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Theory of Superconductors with Overlapping Bands in the Presence of Nonmagnetic Impurities

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The two-band model proposed by Suhl, Matthias, and Walker (SMW) for the superconducting state of pure transition metals is extended to the transition metals in the presence of nonmagnetic impurities. Both the interband and intraband impurity scattering are included in the formulation of the extended model. Two limiting cases of the extended model, namely, the strong intraband phonon-coupling limit and the strong interband phonon-coupling limit, are studied in detail. The basic features of the two limiting cases agree with those of the SMW model. It is found that, in the case of the strong intraband phonon-coupling limit, the lower one of the two transition temperatures is raised from that of the SMW model by the interband impurity scattering, and, in the case of the strong interband phonon-coupling limit, the transition temperature is either raised or lowered from that of the SMW model, depending on whether the interband impurity scattering or the intraband impurity scattering is stronger.

I. INTRODUCTION

A TWO-BAND model was first proposed by Suhl, Matthias, and Walker[1] for the superconducting state of transition metals. It has been known for a long time that there are two overlapping bands in most of the transition metals, an s band and a d band, and the s-d interband impurity scattering contributes considerably to the resistivity of the transition metals in the normal state.[2] Suhl, Matthias, and Walker succeeded in extending the usual one-band BCS theory to the two-band situation. Yet they limited their discussions to the case of pure transition metals. In the two-band model, they introduced an extra phonon-coupling term in the Hamiltonian to take care of the possibility of pair formation of electrons in different bands. (We call this type of phonon coupling the interband phonon coupling, in contrast to the usual BCS-type intraband phonon coupling which causes electrons within the same band to form pairs.) It is noticed that when the interband phonon coupling vanishes, there are two transition temperatures, and when the interband phonon coupling is finite, even if weaker than the intraband phonon coupling, there can only be one transition temperature.

In this paper we shall investigate the influence of nonmagnetic impurities on the critical temperatures of the two-band model. It is fair to remark that we intend to make a model study, with the purpose of understanding better the general properties of the two-band superconductors. The two-spherical-band model which we shall deal with is actually quite different from the band structure of real transition metals. As a matter of fact, we actually know very little about the band structures of the transition metals, for example, vanadium and niobium, on which some superconductivity measurements have been done.

Moreover, we shall only pay attention to two limiting cases of the general problem, namely, the strong intraband phonon-coupling limit, in which the interband phonon coupling is considered to be vanishing, and the strong interband phonon-coupling limit, in which the intraband phonon coupling is considered to be vanishing. In the strong intraband phonon-coupling limit, we obtain two coupled equations relating two order parameters as functions of temperatures, from which two critical temperatures can be obtained. One of the order parameters is identified as due to the s band, another as due to the d band. It is shown that, if the d-band order parameter is much larger than the s-band order parameter, the interband impurity scattering would lead to an s-band critical temperature larger than that of the pure two-band model. In the strong interband phonon-coupling limit, we obtain only one order-parameter equation, and there can be only one critical temperature. The impurity scattering can raise or lower the critical temperature depending on whether the interband scattering is stronger or weaker than the intraband scattering.

II. GENERAL TWO-BAND MODEL

We take the Hamiltonian for the two-band model without impurities to be

$$\begin{aligned}\mathcal{H}_0=&\sum_\sigma\int d^3x\,\psi_{s\sigma}^\dagger(\mathbf{x})\left(-(\nabla^2/2m_s)-\mu\right)\psi_{s\sigma}(\mathbf{x})\\&+\sum_\sigma\int d^3x\,\psi_{d\sigma}^\dagger(\mathbf{x})\left(-(\nabla^2/2m_d)-\mu\right)\psi_{d\sigma}(\mathbf{x})\\&-g_s\int d^3x\,\psi_{s\uparrow}^\dagger(\mathbf{x})\psi_{s\downarrow}^\dagger(\mathbf{x})\psi_{s\downarrow}(\mathbf{x})\psi_{s\uparrow}(\mathbf{x})\\&-g_d\int d^3x\,\psi_{d\uparrow}^\dagger(\mathbf{x})\psi_{d\downarrow}^\dagger(\mathbf{x})\psi_{d\downarrow}(\mathbf{x})\psi_{d\uparrow}(\mathbf{x})\\&-g_{sd}\int d^3x\{\psi_{d\uparrow}^\dagger(\mathbf{x})\psi_{s\downarrow}^\dagger(\mathbf{x})\psi_{s\downarrow}(\mathbf{x})\psi_{d\uparrow}(\mathbf{x})\\&\qquad+\psi_{s\uparrow}^\dagger(\mathbf{x})\psi_{d\downarrow}^\dagger(\mathbf{x})\psi_{d\downarrow}(\mathbf{x})\psi_{s\uparrow}(\mathbf{x})\}.\qquad(1)\end{aligned}$$

[1] H. Suhl, B. T. Matthias, and L. R. Walker, Phys. Rev. Letters **3**, 552 (1959).
[2] A. H. Wilson, *The Theory of Metals* (Cambridge University Press, Cambridge, 1953), 2nd ed.

Here $\psi_{s\sigma}(\mathbf{x})$ and $\psi_{s\sigma}^\dagger(\mathbf{x})$ ($\sigma=\uparrow$ or $\downarrow$) are, respectively, the destruction and creation operators for the s-band electron at position $\mathbf{x}$. Similarly, $\psi_{d\sigma}(\mathbf{x})$ and $\psi_{d\sigma}^\dagger(\mathbf{x})$ are those for the d-band electron. μ is the chemical potential, with respect to which it is convenient to measure single-particle energies. The phonon-induced attractions between electrons are represented by coupling constants, g_s for s-s coupling, g_d for d-d coupling, and g_{sd} for s-d coupling. With the negative signs thus chosen in Eq. (1), g_s, g_d, and g_{sd} are all positive coupling constants.

The presence of impurities will cause transitions of electrons within each band as well as between different bands. The interaction Hamiltonian can be generally written as

$$\begin{aligned}\mathcal{H}_{\text{int}}=&\sum_i\sum_\sigma\int d^3x\, V_s(\mathbf{x}-\mathbf{R}_i)\psi_{s\sigma}^\dagger(\mathbf{x})\psi_{s\sigma}(\mathbf{x})\\&+\sum_i\sum_\sigma\int d^3x\, V_d(\mathbf{x}-\mathbf{R}_i)\psi_{d\sigma}^\dagger(\mathbf{x})\psi_{d\sigma}(\mathbf{x})\\&+\sum_i\sum_\sigma\int d^3x\, V_{sd}(\mathbf{x}-\mathbf{R}_i)\\&\times\{\psi_{s\sigma}^\dagger(\mathbf{x})\psi_{d\sigma}(\mathbf{x})+\psi_{d\sigma}^\dagger(\mathbf{x})\psi_{s\sigma}(\mathbf{x})\}.\end{aligned}\tag{2}$$

