High-Energy Nonlocal Nucleon-Nucleus Optical Potential for a Central Two-Body Force with Second-Order Corrections to the Impulse Approximation*

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A calculation of the high-energy nucleon-nucleus optical potential is carried out for a central two-body force in the energy region 90–310 MeV. We first show that arbitrarily assuming a reduced mass of $0.44m$ for the two-body collision in the nucleus improves the fit to experiment. We then present a novel method for correcting the impulse approximation for terms of second order in the two-body force. Application of this method indicates that the impulse approximation is reliable at 310 MeV. Considerable corrections are found below this energy which substantially decrease the discrepancy between theory and experiment. Off-energy-shell nonlocal effects are of third order in the three-body force and cannot be included with complete consistency. However, there are indications that they might be small. Pair-correlation effects are ignored throughout this work.

I. INTRODUCTION

In this paper, we calculate the high-energy nucleon-nucleus optical potential for a central two-body force in the energy range 90–310 MeV. A novel method of correcting the impulse approximation improves the fit to the experimental data. Nonlocal off-energy-shell effects are of third order in the two-body force and therefore cannot be included with complete consistency. However, there are indications that they might be small.

The high-energy nucleon-nucleus optical potential is nonlocal because the two-body t matrix is a nonlocal operator. Mulligan[1] calculated an energy-dependent local potential which took into account the first moment of this nonlocality. He used a simple central two-body force, calculated the t matrix in the impulse approximation, and neglected pair correlations.[2–5] Mulligan found a considerable improvement in the theoretical fit to the phenomenological optical potential[6] as compared to a similar calculation by Kerman *et al.*,[3] who neglected this nonlocality. We have repeated Mulligan's calculation with several improvements, the most important of which was the fact that we used a two-body force that fitted the forward scattering amplitude in the energy region considered. We found the nonlocal correction to be of opposite sign to that found by Mulligan, and thus his explanation for the discrepancy between Watson's theory and experiment fails for this force.

There is another theory of nucleon-nucleus scattering at high energies—that due to Glauber.[4] The main approximation in this theory is to neglect the excitation energy of all intermediate nuclear states. We shall call this the "diabatic approximation."[7] Glauber's theory solves the infinite set of coupled channels involved in the problem by using the fact that in the diabatic picture there is a diagonal representation, i.e., configuration space. The projectile multiply scatters off the target nucleons which are considered stationary. Because the struck target nucleon is assumed not to recoil, the reduced mass μ of the two-body collision is m as opposed to $\frac{1}{2}m$ in the impulse approximation. If the two-body force is such that at the energy being considered the interaction can be described by the high-energy approximation,[4] then the optical potential turns out to be independent of μ. Thus in the high-energy limit the Watson and Glauber theories should agree. We repeated our calculation, setting μ equal to m. This gave a completely different result from that when μ was $\frac{1}{2}m$, showing that the high-energy approximation could not be applied at these energies to our force. The diabatic approximation gave worse agreement with experiment than did the impulse approximation, and this discrepancy was only worsened by the inclusion of the nonlocal correction.

However, once one has recognized that the difference between the impulse and the diabatic approximations lies in their treatment of the recoil energy of the

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[1] B. Mulligan, Ann. Phys. (N. Y.) **26**, 159 (1964).

[2] K. W. Watson, Phys. Rev. **89**, 575 (1953); N. Francis and K. M. Watson, *ibid.* **92**, 291 (1953); W. B. Riesenfeld and K. M. Watson, *ibid.* **102**, 1157 (1956); K. M. Watson, Rev. Mod. Phys. **30**, 565 (1958); T. K. Fowler, Phys. Rev. **112**, 1325 (1958); H. Bethe, Ann. Phys. (N. Y.) **3**, 190 (1958).

[3] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).

[4] R. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience Publishers, Inc., New York, 1958), Vol. I.

[5] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

[6] A. H. Cromer and J. N. Palmieri, Ann. Phys. (N. Y.) **30**, 32 (1964); A. E. Glassgold, Revs. Mod. Phys. **30**, 419 (1958).

[7] It has been called elsewhere in the literature the "adiabatic approximation" [T. Tamura, Rev. Mod. Phys. **37**, 679 (1965)] and also the quasi-elastic approximation (see Ref. 5).

struck target nucleon *and* that the optical potential is sensitive to this, it is but a short step to treating this energy as a parameter. A crude but simple way of doing this is to allow the recoiling nucleon to have an effective mass m^* different from its free mass.[5,8] It is straightforward to establish limits of uncertainty on this mass. We allowed μ to vary from $0.38m$ to m. An improvement in the fit to experiment could be obtained by setting μ equal to $0.44m$, a value well within the estimated limits. The effect of the nonlocality for this mass was negligibly small.

To eliminate the necessity for this parametrization, one needs a calculation of the two-body t matrix appropriate to describing nucleon-nucleon scattering in a nucleus. We corrected the impulse approximation for terms of second order in the two-body force. The correction was small at 310 MeV. It was appreciable at 90 MeV and improved the fit to experiment. The nonlocal effect is small if we can correctly identify the corrected t matrix with that at $0.44m$.

II. IMPULSE APPROXIMATION: NOTATION AND REVIEW

Nucleon-nucleus scattering at high energies is a relatively old subject. Several excellent review articles are available in the literature, and the reader is referred in particular to Ref. 5. In this section we shall review only what is essential for the understanding of this work.

