also use $c=1$, where c is the velocity of light in a vacuum. In Eq. (1.2), $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, and λ is the cosmological constant. χ denotes the gravitational constant. The $\Gamma_{\alpha\beta}{}^\mu$'s are the well-known Christoffel symbols of the second kind. In Eqs. (1.1) and (1.3), $\mathfrak{N}(x^\rho,u^\rho)$ is the invariant distribution function describing the gas. In Eq. (1.3), m is the mass of a typical particle and the integral is extended to the hyperboloid

$$g_{\mu\nu}(x^\rho)u^\mu u^\nu=1\,,\quad u^0>0 \quad (1.4)$$

except when dealing with particles of vanishing mass. In this latter case the integral is extended to the light cone

$$g_{\mu\nu}(x^\rho)u^\mu u^\nu=0\,,\quad u^0\geqslant 0. \quad (1.4')$$

Finally, we will use the Einstein summation convention, Greek indices running from 0 to 3 and Latin ones from 1 to 3.

Since no explicit solution of Einstein's equations is known in terms of an arbitrary momentum-energy tensor $T_{\mu\nu}$, it was of course impossible to get a Vlasov equation as is commonly done in the case of electromagnetic interactions[2] (i.e., the electromagnetic field is expressed as a functional of the distribution function and next eliminated in the one-particle Liouville equation). Accordingly, we obtained a *linearized* kinetic equation by considering only small deviations of given "background quantities" (i.e., $g_{\mu\nu}$ and $\mathfrak{N}$). It then follows that our previous paper can be mainly applied to problems of stability. We also stressed that the only known relativistic self-gravitating system where *collective* effects are dominant is constituted by the universe as a whole.[3]

In this paper we deal with homogeneous, isotropic cosmological models. It is indeed well known that, in this particular case, Einstein's equations reduce to two differential equations[4] for the pressure p, the mass

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[1] Ph. Droz-Vincent and R. Hakim, Ann. Inst. H. Poincaré (to be published).

[2] See, e.g., S. Gartenhaus, *Elements of Plasma Kinetic Theory* (Holt, Rinehart and Winston, Inc., New York, 1964).

[3] This is not completely true since the *first stages* of the gravitational collapse of a massive star is another such example. However, as the star collapses, the neglect of correlations is less and less valid.

[4] See, e.g., H. Bondi, *Cosmology* (Cambridge University Press, Cambridge, 1961), 2nd ed.; G. C. McVittie, *General Relativity and Cosmology* (Chapman and Hall Ltd., London, 1965); R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1958).