Here we consider only the presence of nonmagnetic impurities which do not cause spin-flipping interactions. $V_s(\mathbf{x}-\mathbf{R}_i)$ is the potential of an impurity atom at position R_i as felt by an s electron at position $\mathbf{x}$, while $V_d(\mathbf{x}-\mathbf{R}_i)$ is the potential as felt by a d electron. $V_{sd}(\mathbf{x}-\mathbf{R}_i)$ will cause interband transitions.[3]

To treat the electrons in the two bands simultaneously, we introduce a 4-component space in which the field operators are defined as

$$\Psi(x)\equiv\begin{pmatrix}\psi_{s\uparrow}(x)\\\psi_{d\uparrow}(x)\\\psi_{s\downarrow}^\dagger(x)\\\psi_{d\downarrow}^\dagger(x)\end{pmatrix},\tag{3}$$

$$\Psi^\dagger(x)\equiv(\psi_{s\uparrow}^\dagger(x),\psi_{d\uparrow}^\dagger(x),\psi_{s\downarrow}(x),\psi_{d\downarrow}(x)).\tag{4}$$

Here, x represents space and time $(\mathbf{x}, t)$. We write matrices in this 4-component space as direct products of 2×2 matrices. For example, we have

$$\boldsymbol{\tau}_3\times\boldsymbol{\sigma}_1=\left(\begin{array}{c|c}\boldsymbol{\sigma}_1&0\\\hline 0&-\boldsymbol{\sigma}_1\end{array}\right)=\left(\begin{array}{cc|cc}0&1&0&0\\1&0&0&0\\\hline 0&0&0&-1\\0&0&-1&0\end{array}\right).$$

Both $\boldsymbol{\tau}_i$ and $\boldsymbol{\sigma}_j$ are Pauli matrices. We let $\boldsymbol{\tau}_i$ denote the Pauli matrices in the larger space, and $\boldsymbol{\sigma}_j$ in the smaller space. Next, we define a 4×4 matrix Green's function

$$\mathcal{G}(x,x')=-i\langle T\Psi(x)\Psi^\dagger(x')\rangle,\tag{5}$$

where T is the time-ordering operator and $\langle\cdots\rangle$ denotes the grand canonical ensemble average. The 4×4 matrix Green's function can be expressed in terms of 2×2 matrix Green's functions

$$\mathcal{G}(x,x')=\begin{pmatrix}\mathbf{G}(x,x')&\mathbf{F}(x,x')\\-\mathbf{F}^\dagger(x,x')&-\mathbf{G}^T(x,x')\end{pmatrix},\tag{6}$$

where

$$\mathbf{G}(x,x')=\begin{pmatrix}G_{ss}(x,x')&G_{sd}(x,x')\\G_{ds}(x,x')&G_{dd}(x,x')\end{pmatrix},\tag{7}$$

$$\mathbf{F}(x,x')=\begin{pmatrix}F_{ss}(x,x')&F_{sd}(x,x')\\F_{ds}(x,x')&F_{dd}(x,x')\end{pmatrix}\tag{8}$$

and

$$[\mathbf{F}^\dagger(x,x')]_{\alpha\beta}=[\mathbf{F}(x',x)]_{\beta\alpha}^*,\tag{9}$$

$$[\mathbf{G}^T(x,x')]_{\alpha\beta}=[\mathbf{G}(x',x)]_{\beta\alpha}.\tag{10}$$

We follow the convention of Baym and Kadanoff[4] for the Fourier transform of the Green's function

$$\begin{aligned}\mathcal{G}(x,x')=&(-i\beta)^{-1}\sum_{z_\nu}\int\frac{d^3p}{(2\pi)^3}\\&\times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-iz_\nu(t-t')]\mathcal{G}(\mathbf{p},z_\nu),\end{aligned}\tag{11}$$

$$\begin{aligned}\mathcal{G}(\mathbf{p},z_\nu)=&\int_0^{-i\beta}d(t-t')\int d^3(\mathbf{x}-\mathbf{x}')\\&\times\exp[-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+iz_\nu(t-t')]\mathcal{G}(x,x').\end{aligned}\tag{12}$$

Here, $z_\nu=i\pi(2\nu+1)/\beta$, with ν equal to any positive or negative integer, and t is taken to be imaginary time, defined in the region $(0, -i\beta)$, just as in the usual finite-temperature Green's-function theory. Equations (9) and (10) in the $(\mathbf{p}, z_\nu)$ space are

$$[\mathbf{F}^\dagger(\mathbf{p},z_\nu)]_{\alpha\beta}=[\mathbf{F}(-\mathbf{p},-z_\nu)]_{\beta\alpha}^*,\tag{13}$$

$$[\mathbf{G}^T(\mathbf{p},z_\nu)]_{\alpha\beta}=[\mathbf{G}(-\mathbf{p},-z_\nu)]_{\beta\alpha}.\tag{14}$$

With the total Hamiltonian $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\text{int}}$ one can obtain the equation of motion for the 4×4 matrix

[3] H. Jones, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956).

[4] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962); also see V. Ambegaokar, in *Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. 2.

Green's function:

$$\begin{pmatrix} i(\partial 1/\partial t)-\mathbf{K}(\mathbf{x})-\mathbf{V}(\mathbf{x}) & -\mathbf{\Delta}(\mathbf{x}) \\ -\mathbf{\Delta}^\dagger(\mathbf{x}) & i(\partial 1/\partial t)+\mathbf{K}(\mathbf{x})+\mathbf{V}(\mathbf{x}) \end{pmatrix} \mathcal{G}(x,x') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{15}$$

where

$$\mathbf{K}(\mathbf{x}) = \begin{pmatrix} -(\nabla^2/2m_s)-\mu & 0 \\ 0 & -(\nabla^2/2m_d)-\mu \end{pmatrix}, \tag{16}$$

$$\mathbf{V}(\mathbf{x}) = \begin{pmatrix} \sum_i V_s(\mathbf{x}-\mathbf{R}_i) & \sum_i V_{sd}(\mathbf{x}-\mathbf{R}_i) \\ \sum_i V_{sd}(\mathbf{x}-\mathbf{R}_i) & \sum_i V_d(\mathbf{x}-\mathbf{R}_i) \end{pmatrix}, \tag{17}$$

and $\mathbf{\Delta}(x)$ is a 2×2 matrix order parameter which can be obtained by self-consistent calculation,

$$\mathbf{\Delta}(x) = \begin{pmatrix} g_s\langle\psi_{s\uparrow}(x)\psi_{s\downarrow}(x)\rangle & g_{sd}\langle\psi_{s\uparrow}(x)\psi_{d\downarrow}(x)\rangle \\ g_{sd}\langle\psi_{d\uparrow}(x)\psi_{s\downarrow}(x)\rangle & g_d\langle\psi_{d\uparrow}(x)\psi_{d\downarrow}(x)\rangle \end{pmatrix}, \tag{18}$$

$$\mathbf{\Delta}^\dagger(x) = \begin{pmatrix} g_s\langle\psi_{s\downarrow}^\dagger(x)\psi_{s\uparrow}^\dagger(x)\rangle & g_{sd}\langle\psi_{s\downarrow}^\dagger(x)\psi_{d\uparrow}^\dagger(x)\rangle \\ g_{sd}\langle\psi_{d\downarrow}^\dagger(x)\psi_{s\uparrow}^\dagger(x)\rangle & g_d\langle\psi_{d\downarrow}^\dagger(x)\psi_{d\uparrow}^\dagger(x)\rangle \end{pmatrix}. \tag{19}$$