The nuclear Hamiltonian H_N describes N target nucleons, position vectors ξ_i, and kinetic energy operators k_i interacting through potentials $U_{ij}(\xi_i-\xi_j)$, i.e.,

$$H_N=\sum_i k_i+\tfrac{1}{2}\sum_{i,j} U_{ij}=-\sum_i \frac{\hbar^2}{2m}\nabla_i^2+\tfrac{1}{2}\sum_{i,j} U_{ij}.$$

We designate the antisymmetrized eigenfunctions of H_N by $\chi_m(\xi_1,\cdots,\xi_N)$, or simply by $|m\rangle$, and the eigenvalues by W_m. We shall completely neglect the c.m. motion of the nucleus, which we will always consider at rest at the origin of our coordinate system .The ground state $|0\rangle$ has energy W_0 equal to zero. The projectile, with position vector $\mathbf{r}$ and kinetic energy operator H_0, interacts with the nucleus through a potential V:

$$H_0=-(\hbar^2/2m)\nabla_r^2,$$
$$V=\sum_i v_i(\mathbf{r}-\xi_i)=\sum_i v_i.$$

We shall denote the eigenfunctions $\exp(i\mathbf{k}\cdot\mathbf{r})$ of H_0 by $|\mathbf{k}\rangle$. We are interested in the problem of determining the elastic scattering amplitude $\langle 0,\mathbf{k}|T(E)|\mathbf{k}_0,0\rangle$, where T is defined by

$$T=V+V(E+i\epsilon-H_N-H_0-V)^{-1}V. \quad (1)$$

The main problem in solving Eq. (1) is that of the treatment of H_N in the propagator. For this purpose it is important to distinguish between a propagator describing the projectile interacting with an individual nucleon and one describing the projectile passing from one nucleon to another. Accordingly, we rewrite Eq. (1) as

$$T=\sum_i T_i+\sum_{\substack{i,j,\\ i\neq j}} T_i(E+i\epsilon-H_N-H_0)^{-1}T_j+\sum_{\substack{i,j,k,\\ i\neq j,j\neq k}} T_i(E+i\epsilon-H_N-H_0)^{-1}\times T_j(E+i\epsilon-H_N-H_0)^{-1}T_k+\cdots, \quad (2)$$

where

$$T_i=v_i+v_i(E+i\epsilon-H_N-H_0)^{-1}T_i. \quad (3)$$

If we have an independent distinct-particle model of the nucleus, then following Watson we can write

$$\begin{aligned}\langle 0|T(E)|0\rangle=&\sum_i \langle 0|T_i|0\rangle\\&+\sum_{\substack{i,j,\\ i\neq j}}\langle 0|T_i|0\rangle(E+i\epsilon-H_0)^{-1}\langle 0|T_j|0\rangle\\&+\sum_{\substack{i,j,k,\\ i\neq j,j\neq k}}\langle 0|T_i|0\rangle(E+i\epsilon-H_0)^{-1}\\&\times\langle 0|T_j|0\rangle(E+i\epsilon-H_0)^{-1}\langle 0|T_k|0\rangle+\cdots\\&+\sum_{\substack{i,j,\\ i\neq j}}\langle 0|T_i(E+i\epsilon-H_N-H_0)^{-1}\\&\times T_j(E+i\epsilon-H_N-H_0)^{-1}T_i|0\rangle+\cdots. \quad (4)\end{aligned}$$

How one treats the last group of terms in Eq. (4) is only important to order N^{-1}.[9] To this order Eq. (4) can be interpreted as the Born series for scattering from an optical potential,

$$V_{\mathrm{opt}}(\mathbf{r},\mathbf{r}')=\frac{-\hbar^2}{(2\pi)^2 m}\int e^{+i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}_0\cdot\mathbf{r}'}\times\sum_i \langle 0,\mathbf{k}|T_i(E)|\mathbf{k}_0,0\rangle d^3k\,d^3k_0. \quad (5)$$

Thus in Eq. (4) we have replaced H_N in the propagator with zero. We shall later show that this is still appropriate even if we have a correlated nuclear wave function. To determine H_N in Eq. (3), we rewrite it as

$$T_i=t_d+t_d(E+i\epsilon-H_0)^{-1}H_N(E+i\epsilon-H_N-H_0)^{-1}T_i, \quad (6)$$

where

$$t_d=v_i+v_i(E+i\epsilon-H_0)^{-1}t_d. \quad (7)$$

We define the replacement of T_i with t_d as the diabatic approximation. We note that once again we have replaced H_N with zero. The t matrix t_d so obtained describes the projectile scattering from a *fixed* potential v_i, i.e., the reduced mass μ of the two-body collision is m. We now estimate the value of H_N by iterating

[8] K. A. Brueckner and W. Wada, Phys. Rev. **103**, 1008 (1956).

[9] J. F. Reading, Phys. Rev. **156**, 1120 (1967).

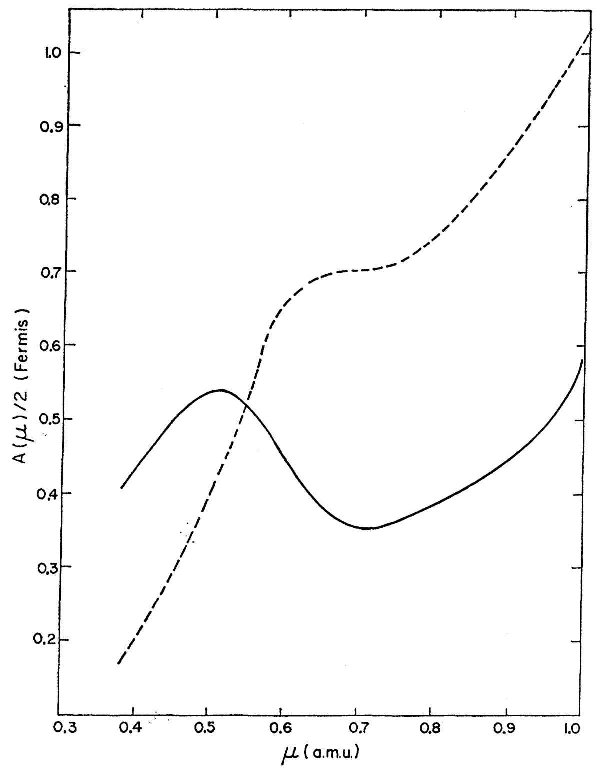

FIG. 1. Plot of $\frac{1}{2}\bar{A}(\mu)$ at 90 MeV. Dashed line: imaginary part of $\frac{1}{2}\bar{A}(\mu)$. Solid line: real part of $\frac{1}{2}\bar{A}(\mu)$. The potential used for all energies was the 90-MeV one (see Appendix A).

Eq. (6) and using the properties of H_N on the ground state.[5]

We write

$$\begin{aligned}\Delta&=\langle 0,\mathbf{k}|T_i|\mathbf{k}_0,0\rangle-\langle 0,\mathbf{k}|t_d|\mathbf{k}_0,0\rangle\\&\approx\langle 0,\mathbf{k}|t_d(E+i\epsilon-H_0)^{-1}H_N(E+i\epsilon-H_0-H_N)^{-1}\\&\quad\times|t_d|\mathbf{k}_0,0\rangle\approx\langle 0,\mathbf{k}|t_d(E+i\epsilon-H_0)^{-2}\\&\quad\times[H_N+H_N^2(E+i\epsilon-H_0)^{-1}]t_d|\mathbf{k}_0,0\rangle.\end{aligned}\tag{8}$$

For large nuclei, we are mainly interested in the region where

$$\mathbf{k}=\mathbf{k}_0=\mathbf{k}_E \quad\text{and}\quad k_E=(2mE/\hbar^2)^{1/2}.$$

For this case,

$$\begin{aligned}\Delta\approx\frac{-\hbar^2}{4m\pi^2}\int d^3k'\langle\mathbf{k}_E|t_d|\mathbf{k}'\rangle\langle\mathbf{k}'|t_d|\mathbf{k}_E\rangle\left(E+i\epsilon-\frac{\hbar^2}{2m}k'^2\right)^{-2}\\\times\left(\frac{\hbar^2}{2m}(\mathbf{k}'-\mathbf{k}_E)^2+\left(\frac{\hbar^2}{2m}\right)^2\left(E+i\epsilon-\frac{\hbar^2}{2m}k'^2\right)^{-1}\left\{(\mathbf{k}'-\mathbf{k}_E)^4\right.\right.\\\left.\left.+4\int[(\mathbf{k}'-\mathbf{k}_E)\cdot\nabla_i\chi_0]^2d^3\xi_1,\cdots,d^3\xi_N\right\}\right).\end{aligned}$$

Writing

$$\mathbf{q}=\mathbf{k}'-\mathbf{k}_E$$

and changing variables gives

$$\begin{aligned}\Delta=\frac{-1}{2\pi^2}\int d^3q\,\langle\mathbf{k}_E|t_d|\mathbf{k}_E+\mathbf{q}\rangle\langle\mathbf{k}_E+\mathbf{q}|t_d|\mathbf{k}_E\rangle\\\times(2\mathbf{k}_E\cdot\mathbf{q}+q^2-i\epsilon)^{-2}\left\{q^2-\left[q^4+4\int(\mathbf{q}\cdot\nabla_i\chi_0)^2\right.\right.\\\left.\left.\times d^3\xi_1,\cdots,d^3\xi_N\right](2\mathbf{k}_E\cdot\mathbf{q}+q^2-i\epsilon)^{-1}\right\}.\end{aligned}\tag{9}$$

Thus for some momentum transfer q the diabatic approximation replaces the propagator $[-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon-(2m/\hbar^2)H_N]^{-1}$ with $(-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon)^{-1}$, whereas we can read off from Eqs. (8) and (9) that

$$(2m/\hbar^2)\langle H_N\rangle=q^2,$$

$$\left(\frac{2m}{\hbar^2}\right)^2\langle H_N^2\rangle=q^4+\int 4[\mathbf{q}\cdot\nabla_i\chi_0]^2d^3\epsilon_1,\cdots,d^3\epsilon_N.$$

We therefore deduce that we can replace H_N with

$$H_N=(\hbar^2/2m)q^2[1\pm O(a/r_0)],\tag{10}$$

where a is the range of the nucleon-nucleon force and r_0 is the distance over which the bound-state wave function (not the nuclear density) varies appreciably. We can estimate a/r_0 for a light nucleus as being somewhat less than unity and greater than 0.25. In the limit that

$$a\ll r_0,$$

as in the deuteron, for example, H_N can be replaced to a good approximation by $\hbar^2q^2/2m$, i.e.,

$$[-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon-(2m/\hbar^2)H_N]^{-1}\approx 2^{-1}[-2(\tfrac{1}{2}\mathbf{k}_E\cdot\mathbf{q})-q^2+i\epsilon]^{-1}.$$

If we insert this into Eq. (3), then

$$\langle 0,\mathbf{k}_E|T_i|\mathbf{k}_E,0\rangle\approx A_i(\mu,E)=2t_i(\tfrac{1}{2}m,\tfrac{1}{2}E,\tfrac{1}{2}k_E),\tag{11}$$

where $t_i(m,E,k_E)$ is the forward scattering amplitude for a mass m scattering from a fixed potential v_i at energy E. In this limit, then, we have the impulse approximation; T_i is replaced by the free two-body t matrix. We are now in a position to understand why, if the high-energy approximation can be applied to v_i it is irrelevent whether we use the impulse or diabatic approximation for T_i. In the high-energy limit,[4] $(-2\mathbf{k}_E\cdot\mathbf{q}-2q^2+i\epsilon)^{-1}$ and $(-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon)^{-1}$ are both replaced by $(-2\mathbf{k}_E\cdot\mathbf{q}+i\epsilon)^{-1}$ and the energy given to the target nucleon has no effect, whether it recoils freely or not.[10]

[10] The corollary of this statement, at least in the small a/r_0 limit, is that the error to be expected in A_i from making the diabatic approximation will be of the same order as the error to be expected in t if we calculate it from v_i using the high-energy approximation.

If the high-energy approximation cannot be applied to the two-body force, then one must be very careful how one treats the recoil. We have investigated the sensitivity of T_i to this by writing H_N as $(\hbar^2 q^2/2m^*)$ and allowing m^*, the effective mass of the struck nucleon, to vary.[5,8] We then have

$$[-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon-(2m/\hbar^2)H_N]^{-1} \approx[-2\mathbf{k}_E\cdot\mathbf{q}-q^2+i\epsilon-(m/m^*)q^2]^{-1}.$$

This gives, on substitution into Eq. (3), that

$$\langle 0,\mathbf{k}_E|T_i|\mathbf{k}_E,0\rangle\approx A_i(\mu,E,k_E) =(m/\mu)t_i(\mu,E\mu/m,\mathbf{k}_E\mu/m)\,, \quad (12)$$

where

$$\mu=mm^*/(m+m^*)\,.$$

We allowed μ to vary from $0.38m$ to m.[11] A typical variation of A with μ is shown in Fig. 1. A value of μ/m greater than 0.5 could be interpreted as the target nucleon being held to some extent by the nuclear forces. A value of μ/m less than 0.5 might be symptomatic of the nonlocality of the nuclear potential. From Fig. 1 we see that there is a striking variation in the imaginary part of $A(\mu)$ and that the real part is roughly constant. Since the imaginary part of $A(\mu)$ is responsible for the absorption out of the elastic channel into the inelastic channel, we might have expected

$$\mathrm{Im}A(0.5m)<\mathrm{Im}A(m)\,.$$

This is presumably because it is easier to excite a nucleus whose states are all degenerate at zero energy than it is to excite one that has states with energy appropriate to a freely recoiling nucleon.