The problem now is to solve the set of Eqs. (15)–(19) self-consistently. Once the 4×4 matrix Green's function is obtained, one can calculate all the thermodynamic quantities through the usual BCS theory. The particular case with no impurities involved has been solved by Suhl, Matthias, and Walker.[1] They have shown that without impurities, there are two transition temperatures in the limit $g_{sd}=0$ (that is, there is only intraband phonon coupling); on the other hand, there can be only one transition temperature when $g_{sd}\neq 0$. It will be shown that, with the inclusion of nonmagnetic impurity scattering, the general features of their theory still exist. Due to mathematical complication involved in the general case (with all g_s, g_d, $g_{sd}\neq 0$), we shall pay attention to two limiting cases, namely, (1) $g_{sd}=0$, the intraband phonon-coupling limit, and (2) $g_s=g_d=0$, the interband phonon-coupling limit. Through solving these two cases, some important features of the influence of the nonmagnetic impurities on the two-band model are revealed.

III. INTRABAND PHONON-COUPLING LIMIT

In this limiting case, the 2×2 matrix order parameter is diagonal:

$$\mathbf{\Delta}(x) = \mathbf{\Delta}_1(x) = \begin{pmatrix} g_s\langle\psi_{s\uparrow}(x)\psi_{s\downarrow}(x)\rangle & 0 \\ 0 & g_d\langle\psi_{d\uparrow}(x)\psi_{d\downarrow}(x)\rangle \end{pmatrix}. \tag{20}$$

It is convenient to solve the problem in the $(\mathbf{p}, z_\nu)$ space and to define a 4×4 matrix self-energy $\Sigma(\mathbf{p}, z_\nu)$ in a formally identical fashion as one does in a one-component case:

$$\mathcal{G}^{-1}(\mathbf{p}, z_\nu) = z_\nu \mathbf{1}\times\mathbf{1} - \boldsymbol{\tau}_3\times\boldsymbol{\varepsilon}_\mathrm{p} - \Sigma(\mathbf{p}, z_\nu)$$
$$\equiv \mathcal{G}_0^{-1}(\mathbf{p}, z_\nu) - \Sigma(\mathbf{p}, z_\nu). \tag{21}$$

Here, $\mathcal{G}_0^{-1}(\mathbf{p}, z_\nu)$ is the corresponding 4×4 matrix Green's function for the two-band model in the normal state and $\boldsymbol{\varepsilon}_\mathrm{p}$ is now

$$\boldsymbol{\varepsilon}_\mathrm{p} = \begin{pmatrix} \epsilon_{s\mathrm{p}} & 0 \\ 0 & \epsilon_{d\mathrm{p}} \end{pmatrix}, \tag{22}$$

where $\epsilon_{s\mathrm{p}}$ and $\epsilon_{d\mathrm{p}}$ are single-particle kinetic energies measured with respect to the Fermi energy level. The matrix self-energy $\Sigma(\mathbf{p}, z_\nu)$ is to be determined self-consistently by considering the lowest-order diagrams contributing to the self-energy. Similar to the case treated by Markowitz and Kadanoff,[5] the 4×4 matrix self-energy is the sum of two types of diagrams:

$$\Sigma(\mathbf{p}, z_\nu) = \Sigma_{\mathrm{sup}}(\mathbf{p}, z_\nu) + \Sigma_{\mathrm{imp}}(\mathbf{p}, z_\nu). \tag{23}$$

The interaction leading to the superconductivity gives rise to components (off-diagonal in the larger space but diagonal in the smaller space) of

$$\Sigma_{\mathrm{sup}}(\mathbf{p}, z_\nu) = -i(1/-i\beta) \times \sum_\mu \int \frac{d^3k}{(2\pi)^3} \boldsymbol{\tau}_3 \times \mathbf{g}\mathcal{G}(\mathbf{p}-\mathbf{k}, z_\nu - z_\mu)\boldsymbol{\tau}_3 \times \mathbf{1}, \tag{24}$$

or

$$\Sigma_{\mathrm{sup}}\Big|_{\mathrm{o.d.}\times\mathrm{d.}} = -1/\beta \sum_\mu \int \frac{d^3k}{(2\pi)^3} \mathbf{1}\times\mathbf{g}\mathcal{G}(\mathbf{k}, z_\mu)\Big|_{\mathrm{o.d.}\times\mathrm{d.}}, \tag{25}$$

[5] D. Markowitz and L. P. Kadanoff, Phys. Rev. **131**, 563 (1963).

where

$$\mathbf{g}=\begin{pmatrix} g_s & 0 \\ 0 & g_d \end{pmatrix}. \tag{26}$$

We remark that this will turn out to be the following:

$$\Sigma_{\text{sup}}\,|_{\text{o.d.}\times\text{d.}}=\boldsymbol{\tau}_1\times\bar{\boldsymbol{\Delta}}, \tag{27}$$

with

$$\bar{\boldsymbol{\Delta}}=\begin{pmatrix} \bar{\Delta}_s & 0 \\ 0 & \bar{\Delta}_d \end{pmatrix}. \tag{28}$$

Since the impurities are randomly distributed throughout the superconductor, the self-energy due to the electron impurity is obtained by taking the usual second-order Born approximation and averaging the positions of the impurities over the entire superconductor. We write Eq. (17) as

$$\mathbf{V}(\mathbf{x})=\mathbf{V}_1(\mathbf{x})+\mathbf{V}_2(\mathbf{x}), \tag{29}$$

with

$$\mathbf{V}_1(\mathbf{x})=\begin{pmatrix} \sum_i V_s(\mathbf{x}-\mathbf{R}_i) & 0 \\ 0 & \sum_i V_d(\mathbf{x}-\mathbf{R}_i) \end{pmatrix}, \tag{30}$$

$$\mathbf{V}_2(\mathbf{x})=\boldsymbol{\sigma}_1\sum_i V_{sd}(\mathbf{x}-\mathbf{R}_i). \tag{31}$$

The self-energy due to electron-impurity scattering is

$$\Sigma_{\text{imp}}(\mathbf{p},z_\nu)=n_i\int\frac{d^3k}{(2\pi)^3}\boldsymbol{\tau}_3\times\mathbf{V}_1(\mathbf{k})\mathcal{G}(\mathbf{p}-\mathbf{k},z_\nu)\boldsymbol{\tau}_3\times\mathbf{V}_1(\mathbf{k})$$
$$+n_i\int\frac{d^3k}{(2\pi)^3}\boldsymbol{\tau}_3\times\boldsymbol{\sigma}_1\mathcal{G}(\mathbf{p}-\mathbf{k},z_\nu)\boldsymbol{\tau}_3\times\boldsymbol{\sigma}_1 V_{sd}^2(\mathbf{k}), \tag{32}$$

where $\mathbf{V}_1(\mathbf{k})$ and $V_{sd}(\mathbf{k})$ are the Fourier transforms of $\mathbf{V}_1(\mathbf{x})$ and $V_{sd}(\mathbf{x}-\mathbf{R}_i)$, respectively. n_i is the density of the impurities. Because of the nature of the symmetry of the components of the matrix Green's function, Eq. (6), we have

$$\Sigma_{\text{imp}}=\begin{pmatrix} \boldsymbol{\Sigma}_{\text{imp}} & \boldsymbol{\Xi}_{\text{imp}} \\ -\boldsymbol{\Xi}_{\text{imp}}^\dagger & -\boldsymbol{\Sigma}_{\text{imp}}^T \end{pmatrix}. \tag{33}$$

The nonvanishing components of $\boldsymbol{\Sigma}_{\text{imp}}$ are

$$\Sigma_{\text{imp},ss}(\mathbf{p},z_\nu)=n_i\int\frac{d^3k}{(2\pi)^3}V_s^2(\mathbf{k})G_{ss}(\mathbf{p}-\mathbf{k},z_\nu)$$
$$+n_i\int\frac{d^3k}{(2\pi)^3}V_{sd}^2(\mathbf{k})G_{dd}(\mathbf{p}-\mathbf{k},z_\nu), \tag{34}$$

$$\Sigma_{\text{imp},dd}(\mathbf{p},z_\nu)=n_i\int\frac{d^3k}{(2\pi)^3}V_d^2(\mathbf{k})G_{dd}(\mathbf{p}-\mathbf{k},z_\nu)$$
$$+n_i\int\frac{d^3k}{(2\pi)^3}V_{sd}^2(\mathbf{k})G_{ss}(\mathbf{p}-\mathbf{k},z_\nu). \tag{35}$$

The nonvanishing components of $\boldsymbol{\Xi}_{\text{imp}}$ are

$$\Xi_{\text{imp},ss}(\mathbf{p},z_\nu)=-n_i\int\frac{d^3k}{(2\pi)^3}V_s^2(\mathbf{k})F_{ss}(\mathbf{p}-\mathbf{k},z_\nu)$$
$$-n_i\int\frac{d^3k}{(2\pi)^3}V_{sd}^2(\mathbf{k})F_{dd}(\mathbf{p}-\mathbf{k},z_\nu), \tag{36}$$

$$\Xi_{\text{imp},dd}(\mathbf{p},z_\nu)=-n_i\int\frac{d^3k}{(2\pi)^3}V_d^2(\mathbf{k})F_{dd}(\mathbf{p}-\mathbf{k},z_\nu)$$
$$-n_i\int\frac{d^3k}{(2\pi)^3}V_{sd}^2(\mathbf{k})F_{ss}(\mathbf{p}-\mathbf{k},z_\nu). \tag{37}$$

In this case the 4×4 matrix Green's function is diagonal in the smaller 2×2 space, and, therefore, the 4×4 equation of motion, Eq. (15), can be split into two simultaneous 2×2 equations in the $(\mathbf{p},z_\nu)$ space:

$$\begin{pmatrix} z_\nu-\epsilon_{sp}-\Sigma_{\text{imp},ss}(\mathbf{p},z_\nu) & -\bar{\Delta}_s-\Xi_{\text{imp},ss}(\mathbf{p},z_\nu) \\ -\bar{\Delta}_s^*+\Xi_{\text{imp},ss}^*(-\mathbf{p},-z_\nu) & z_\nu+\epsilon_{sp}+\Sigma_{\text{imp},ss}(-\mathbf{p},-z_\nu) \end{pmatrix}\mathbf{G}_s(\mathbf{p},z_\nu)=\mathbf{1}, \tag{38}$$

$$\begin{pmatrix} z_\nu-\epsilon_{dp}-\Sigma_{\text{imp},dd}(\mathbf{p},z_\nu) & -\bar{\Delta}_d-\Xi_{\text{imp},dd}(\mathbf{p},z_\nu) \\ -\bar{\Delta}_d^*+\Xi_{\text{imp},dd}^*(-\mathbf{p},-z_\nu) & z_\nu+\epsilon_{dp}+\Sigma_{\text{imp},dd}(-\mathbf{p},-z_\nu) \end{pmatrix}\mathbf{G}_d(\mathbf{p},z_\nu)=\mathbf{1}, \tag{39}$$

where

$$\mathbf{G}_s(\mathbf{p},t_\nu)=\begin{pmatrix} G_{ss}(\mathbf{p},t_\nu) & F_{ss}(\mathbf{p},t_\nu) \\ -F_{ss}^*(-\mathbf{p},-t_\nu) & -G_{ss}(-\mathbf{p},-t_\nu) \end{pmatrix}, \tag{40}$$

$$\mathbf{G}_d(\mathbf{p},z_\nu)=\begin{pmatrix} G_{dd}(\mathbf{p},z_\nu) & F_{dd}(\mathbf{p},z_\nu) \\ -F_{dd}^*(-\mathbf{p},-z_\nu) & -G_{dd}(-\mathbf{p},-z_\nu) \end{pmatrix}, \tag{41}$$

and, from Eqs. (25) and (28), we have

$$\bar{\Delta}_s=-\frac{g_s}{\beta}\sum_\mu\int'\frac{d^3k}{(2\pi)^3}F_{ss}(\mathbf{k},z_\mu), \tag{42}$$

$$\bar{\Delta}_d=-\frac{g_d}{\beta}\sum_\mu\int'\frac{d^3k}{(2\pi)^3}F_{dd}(\mathbf{k},z_\mu). \tag{43}$$

Here the prime on the integral signs is to denote the necessary cutoff in the momentum space, since we must limit the virtual-phonon energy exchange to the range

$(-\omega_D, \omega_D)$, where ω_D is the Debye frequency. It can be shown later that $\bar{\Delta}_s$ and $\bar{\Delta}_d$ are real quantities, the order parameters.