In the same way that we investigated H_N in Eq. (3) we can now investigate H_N in Eq. (2). If we approximate T_i as in Eq. (12), then T_i is a single-body operator. In this case, the appropriate quantity $\langle H_N\rangle$ is given by

$$\langle H_N\rangle=\langle 0|\exp[-i(\mathbf{k}-\mathbf{k}')\cdot\xi_i]\left[-\frac{\hbar^2}{2m}\nabla_j^2-\frac{\hbar^2}{2m}\nabla_i^2+\sum_{k,k\neq i\neq j}-\frac{\hbar^2}{2m}\nabla_k^2+\tfrac{1}{2}\sum_{k,l}u(\xi_k-\xi_l)\right]\exp[-i(\mathbf{k}'-\mathbf{k}_0)\cdot\xi_j]|0\rangle$$
$$=\langle 0|\exp[-i(\mathbf{k}-\mathbf{k}')\cdot\xi_i][-(\hbar^2/2m)\nabla_j^2]\exp[-i(\mathbf{k}'-\mathbf{k}_0)\cdot\xi_j]|0\rangle-\langle 0|\exp[-i(\mathbf{k}-\mathbf{k}')\cdot\xi_j]$$
$$\times\exp[-i(\mathbf{k}'-\mathbf{k}_0)\cdot\xi_j][-(\hbar^2/2m)\nabla_j^2]|0\rangle=0\,.$$

Thus the mean excitation energy of the nucleus as the projectile passes from one nucleon to another is zero, confirming our intuitive result obtained from an independent-particle model. We can work out the standard deviation from this mean value in a similar way. But we shall leave that for a later calculation involving the correlation correction. Sufficient to say here that one can expect the uncertainty in H_N to play as great a role in the propagation between different nucleons as it does when the projectile interacts with an individual nucleon.

So far, then, we have established the sensitivity of $A(\mu)$ to the effective mass of the struck target nucleon. In Sec. III, we show the variation of the off-energy-shell characteristics of A with this mass.

III. NONLOCALITY OF OPTICAL POTENTIAL

Whatever value of μ we chose in Eq. (12), the optical potential given by Eq. (5) will be nonlocal, since $\langle 0,\mathbf{k}|T_i|\mathbf{k}_0,0\rangle$ is a function not only of $(\mathbf{k}-\mathbf{k}_0)$ but also of $(\mathbf{k}+\mathbf{k}_0)$. We can write a local energy-dependent potential V_e which takes into account the first moment of this nonlocality[1,12]

$$V_e(\mu,r)\approx-(2\pi\hbar^2/\mu)N\rho(r)\bar{t}(\mu,E\mu/m,k_E\mu/m) \times[1-2\pi k_E^{-1}N\rho(r)\bar{t}'(\mu,E\mu/m,k_E\mu/m)]^{-1}\,, \quad (13)$$

where

$$\bar{t}'(\mu,E,k)=\frac{\partial}{\partial k}\bar{t}(\mu,E,k)\equiv\frac{\partial}{\partial k}\langle\mathbf{k}|\bar{t}(E)|\mathbf{k}\rangle \quad (14)$$

and $\rho(r)$ is the nuclear density. In Eq. (13), we have replaced a sum over nucleon quantum numbers[3] by an appropriate average $N\bar{t}$. We shall use this notation throughout this work. Note that $\bar{t}'$ is of second order in the nucleon-nucleon force.[12] Thus the nonlocal correction is a third-order effect. Because of this, for a calculation which is only accurate to second order in the t matrix such as we shall present in Sec. V, there is a certain amount of ambiguity in determining the nonlocal correction. In Eq. (13), we are only using one of the many possible corrections.

For the purpose of illustrating the effect of the nonlocality, we define $V_l(\mu,r)$ as that value of $V_e(\mu,r)$ obtained by setting $\bar{t}'$ to zero. It will be sufficient to only consider $V_l(\mu)$ and $V_e(\mu)$ as the values of $V_l(\mu,0)$ and $V_e(\mu,0)$. It turns out that for the impulse approximation, $-V_e(0.5m)$ is on the whole larger than the experimental points found for it from an analysis of the data.[6] Mulligan[1] calculated only the real part of $\bar{t}(0.5m)$ and found it to be negative. He assumed that the imaginary part of $\bar{t}'(0.5m)$ was zero. The effect of the nonlocality then is to make $-V_e(0.5m)$ smaller than $-V_l(0.5m)$ as can be seen by substituting a negative real value of $\bar{t}'$ into Eq. (13). Mulligan thus found a substantial change in the optical potential about 30%, which was of the right sign to improve considerably the fit to experiment.[1] However, he used a force which bore only a qualitative resemblance to the real force.

We have repeated Mulligan's calculation for both the real and imaginary parts of $\bar{t}'(0.5m)$, which we call $t_M'(0.5m)$. We then calculated $\bar{t}'(0.5m)$, using an im-

[11] There is no reason why m^* should be real, but we did not investigate complex values in this paper.
[12] J. F. Reading, Phys. Rev. 156, 1116 (1967).

TABLE I. Spin-averaged forward scattering amplitude.[a]

Lab energy (MeV)	$\bar{t}_G(0.5m)$ (F)	$\bar{t}(0.5m)$ (F)	$t_M(0.5m)$[b] (F)
90	$0.59+i0.44$	$0.59+i0.42$	$0.65+i0.16$
156	$0.48+i0.40$	$0.49+i0.37$	$0.62+i0.14$
310	$0.14+i0.48$	$0.16+i0.50$	$0.57+i0.38$

[a] The full Gammel-Thaler result is $\bar{t}_G(0.5m)$, our modified central force gives $\bar{t}(0.5m)$ and Mulligan's force is $t_M(0.5m)$.
[b] There is no spin-isospin dependence with Mulligan's force.

proved force, but one that still had no tensor component. This force is essentially the Gammel-Thaler force[13] with the triplet n-p part modified (see Appendix A). It gives the correct forward scattering amplitude in the 90–300 MeV region, as is shown in Table I.

The calculation of $\bar{t}'$ involved an improvement on the perturbation approach used by Mulligan. We must calculate the derivative of the t matrix to go off the energy shell, but in (13) we only need this derivative evaluated on the energy shell.[12]

In Appendix B, we indicate how to prove that

$$\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}\rangle=2\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle \qquad (15)$$

if the derivative is to be evaluated at k equal to k_E. Here

$$\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle=-\frac{2\mu}{\hbar^2}\frac{1}{4\pi}\int e^{-i\mathbf{k}\cdot\mathbf{r}}v(r)\psi(r)d^3r,$$

where $\psi(r)$ is the usual on-energy-shell wave function. Thus all we have to do to find t' is to evaluate such expressions as

$$\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle=-\sum_l e^{i\delta_l}(2l+1)\times\left(\frac{2\mu}{\hbar^2}\right)\int_0^\infty\frac{\partial}{\partial k}j_l(kr)v(r)R_l(r)r^2dr. \qquad (16)$$

The only remaining difficulty is that of dealing with hard cores, since the integral in Eq. (16) is not clearly

TABLE II. Forward scattering amplitude $\bar{A}(\mu)$ for various values of μ.[a,b]

Lab energy (MeV)	$\frac{1}{2}\bar{A}(0.44m)$ (F)	$\frac{1}{2}\bar{A}(0.5m)$ (F)	$\frac{1}{2}\bar{A}(m)$ (F)
90	$0.53+i0.28$	$0.59+i0.42$	$0.62+i1.09$
156	$0.43+i0.25$	$0.49+i0.37$	$0.01+i1.43$
310	$0.07+i0.40$	$0.16+ 0.50$	$-0.11+i1.19$

[a] We have antisymmetrized in a way appropriate to free two-body scattering for all μ. G. Takeda and K. M. Watson [Phys. Rev. **97**, 1336 (1955)] have justified this for the impulse approximation.
[b] Our potential has a slight energy dependence. We took the 156-MeV potential for $\bar{A}(m)$ at 90 MeV and the 310-MeV potential for $\bar{A}(m)$ at 156 and 310 MeV.

[13] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957); **107**, 1337 (1957).

TABLE III. Derivative of the forward scattering amplitude to go off the energy shell.

Lab energy (MeV)	$\bar{t}'(0.44m)$ (F^2)	$\bar{t}'(0.5m)$ (F^2)	$\bar{t}'(m)$ (F^2)	$t_M'(0.5m)$
90	$0.10+i0.04$	$0.36+i0.26$	$1.80+i1.34$	$-0.36+i0.32$
156	$0.08+i0.08$	$0.24+i0.16$	$-0.08+i1.80$	$-0.15+i0.08$
310	$0.12+i0.02$	$0.22+i0.02$	$0.26+i0.58$	$-0.04+i0.32$

defined in that case. We show in Appendix B that

$$\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle=-\sum_l e^{i\delta_l}(2l+1)\left[R_l'(c_0)\frac{\partial}{\partial k}j_l(kc_0)c_0^2+\left(\frac{2\mu}{\hbar^2}\right)\int_{c_0}^\infty v(r)R_l(r)r^2\frac{\partial}{\partial k}j_l(kr)dr\right], \qquad (17)$$

where c_0 is the hard-core radius.

In Eqs. (16) and (17), the radial wave function $R_l(r)$ is such that at infinity

$$R_l(r)\sim\cos\delta_l(kr)-\sin\delta_l n_l(kr).$$

IV. RESULTS

In Table II, we plot $\bar{A}(\mu)$ for μ equal to $0.5m$, m, and $0.44m$. There is a wide variation of $A(\mu)$ for this range of μ which would not be present if the high-energy limit obtained. In Table III, we plot $\bar{t}'(\mu)$ for the same values of μ. Once again there is a wide variation; the nonlocality is large in the diabatic limit and small for μ equal to $0.44m$. For the impulse approximation, $\bar{t}'(0.5m)$ has a positive real part, so we might expect $-V_e(0.5m)$ to be greater than $-V_l(0.5m)$. This is so, as can be seen from Table IV, though the imaginary part of $\bar{t}'$ also plays a role. We do not get Mulligan's result that

$$-V_e(0.5m)<-V_l(0.5m).$$

The nonlocality for this force makes both the real and imaginary parts of V_e more negative. The diabatic limit produces an optical potential which seems to bear no relationship at all to the experimental results (Table IV). A reasonable fit can be obtained with $0.44m$ for μ; the nonlocality for this case is negligibly small. In Table IV, we plot $V_l(0.44m)$ against some optical potentials obtained from an analysis of experimental data.[6] A reasonable fit to the data is obtained.

Of course, a value of μ obtained in this way does not have much meaning. It is apparent that both the nonlocality and variation of $\bar{A}(\mu)$ will be very force-dependent. This is not necessarily a bad thing, since it is hoped that we may be able to distinguish between two-body forces in this way. However, a cautionary note should be sounded here.

Both the effect of the nonlocality and the presence of H_N in the propagator are the result of the binding forces the struck target nucleon experiences because of the other nucleons in the nucleus. The nonlocality is

TABLE IV. Optical potentials $v_l(\mu)$ and $v_e(\mu)$ (in MeV)[a] compared with some experimentally determined values v_{expt}.[b]

Lab energy (MeV)	$v_l(0.5m)$	$v_e(0.5m)$	$v_l(m)$	$v_e(m)$	$v_l(0.44m)$	v_{expt}
90	$-37-i26$	$-39-i35$	$-36-i64$	$29-i112$	$-33-i17$	$-27\pm3-i10\pm5$
156	$-29-i22$	$-30-i25$	$0.0-i78$	$27-i65$	$-25-i14$	$-28\pm3-i14\pm2$
310	$-8-i27$	$-8-i27$	$6-i64$	$0-i69$	$-4-i21$	$\pm3-i15\pm5$

[a] v_l and v_e depend on a choice of $\rho(r)$ and how one includes the finite range of the nucleon-nucleon interaction. The potentials were chosen such as to be normalized to the results of Ref. 3.
[b] The values of v_{expt} were taken from Ref. 6.

intimately connected to the overlap of the target-nucleon potential with that of the next particle with which the projectile interacts.[14] It can be shown that if there is no overlap of these potentials, then absolutely no off-energy-shell information is obtained from the experiment.[15] The projectile propagates between the two nucleons on its mass shell. The amount of overlap that any given pair of nucleons have is determined by the pair-correlation function the effect of which we have completely neglected. Thus the correlation correction and the nonlocal correction are inextricably tried together. Fortunately, a deuteron experiment can provide an estimate of correlation correction,[16] and we calculate the nonlocal effect.