To solve the simultaneous equations (38) and (39) we take the ansatz

$$\mathbf{G}_s^{-1}(\mathbf{p}, z_\nu) = \tilde{z}_{s\nu} - \epsilon_{s\mathbf{p}}\boldsymbol{\tau}_3 - \tilde{\Delta}_{s\nu}\boldsymbol{\tau}_1, \tag{44}$$

$$\mathbf{G}_d^{-1}(\mathbf{p}, z_\nu) = \tilde{z}_{d\nu} - \epsilon_{d\mathbf{p}}\boldsymbol{\tau}_3 - \tilde{\Delta}_{d\nu}\boldsymbol{\tau}_1, \tag{45}$$

or

$$\mathbf{G}_s(\mathbf{p}, z_\nu) = \frac{\tilde{z}_{s\nu} + \epsilon_{s\mathbf{p}}\boldsymbol{\tau}_3 + \tilde{\Delta}_{s\nu}\boldsymbol{\tau}_1}{\tilde{z}_{s\nu}^2 - \epsilon_{s\mathbf{p}}^2 - \tilde{\Delta}_{s\nu}^2}, \tag{46}$$

$$\mathbf{G}_d(\mathbf{p}, z_\nu) = \frac{\tilde{z}_{d\nu} + \epsilon_{d\mathbf{p}}\boldsymbol{\tau}_3 + \tilde{\Delta}_{d\nu}\boldsymbol{\tau}_1}{\tilde{z}_{d\nu}^2 - \epsilon_{d\mathbf{p}}^2 - \tilde{\Delta}_{d\nu}^2}. \tag{47}$$

The ansatz gives the equations of the order parameters as

$$\bar{\Delta}_s = -\frac{g_s}{\beta}\sum_\mu \int' \frac{d^3k}{(2\pi)^3} \frac{\tilde{\Delta}_{s\mu}}{\tilde{z}_{s\mu}^2 - \epsilon_{s\mathbf{k}}^2 - \tilde{\Delta}_{s\mu}^2}, \tag{48}$$

$$\bar{\Delta}_d = -\frac{g_d}{\beta}\sum_\mu \int' \frac{d^3k}{(2\pi)^3} \frac{\tilde{\Delta}_{d\mu}}{\tilde{z}_{d\mu}^2 - \epsilon_{d\mathbf{k}}^2 - \tilde{\Delta}_{d\mu}^2}. \tag{49}$$

Based on the same approximations, as used by Markowitz and Kadanoff[5] in treating the impurity effects in one-band superconductors, we have from the ansatz and Eqs. (34)–(37)

$$\Sigma_{\mathrm{imp},ss}(\mathbf{p}, z_\nu) \cong \Sigma_{\mathrm{imp},ss}(z_\nu) = -\frac{i}{2\tau_s}\frac{\tilde{z}_{s\nu}}{(\tilde{z}_{s\nu}^2 - \tilde{\Delta}_{s\nu}^2)^{1/2}} - \frac{i}{2\tau_{sd}}\frac{\tilde{z}_{d\nu}}{(\tilde{z}_{d\nu}^2 - \tilde{\Delta}_{d\nu}^2)^{1/2}}, \tag{50}$$

$$\Sigma_{\mathrm{imp},dd}(\mathbf{p}, z_\nu) \cong \Sigma_{\mathrm{imp},dd}(z_\nu) = -\frac{i}{2\tau_d}\frac{\tilde{z}_{d\nu}}{(\tilde{z}_{d\nu}^2 - \tilde{\Delta}_{d\nu}^2)^{1/2}} - \frac{i}{2\tau_{ds}}\frac{\tilde{z}_{s\nu}}{(\tilde{z}_{s\nu}^2 - \tilde{\Delta}_{s\nu}^2)^{1/2}}, \tag{51}$$

$$\Xi_{\mathrm{imp},ss}(\mathbf{p}, z_\nu) \cong \Xi_{\mathrm{imp},ss}(z_\nu) = \frac{i}{2\tau_s}\frac{\Delta_{s\nu}}{(\tilde{z}_{s\nu}^2 - \tilde{\Delta}_{s\nu}^2)^{1/2}} + \frac{i}{2\tau_{sd}}\frac{\tilde{\Delta}_{d\nu}}{(\tilde{z}_{d\nu}^2 - \tilde{\Delta}_{d\nu}^2)^{1/2}}, \tag{52}$$

$$\Xi_{\mathrm{imp},dd}(\mathbf{p}, z_\nu) \cong \Xi_{\mathrm{imp},dd}(z_\nu) = \frac{i}{2\tau_d}\frac{\Delta_{d\nu}}{(\tilde{z}_{d\nu}^2 - \tilde{\Delta}_{d\nu}^2)^{1/2}} + \frac{i}{2\tau_{ds}}\frac{\tilde{\Delta}_{s\nu}}{(\tilde{z}_{s\nu}^2 - \tilde{\Delta}_{s\nu}^2)^{1/2}}, \tag{53}$$

where

$$1/2\tau_s = n_i\pi N_s(0)\langle V_s^2(\mathbf{p})\rangle_\Omega, \tag{54}$$

$$1/2\tau_d = n_i\pi N_d(0)\langle V_d^2(\mathbf{p})\rangle_\Omega, \tag{55}$$

$$1/2\tau_{sd} = n_i\pi N_d(0)\langle V_{sd}^2(\mathbf{p})\rangle_\Omega, \tag{56}$$

$$1/2\tau_{ds} = n_i\pi N_s(0)\langle V_{sd}^2(\mathbf{p})\rangle_\Omega. \tag{57}$$

The brackets $\langle\ \rangle_\Omega$ on the impurity potential functions denote the solid-angle average of the functions. $N_s(0) = m_s p_{Fs}/2\pi^2$ is the density of states of the s band at the Fermi energy level, and $N_d(0) = m_d p_{Fd}/2\pi^2$ is the density of states of the d band at the Fermi level. p_{Fs} and p_{Fd} are, respectively, the Fermi momenta for s band and d band, as we are going to treat the case of two spherical Fermi surfaces.

Further, we obtain some important relations between the quantities given in the ansatz and the already defined quantities. It is convenient to write the relations in terms of $\omega_\nu \equiv (1/i)z_\nu$, $\tilde{\omega}_{s\nu} \equiv (1/i)\tilde{z}_{s\nu}(i\omega_\nu)$, and $\tilde{\omega}_{d\nu} \equiv (1/i)\tilde{z}_{d\nu}(i\omega_\nu)$:

$$\tilde{\omega}_{s\nu} = \omega_\nu + (2\tau_s)^{-1}\frac{\tilde{\omega}_{s\nu}}{(\tilde{\omega}_{s\nu}^2 + \tilde{\Delta}_{s\nu}^2)^{1/2}} + (2\tau_{sd})^{-1}\frac{\tilde{\omega}_{d\nu}}{(\tilde{\omega}_{d\nu}^2 + \tilde{\Delta}_{d\nu}^2)^{1/2}}, \tag{58}$$

$$\tilde{\Delta}_{s\nu} = \bar{\Delta}_s + (2\tau_s)^{-1}\frac{\tilde{\Delta}_{s\nu}}{(\tilde{\omega}_{s\nu}^2 + \tilde{\Delta}_{s\nu}^2)^{1/2}} + (2\tau_{sd})^{-1}\frac{\tilde{\Delta}_{d\nu}}{(\tilde{\omega}_{d\nu}^2 + \tilde{\Delta}_{d\nu}^2)^{1/2}}, \tag{59}$$

$$\tilde{\omega}_{d\nu} = \omega_\nu + (2\tau_d)^{-1}\frac{\tilde{\omega}_{d\nu}}{(\tilde{\omega}_{d\nu}^2 + \tilde{\Delta}_{d\nu}^2)^{1/2}} + (2\tau_{ds})^{-1}\frac{\tilde{\omega}_{s\nu}}{(\tilde{\omega}_{s\nu}^2 + \tilde{\Delta}_{s\nu}^2)^{1/2}}, \tag{60}$$

$$\tilde{\Delta}_{d\nu} = \bar{\Delta}_d + (2\tau_d)^{-1}\frac{\tilde{\Delta}_{d\nu}}{(\tilde{\omega}_{d\nu}^2 + \tilde{\Delta}_{d\nu}^2)^{1/2}} + (2\tau_{ds})^{-1}\frac{\tilde{\Delta}_{s\nu}}{(\tilde{\omega}_{s\nu}^2 + \tilde{\Delta}_{s\nu}^2)^{1/2}}. \tag{61}$$

Now we can rewrite Eqs. (42) and (43) as

$$\bar{\Delta}_s = g_s N_s(0) k_B T \sum_{\nu=-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} d\epsilon_s \frac{\tilde{\Delta}_{s\nu}}{\tilde{\omega}_{s\nu}^2 + \epsilon_s^2 + \tilde{\Delta}_{s\nu}^2}, \tag{62}$$

$$\bar{\Delta}_d = g_d N_d(0) k_B T \sum_{\nu=-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} d\epsilon_d \frac{\tilde{\Delta}_{d\nu}}{\tilde{\omega}_{d\nu}^2 + \epsilon_d^2 + \tilde{\Delta}_{d\nu}^2}. \tag{63}$$

Making use of the well-known identity in the BCS theory,

$$\ln\left(\frac{2\gamma\omega_D}{\pi k_B T}\right) = k_B T \sum_{\nu=-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} \frac{dx}{\omega_\nu^2 + x^2}, \tag{64}$$

where $\ln\gamma =$ Euler's constant, and also of the equations for the critical temperatures of the two bands, when the superconductor is pure,

$$1/g_s N_s(0) = \ln(2\gamma\omega_D/\pi T_{cs}^{(0)}), \tag{65}$$

$$1/g_d N_d(0) = \ln(2\gamma\omega_D/\pi T_{cd}^{(0)}), \tag{66}$$

we obtain the following equations:

$$\ln\frac{T}{T_{cs}^{(0)}}=\frac{k_BT}{\bar{\Delta}_s}\sum_{\nu=-\infty}^{\infty}\int_{-\omega_D}^{\omega_D}d\epsilon_s\left(\frac{\tilde{\Delta}_{s\nu}}{\tilde{\omega}_{s\nu}^2+\epsilon_s^2+\tilde{\Delta}_{s\nu}^2}-\frac{\bar{\Delta}_s}{\omega_\nu^2+\epsilon_s^2}\right),\quad(67)$$

$$\ln\frac{T}{T_{cd}^{(0)}}=\frac{k_BT}{\bar{\Delta}_d}\sum_{\nu=-\infty}^{\infty}\int_{-\omega_D}^{\omega_D}d\epsilon_d\left(\frac{\tilde{\Delta}_{d\nu}}{\tilde{\omega}_{d\nu}^2+\epsilon_d^2+\tilde{\Delta}_{d\nu}^2}-\frac{\bar{\Delta}_d}{\omega_\nu^2+\epsilon_d^2}\right).\quad(68)$$

The cutoff frequency now can be taken to be infinity, as we no longer have the divergence difficulty. After carrying out the integration, we have

$$\ln\frac{T}{T_{cs}^{(0)}}=\frac{2\pi k_BT}{\bar{\Delta}_s}\sum_{\nu=0}^{\infty}\left(\frac{\tilde{\Delta}_{s\nu}}{(\tilde{\omega}_{s\nu}^2+\tilde{\Delta}_{s\nu}^2)^{1/2}}-\frac{\bar{\Delta}_s}{\omega_\nu}\right),\quad(69)$$

$$\ln\frac{T}{T_{cd}^{(0)}}=\frac{2\pi k_BT}{\bar{\Delta}_d}\sum_{\nu=0}^{\infty}\left(\frac{\tilde{\Delta}_{d\nu}}{(\tilde{\omega}_{d\nu}^2+\tilde{\Delta}_{d\nu}^2)^{1/2}}-\frac{\bar{\Delta}_d}{\omega_\nu}\right).\quad(70)$$

These two equations are valid for all temperatures.

It is convenient to introduce the following dimensionless quantities:

$$u_{s\nu}=\tilde{\omega}_{s\nu}/\tilde{\Delta}_{s\nu},\qquad u_{d\nu}=\tilde{\omega}_{d\nu}/\tilde{\Delta}_{d\nu},\quad(71)$$

$$v_{s\nu}=\omega_\nu/\bar{\Delta}_s=(2\nu+1)\pi k_BT/\bar{\Delta}_s,$$

$$v_{d\nu}=\omega_\nu/\bar{\Delta}_d=(2\nu+1)\pi k_BT/\bar{\Delta}_d.\quad(72)$$

With the new quantities, Eqs. (69) and (70) become

$$\ln\frac{T}{T_{cs}^{(0)}}=2v_{s0}\sum_{\nu=0}^{\infty}\left((1+u_{s\nu}^2)^{-1/2}-(v_{s\nu})^{-1}\right),\quad(73)$$

$$\ln\frac{T}{T_{cd}^{(0)}}=2v_{d0}\sum_{\nu=0}^{\infty}\left((1+u_{d\nu}^2)^{-1/2}-(v_{d\nu})^{-1}\right).\quad(74)$$

The general problem about the temperature dependence of the order parameters in the present limiting case can, in principle, be solved from Eqs. (73) and (74), though the mathematics involved are quite complicated. To show the effects induced by the impurity scattering, we consider a simpler situation, some aspects of which have been investigated by Sung and Wong.[6] We assume

$$N_d(0)\gg N_s(0),\qquad \bar{\Delta}_d\gg\bar{\Delta}_s,\quad(75)$$

and thus, in the pure limit,

$$T_{cd}^{(0)}\gg T_{cs}^{(0)}.\quad(76)$$

In this case, for d band, we have approximately

$$u_{d\nu}\cong v_{d\nu}.\quad(77)$$

Equation (74) becomes the equation obtained by the BCS theory for pure one-band superconductors:

$$\ln\frac{T}{T_{cd}^{(0)}}\cong 2v_{d0}\sum_{\nu=0}^{\infty}\left((1+v_{d\nu}^2)^{-1/2}-(v_{d\nu})^{-1}\right).\quad(78)$$