The same thing can be said about determining the effect of H_N. Here we can solve the problem, given the nuclear wave function χ_0. And, in fact, we only need single-particle wave functions to evaluate the integral in Eq. (9). However, this means that any off-energy-shell information about the two-body force obtained from these experiments can only be as reliable as our microscopic description of the nucleus. Fortunately, as we have demonstrated, all the information that we need is contained in the *ground-state* wave function. There has been a considerable improvement in the calculation of this function in recent years,[17] and a reliable calculation of the effects which we have only estimated so far should now be possible. In Sec. V, we present a calculation of the second-order correction to T_i, using a crude estimate of r_0.

V. SECOND-ORDER CORRECTION TO THE IMPULSE APPROXIMATION

The impulse approximation treats the target nucleon as if it were free. In the previous sections, we tried to mock up the effect of the binding forces by allowing the nucleon mass to vary. In this section, we present a perturbation approach to the problem of determining T_i. It turns out that indeed the variation of $A(\mu)$ with μ does play a fundamental role.

[14] We wish to thank Professor R. E. Peierls for an illuminating discussion on this point.
[15] L. Eyges, Ann. Phys. (N. Y.) **2**, 101 (1957).
[16] J. F. Reading, Phys. Rev. **156**, 1110 (1967).
[17] T. T. S. Kuo and G. E. Brown, Nucl. Phys. **85**, 40 (1966); R. L. Becker and A. D. MacKellar, Phys. Letters **21**, 101 (1966); C. M. Shakin, Y. R. Waghmare, and M. H. Hull, Jr., Phys. Rev. **161**, 1006 (1967).

Using the results derived in Sec. II, we return to Eq. (3) and start a perturbation expansion, not in H_N, but in $(H-\langle H_N\rangle)$. We write

$$\begin{aligned}\langle 0,\mathbf{k}|T_i(E)|\mathbf{k}_0,0\rangle &= \langle 0,\mathbf{k}|T_i(m,E)|\mathbf{k}_0,0\rangle \\ &+\langle 0,\mathbf{k}|\tau_i(m)(E+i\epsilon-H_0-\langle H_N\rangle)^{-1}(H_N-\langle H_N\rangle) \\ &\times(E+i\epsilon-H_0-H_N)^{-1}T_i|\mathbf{k}_0,0\rangle, \quad (18)\end{aligned}$$

where

$$\begin{aligned}\langle 0,\mathbf{k}|\tau_i(m^*,E)|\mathbf{k}_0,0\rangle &= \langle 0,\mathbf{k}|v_i|\mathbf{k}_0,0\rangle+\langle 0,\mathbf{k}|v_i \\ &\times(E-H_0-\langle H_N\rangle m/m^*)^{-1}\tau_i(m^*)|\mathbf{k}_0,0\rangle\end{aligned}$$

and

$$\langle H_N\rangle=(\hbar^2/2m)(i\nabla_r-\mathbf{k}_0)(i\nabla_r-\mathbf{k}).$$

We first of all establish a relationship between $\tau_i(m^*,E)$ and two-body t matrix t_i. If $\mathbf{k}=\mathbf{k}_0$, then

$$\begin{aligned}&\langle 0,\mathbf{k}_0|\tau_i(m^*,E)|\mathbf{k}_0,0\rangle \\ &=\frac{m+m^*}{m^*}t_i\left(\frac{mm^*}{m+m^*},\ E-\frac{\hbar^2k_0^2}{2(m+m^*)},\ \frac{m^*}{m+m^*}k_0\right), \quad (19)\end{aligned}$$

where $t_i(\mu,E,k_0)$ is the off-energy-shell forward scattering amplitude of a particle of mass μ and energy E interacting with a fixed potential v_i. Thus $\bar{\tau}$ is a generalization of $\bar{A}(\mu)$. Since t_i is a single-body operator,

$$\begin{aligned}\langle 0,\mathbf{k}|\tau_i(m,E)|\mathbf{k}_0,0\rangle &\approx F(\mathbf{k}\sim\mathbf{k}_0)2t_i \\ &\times(0.5m,\ E-(\hbar^2/4m)k_0^2,\ \tfrac{1}{2}k_0),\end{aligned}$$

where $F(\mathbf{k}-\mathbf{k}_0)$ is the nuclear form factor and we have made the zero-range approximation for the two-body force. If we are on the energy shell, i.e.,

$$|\mathbf{k}_0|=k_0=k_E,$$

we have the impulse approximation. Then

$$\begin{aligned}\langle 0,\mathbf{k}|\tau_i(m,E)|\mathbf{k}_0,0\rangle &= F(\mathbf{k}-\mathbf{k}_0)2t_i(0.5m,\ \tfrac{1}{2}E,\ \tfrac{1}{2}k_E) \\ &= F(\mathbf{k}-\mathbf{k}_0)A_i(0.5m).\end{aligned}$$

We must know two things to determine the scattering:

$$\langle 0,\mathbf{k}_E|\bar{T}|\mathbf{k}_E,0\rangle \quad\text{and}\quad \langle 0,\mathbf{k}_E|\bar{T}'|\mathbf{k}_E,0\rangle.$$

We shall first of all calculate Δ_i, the second-order correction to T_i, where

$$\begin{aligned}\Delta_i &\approx \langle 0,\mathbf{k}_E|T_i|\mathbf{k}_E,0\rangle-2t_i(\tfrac{1}{2}m,\ \tfrac{1}{2}E,\ \tfrac{1}{2}k_E) \\ &\approx \langle 0,\mathbf{k}_E|\tau_i(m)(E+i\epsilon-H_0-\langle H_N\rangle)^{-3} \\ &\times[H_N-\langle H_N\rangle]^2\tau_i(m)|0,\mathbf{k}_E\rangle.\end{aligned}$$

TABLE V. Second-order correction to the impulse approximation $\bar{\Delta}$, with the corresponding local potential $v_l^{(2)}$.

Energy lab	$\frac{1}{2}\bar{\Delta}$	$t^{(2)}$	$v_l^{(2)}$	v_{expt}
90	$-0.02-i0.19$	$0.57+i0.23$	$-36-i14$	$-27\pm3-i10\pm5$
156	$-0.12-i0.10$	$0.37+i0.27$	$-25-i15$	$-28\pm3-i14\pm2$
310	$0.03-i0.00$	$0.19+i0.50$	$-9-i27$	$\pm3-i15\pm5$

Using the results of Sec. II, we have

$$\Delta_i=\frac{\langle\mathbf{k}_E|\tau_i(m)(i\nabla_r-\mathbf{k}_E)^2\tau_i(m)|\mathbf{k}_E\rangle}{[E+i\epsilon-H_0-(\hbar^2/2m)(i\nabla_r-\mathbf{k}_E)^2]^3}\left(\frac{\hbar^2}{2m}\right)^2\frac{1}{r_0^2},$$

where

$$\frac{1}{r_0^2}=4\int\left[\frac{\partial}{\partial\xi_i}\chi_0\right]^2 d^3\xi_1,\,\cdots,\,d^3\xi_N.$$

We could evaluate Δ_i by substituting for τ and performing the required integration. However, for a second-order calculation, this is unnecessary, since we already have sufficient information to calculate Δ_i directly to this order.

We have

$$\begin{aligned}\langle\mathbf{k}_0|\tau_i(m^*,E)|\mathbf{k}_0\rangle=&\langle\mathbf{k}_0|\tau_i(m,E)|\mathbf{k}_0\rangle+\langle\mathbf{k}_0|\tau_i(m,E)\\&\times(E+i\epsilon-H_0-\langle H_N\rangle)^{-1}\langle H_N\rangle(m/m^*-1)\\&\times[E+i\epsilon-H_0-(m/m^*)\langle H_N\rangle]^{-1}\tau_i(m^*,E)|\mathbf{k}_0\rangle.\end{aligned}\quad(20)$$

Differentiating Eq. (20) with respect to m^* and evaluating the derivative at m, we have

$$\frac{\partial}{\partial m^*}\langle\mathbf{k}_0|\tau_i(m^*,E)|\mathbf{k}_0\rangle=-\frac{\langle\mathbf{k}_0|\tau_i(m,E)[\langle H_N\rangle|m]\tau_i(m,E)|\mathbf{k}_0\rangle}{(E+i\epsilon-H_0-\langle H_N\rangle)^2}.$$

Differentiating again with respect to E, we have [neglecting $(\partial\tau_i/\partial E)$, which is already of second order in the two-body force] the result

$$\frac{\partial^2}{\partial E\partial m^*}\langle\mathbf{k}_0|\tau_i(m^*,E)|\mathbf{k}_0\rangle\sim\frac{\langle\mathbf{k}_0|\tau_i(m,E)[2\langle H_N\rangle|m]\tau_i(m,E)|\mathbf{k}_0\rangle}{(E+i\epsilon-H_0-\langle H_N\rangle)^3}.$$

If we evaluate this derivative on the energy shell, then

$$\Delta_i\sim\frac{\hbar^2}{4r_0^2}\frac{\partial^2}{\partial E\partial m^*}\langle\mathbf{k}_0|\tau_i(m^*,E)|\mathbf{k}_0\rangle\sim\frac{\hbar^2}{4r_0^2}\frac{\partial^2}{\partial E\partial m^*}\times\left\{\frac{m+m^*}{m^*}t_i\left[\frac{mm^*}{m+m^*},\,E-\frac{\hbar^2k_0^2}{2(m+m^*)},\,\frac{m^*}{m+m^*}k_0\right]\right\},$$

i.e.,

$$\bar{\Delta}\sim\frac{\hbar^2}{4r_0^2}\frac{\partial^2}{\partial E\partial m^*}\left[\bar{A}(\mu)\frac{m+m^*}{m^*}\right]\sim\frac{\hbar^2}{4r_0^2}\frac{\partial}{\partial m^*}\times\left(\frac{d}{dE}-\frac{dk_E}{dE}\frac{\partial}{\partial k_E}\right)\left[\bar{A}(\mu)\frac{m+m^*}{m^*}\right].\quad(21)$$

We can find all the quantities in Eq. (21) by working out $\bar{A}(\mu)$ at various values of m^* and E and calculating the derivatives. The results are shown in Table V. We calculated the energy derivative by evaluating $\partial\bar{A}/\partial m^*$ at 90, 120, and 156 MeV and then using linear extrapolation. One difficulty was the energy dependence of our potential. We used the average of the parameters at 90 and 156 MeV for the determination at 120 MeV. A considerable error could be introduced by this procedure, so $\bar{\Delta}$ in Table V should be regarded only as an indication of the magnitude of the expected effect. With r_0 estimated as 1F, a considerable improvement in the agreement between the local optical potential $V_l^{(2)}$ and V_{expt} was observed. The change at 310 MeV worked out from $\partial\bar{A}/\partial m^*$ evaluated at 280 and 340 MeV was quite small. This indicated that the impulse approximation was reasonably good at this energy, at least for this force.

We cannot include the nonlocal correction with complete consistency, but some remarks on its possible effects may not be completely out of place. The first thing to notice is that with our definition of τ_i in Eq. (19), the nonlocal parametric dependence on k_0 appears not only in the momentum variable, but also in the energy variable. This can result in much larger values of $[\partial\langle 0,\mathbf{k}_0|\bar{\tau}(m^*,E)|\mathbf{k}_0,0\rangle/\partial k_0]$ than estimated from $\bar{t}'$. In fact, neglecting the total energy dependence of $\bar{\tau}$, we get

$$[\partial\langle 0,\mathbf{k}_0|\bar{\tau}(m,E)|\mathbf{k}_0,0\rangle/\partial k_0]\sim 2\bar{t}'.$$

If we applied the impulse approximation, this nonlocal effect would be large enough to destroy the agreement obtained in Table V. But, as we have seen, $\bar{t}'$ is very dependent on a choice of m^*. If one compares $t^{(2)}$ in Table V at 90 MeV with $\bar{A}(0.44m)$ in Table II, one sees they are not all that different. Since the nonlocal correction is negligibly small, with μ equal to $0.44m$, it is possible that the same is true for $t^{(2)}$. However, only a third-order calculation can resolve this point, and we can say nothing conclusive at the moment. On the whole, however, Table V, which contains no adjustable parameters, must be regarded as encouraging. It gives corrections with the right sign and magnitude to remove the discrepancy.