It is easy to show that $T_{cd}=T_{cd}^{(0)}$. However, we still have to be careful in treating the s band. The approximate relation between $u_{s\nu}$ and $v_{s\nu}$ is now

$$v_{s\nu}\cong u_{s\nu}\{1+[\alpha_{sd\nu}/(1+v_{d\nu}^2)^{1/2}]\},\quad(79)$$

where

$$\alpha_{sd\nu}\cong(2\tau_{sd})^{-1}(\tilde{\Delta}_{s\nu}^{-1}-\tilde{\Delta}_{d\nu}^{-1}).\quad(80)$$

We rewrite Eq. (73) as

$$\ln\frac{T}{T_{cs}^{(0)}}=2v_{s0}\sum_{\nu=0}^{\infty}\left((1+u_{d\nu}^2)^{-1/2}-(u_{s\nu})^{-1}\right)-2v_{s0}\sum_{\nu=0}^{\infty}(v_{s\nu}^{-1}-u_{s\nu}^{-1}).\quad(81)$$

When $T\lesssim T_{cs}$, we have $\bar{\Delta}_s\sim 0$, $u_{s\nu}\gg 1$. Then we have approximately

$$\ln\frac{T}{T_{cs}^{(0)}}\cong 2v_{s0}\sum_{\nu=0}^{\infty}(u_{s\nu}^{-1}-{}^1v_{s\nu}^{-1}).\quad(82)$$

In the limit $T\to T_{cs}$, $\bar{\Delta}_s\to 0$. From Eq. (59) we know that $\tilde{\Delta}_{s\nu}$ is small but does not approach zero in this temperature limit. Then, at $T=T_{cs}$, $v_{s\nu}>u_{s\nu}$ and Eq. (82) gives $T_{cs}>T_{cs}^{(0)}$. Here we have only given a qualitative discussion about this interesting effect. One would notice that such an effect is due to the nonvanishing of $(2\tau_{sd})^{-1}$. We can therefore draw a qualitative conclusion that the interband impurity scattering leads to an enhancement of the s-band pair formation, and thus leads to an s-band critical temperature which is higher than that of the corresponding pure two-band superconductor. It is conjectured that perhaps a detailed investigation of Eqs. (73) and (74) will show that the s-d interband impurity scattering would lead to the increase of the s-band critical temperature and simultaneously to the decrease of the d-band critical temperature if $\bar{\Delta}_d>\bar{\Delta}_s$.

IV. INTERBAND PHONON-COUPLING LIMIT

In this limiting case, the 2×2 matrix order parameter is off-diagonal:

$$\mathbf{\Delta}(x)=\mathbf{\Delta}_2(x)=\begin{pmatrix}0 & g_{sd}\langle\psi_{s\uparrow}(x)\psi_{d\downarrow}(x)\rangle\\ g_{sd}\langle\psi_{d\uparrow}(x)\psi_{s\downarrow}(x)\rangle & 0\end{pmatrix}=g_{sd}\langle\psi_{s\uparrow}(x)\psi_{d\downarrow}(x)\rangle\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.\quad(83)$$

We also assume a 4×4 matrix self-energy $\Sigma(\mathbf{p},z_\nu)$, just as we did in the previous case. It is noticed that, in the present case, the 4×4 matrix Green's function can only have diagonal components and off-diagonal components, similar to the 4×4 matrix Green's function

[6] C. C. Sung and V. K. Wong, J. Phys. Chem. Solids **28**, 1933 (1967).

one meets in treating the paramagnetic impurities in a one-band superconductor. See Appendix A of a paper by Ambegoakar and Griffin.[7]

The part of the self-energy which contributes to the superconductivity is

$$\Sigma_{\rm sup}(\mathbf{p}, z_\nu) = -i(-i\beta)^{-1} g_{sd} \times \sum_\mu \int \frac{d^3k}{(2\pi)^3} \boldsymbol{\tau}_3 \times \mathbf{1}\mathcal{G}(\mathbf{p}-\mathbf{k}, z_\nu - z_\mu)\boldsymbol{\tau}_3 \times \mathbf{1} \quad (84)$$

or

$$\Sigma_{\rm sup}\Big|_{\rm o.d.} = g_{sd}(-\beta^{-1}) \sum_\mu \int \frac{d^3k}{(2\pi)^3} \mathcal{G}(\mathbf{k}, z_\mu)\Big|_{\rm o.d.}. \quad (85)$$

We remark that this will turn out to be the following:

$$\Sigma_{\rm sup}\,|_{\rm o.d.} = -\bar{\Delta}_{sd}\boldsymbol{\tau}_2 \times \boldsymbol{\sigma}_2. \quad (86)$$

The self-energy due to electron-impurity scattering can be calculated from Eq. (32). The nonvanishing components of $\Sigma_{\rm imp}$ can be shown explicitly:

$$\Sigma_{\rm imp}(\mathbf{p}, z_\nu) = \begin{pmatrix} \Sigma_{{\rm imp},ss}(\mathbf{p}, z_\nu) & 0 & 0 & \Xi_{{\rm imp},sd}(\mathbf{p}, z_\nu) \\ 0 & \Sigma_{{\rm imp},dd}(\mathbf{p}, z_\nu) & \Xi_{{\rm imp},ds}(\mathbf{p}, z_\nu) & 0 \\ 0 & -\Xi_{{\rm imp},ds}^*(-\mathbf{p}, -z_\nu) & -\Sigma_{{\rm imp},ss}(-\mathbf{p}, -z_\nu) & 0 \\ -\Xi_{{\rm imp},sd}^*(-\mathbf{p}, -z_\nu) & 0 & 0 & -\Sigma_{{\rm imp},dd}(-\mathbf{p}, -z_\nu) \end{pmatrix}, \quad (87)$$

where $\Sigma_{{\rm imp},ss}(\mathbf{p}, z_\nu)$ and $\Sigma_{{\rm imp},dd}(\mathbf{p}, z_\nu)$ are the same as Eqs. (34) and (35), and

$$\Xi_{{\rm imp},sd}(\mathbf{p}, z_\nu) = -n_i \int \frac{d^3k}{(2\pi)^3} [V_s(\mathbf{k})V_d(\mathbf{k})F_{sd}(\mathbf{p}-\mathbf{k}, z_\nu) + V_{sd}^2(\mathbf{k})F_{ds}(\mathbf{p}-\mathbf{k}, z_\nu)], \quad (88)$$

$$\Xi_{{\rm imp},ds}(\mathbf{p}, z_\nu) = -n_i \int \frac{d^3k}{(2\pi)^3} [V_s(\mathbf{k})V_d(\mathbf{k})F_{ds}(\mathbf{p}-\mathbf{k}, z_\nu) + V_{sd}^2(\mathbf{k})F_{sd}(\mathbf{p}-\mathbf{k}, z_\nu)]. \quad (89)$$