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APPENDIX A

We have obtained a central modified Gammel-Thaler[13] force to fit the forward scattering amplitude. We use the word central in a loose sense, meaning that there is no coupling of the radial equations

$$\left[\frac{d^2}{dx^2}+k^2-\frac{J(J-1)}{x^2}-v_c(x)+\frac{2(J-1)}{(2J+1)}v_T(x)\right.$$
$$\left.-(J-1)v_{LS}(x)-(J-1)v_{LL}(x)\right]u_j(x)$$
$$=\frac{6[J(J+1)]^{1/2}}{(2J+1)}v_T(x)w_j(x)\,,$$

$$\left[\frac{d^2}{dx^2}+k^2-\frac{(J+1)(J+2)}{x^2}-v_c(x)+\frac{2(J+2)}{(2J+1)}\right.$$
$$\left.+(J+2)v_{LS}(x)+(J+2)v_{LL}(x)\right]w_j(x)$$
$$=\frac{6[J(J+1)]^{1/2}}{2J+1}v_T(x)u_j(x)\,.$$

The modification had the form that the coupling term was set to zero, and the diagonal v_T term was changed for the n-p triplet force only, to be of the form

$$\left\{\frac{2(J-1)}{(2J+1)}+\frac{6C[J(J+1)]^{1/2}}{(2J+1)}\right\}v_T'\,,$$

where v_T' differed from v_T by a constant of multiplication. There is a slight energy dependence of v_T' and C, as well as the suggested energy dependence of the full Gammel-Thaler force. In Table VI, we give the parameters of the force at 90 MeV, with the modifications that are to be made at 156 and 310 MeV.

TABLE VI. A central force[a] at 90 MeV (lab).[b]

$np,S=0$	$np,S=1$		$pp,S=0$	$pp,S=1$
${}^1v_c^+=425.5$	${}^3v_c^+=100.7$	${}^3v_T^+=400$	${}^1v_c^+=425.5$	${}^3v_c=0$
${}^1\mu_c^+=1.45$	${}^3\mu_c^+=1.23$	${}^3\mu_T^+=1.203$	${}^1\mu_c^+=1.45$	${}^3v_{LS}^+=0$
${}^1v_c^-=-100$	${}^3v_c^-=60$	${}^3v_T^-=0$	${}^1v_c^-=0$	${}^3v_T^+=0$
${}^1\mu_c^-=1.0$	${}^3\mu_c^-=1.5$	${}^3\mu_T^-=0.8$	${}^1v_{LS}=0$	${}^3v_{LS}^-=7122.5$
$r_c^-=0.5$	${}^3v_{LS}^+=5000$	$r_c^+=0.4$	${}^1v_T=0$	${}^3\mu_{LS}=3.7$
${}^1v_T=0$	${}^3\mu_{LS}^+=3.7$	$r_c^-=0.4125$	$r_c^+=0.4$	${}^3v_T^-=-26$
${}^1v_{LS}=0$	${}^3v_{LS}^-=7122.5$	$C=0.083$	$r_c^-=0.5$	${}^3\mu_T^-=0.8$
$r_c^+=0.4$	${}^3\mu_{LS}^-=3.7$			$r_c^+=0.4$
				$r_c^-=0.4125$

[a] We use the conventional notation of Ref. 13 and the tensor force described there which is zero at the core radius.
[b] At 156 MeV, ${}^3v_c^-$ is 20, ${}^3v_T^+$ is 290, C is 0.133. At 310 MeV, ${}^1v_c^-$ is -150, ${}^3v_c^+$ is 60, ${}^3v_c^-=-5$, ${}^3v_T^+$ is 175, $C=0$.

Mulligan used a force

$$\begin{aligned} v(r)&=v_0(e^{-\mu r}/r) && \text{even } l,\ r>c_0 \\ &=0, && \text{odd } l,\ r>c_0 \\ v(r)&=\infty\,, && r<c_0\,, \end{aligned}$$

where $c_0=0.5$ F, $\mu=1.7$ F^{-1}, and $v_0=-900$ MeV.

APPENDIX B

We wish to prove that

$$\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}\rangle=2\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle$$

when $\mathbf{k}=\mathbf{k}_E$.

The proof trivially follows for a potential $v(|\mathbf{r}|)$ by writing down the Born series for $\langle\mathbf{k}|t(E)|\mathbf{k}\rangle$, differentiating, and comparing with $\partial\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle/\partial k$. For parity-dependent, spin-orbit potentials, etc., the proof most easily follows if one performs the partial-wave decomposition first and then uses the Born series for $\langle k|t_l(E)|k\rangle$. For tensor forces, because we average over the spins of the target nucleus for an assumed closed-shell nucleus and we only look at forward angles, once again the theorem is true.

To use the theorem for a hard core, we must work a little, since the integral $\langle k|t_l(E)|k\rangle$ is not clearly defined. However, Mulligan[1] showed that

$$\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle=-\sum_l e^{i\delta_l}(2l+1)\left[j_l(kc_0)R_l'(k_E,c_0)c_0^2+\left(\frac{2\mu}{\hbar^2}\right)\int_{c_0}^{\infty}j_l(kr)v(r)R_l(k_E,r)r^2dr\right],$$

so that

$$\frac{\partial}{\partial k}\langle\mathbf{k}|t(E)|\mathbf{k}_E\rangle=-\sum_l e^{i\delta_l}(2l+1)\left[\frac{\partial j_l(kc_0)}{\partial k}R_l'(k_E,c_0)c_0^2+\left(\frac{2\mu}{\hbar^2}\right)\int_{c_0}^{\infty}\frac{\partial j_l(kr)}{\partial k}v(r)R_l(k_E,r)r^2dr\right].$$