Now we write the nonvanishing components of the 4×4 matrix Green's function explicitly:

$$\mathcal{G}(\mathbf{p}, z_\nu) = \begin{pmatrix} G_{ss}(\mathbf{p}, z_\nu) & 0 & 0 & F_{sd}(\mathbf{p}, z_\nu) \\ 0 & G_{dd}(\mathbf{p}, z_\nu) & F_{ds}(\mathbf{p}, z_\nu) & 0 \\ 0 & -F_{ds}^*(-\mathbf{p}, -z_\nu) & -G_{ss}(-\mathbf{p}, -z_\nu) & 0 \\ -F_{sd}^*(-\mathbf{p}, -z_\nu) & 0 & 0 & -G_{dd}(-\mathbf{p}, -z_\nu) \end{pmatrix}. \quad (90)$$

The 4×4 matrix equation of motion, Eq. (15), can be split into two essentially identical 2×2 matrix equations in the $(\mathbf{p}, z_\nu)$ space. One of them is

$$\begin{pmatrix} z_\nu - \epsilon_{sp} - \Sigma_{{\rm imp},ss}(\mathbf{p}, z_\nu) & -\bar{\Delta}_{sd} - \Xi_{{\rm imp},sd}(\mathbf{p}, z_\nu) \\ -\bar{\Delta}_{sd}^* + \Xi_{{\rm imp},sd}^*(-\mathbf{p}, -z_\nu) & z_\nu + \epsilon_{dp} + \Sigma_{{\rm imp},dd}(-\mathbf{p}, -z_\nu) \end{pmatrix} \mathbf{G}_{sd}(\mathbf{p}, z_\nu) = \mathbf{1}. \quad (91)$$

Since another equation is essentially identical to this one, we can only obtain one order parameter, and therefore there can be only one critical temperature. The order-parameter equation is now

$$\bar{\Delta}_{sd} = -\frac{g_{sd}}{\beta} \sum_\mu \int' \frac{d^3k}{(2\pi)^3} F_{sd}(\mathbf{k}, z_\mu). \quad (92)$$

The prime on the integral sign bears the same meaning as those in Eqs. (42) and (43). $\bar{\Delta}_{sd}$ will be shown to be real.

We take the ansatz

$$\mathbf{G}_{sd}^{-1}(\mathbf{p}, z_\nu) = \begin{pmatrix} \tilde{z}_{s\nu} - \epsilon_{sp} & -\tilde{\Delta}_{sd,\nu} \\ -\tilde{\Delta}_{sd\nu} & \tilde{z}_{d\nu} + \epsilon_{dp} \end{pmatrix} \quad (93)$$

or

$$\mathbf{G}_{sd}(\mathbf{p}, z_\nu) = [(\tilde{z}_{s\nu} - \epsilon_{sp})(\tilde{z}_{d\nu} + \epsilon_{dp}) - \Delta_{sd\nu}^2]^{-1} \times \begin{pmatrix} \tilde{z}_{d\nu} + \epsilon_{dp} & \Delta_{sd\nu} \\ \tilde{\Delta}_{sd\nu} & \tilde{z}_{s\nu} - \epsilon_{sp} \end{pmatrix}. \quad (94)$$

[7] V. Ambegoakar and A. Griffin, Phys. Rev. **137**, A1151 (1965).

With the ansatz, Eq. (92) becomes

$$\bar{\Delta}_{sd}=-\frac{g_{sd}}{\beta}\sum_{\mu}\int'\frac{d^3k}{(2\pi)^3}\frac{\tilde{\Delta}_{sd\nu}}{(\tilde{z}_{s\nu}-\epsilon_{sp})(\tilde{z}_{d\nu}+\epsilon_{dp})-\tilde{\Delta}_{sd\nu}^2}. \quad (95)$$

To further our calculation, we pause to take a look at the condition under which the s-d interband phonon coupling is likely to be the strongest. The BCS theory requires that, for two electrons to form a Cooper pair via virtual phonon exchange, the two electrons should have opposite momenta and opposite spins. Since the electrons which can be efficiently couplied via virtual phonon exchange lie on the Fermi surface, it is not hard to visualize that, for strong s-d interband phonon coupling to occur, parts of the Fermi surfaces of the two bands should be common. If we limit our discussion to the case of two spherical Fermi surfaces, then the radii of the two Fermi spheres should be equal, $p_{Fs}=p_{Fd}=p_F$. In this case, the calculation is very much simplified. Now, we have

$$\epsilon_{sp}=(2m_s)^{-1}(p^2-p_F^2), \quad (96)$$

$$\epsilon_{dp}=(2m_d)^{-1}(p^2-p_F^2). \quad (97)$$

It is convenient to introduce a variable x:

$$x=[1/2(m_sm_d)^{1/2}](p^2-p_F^2). \quad (98)$$

It follows that

$$d^3p=N_{sd}(0)\,dx, \quad (99)$$

where

$$N_{sd}(0)=(N_s(0)N_d(0))^{1/2}. \quad (100)$$

With $\tilde{z}_{s\nu}\equiv i\tilde{\omega}_{s\nu}$ and $\tilde{z}_{d\nu}\equiv i\tilde{\omega}_{d\nu}$, Eq. (95) can be written as

$$\bar{\Delta}_{sd}=(g_{sd}/\beta)N_{sd}(0)\sum_{\nu=-\infty}^{\infty}\int_{-\omega_D}^{\omega_D}dx\frac{\tilde{\Delta}_{sd\nu}}{\tilde{\Delta}_{sd\nu}^2+[\tilde{\omega}_{s\nu}+i(m_d/m_s)^{1/2}x][\tilde{\omega}_{d\nu}-i(m_s/m_d)^{1/2}x]}. \quad (101)$$

The following integral is useful:

$$\int_{-\infty}^{\infty}\frac{dx}{\tilde{\Delta}_{sd\nu}^2+[\tilde{\omega}_{s\nu}+i(m_d/m_s)^{1/2}x][\tilde{\omega}_{d\nu}-i(m_s/m_d)^{1/2}x]}=\frac{\pi}{(\tilde{\Delta}_{sd\nu}^2+\tilde{\omega}_{sd\nu}^2)^{1/2}}, \quad (102)$$

where

$$\tilde{\omega}_{sd\nu}=\tfrac{1}{2}[(m_s/m_d)^{1/2}\tilde{\omega}_{s\nu}+(m_d/m_s)^{1/2}\tilde{\omega}_{d\nu}]. \quad (103)$$

Substituting the ansatz, Eq. (93), into Eqs. (34), (35), (88), and (89) and making use of Eq. (102), we obtain

$$\tilde{\omega}_{s\nu}=\omega_\nu+\left(\frac{\tilde{\omega}_{d\nu}}{2\tau_s'}+\frac{\tilde{\omega}_{s\nu}}{2\tau_{sd}'}\right)(\tilde{\Delta}_{sd\nu}^2+\tilde{\omega}_{sd\nu}^2)^{-1/2}, \quad (104)$$

$$\tilde{\omega}_{d\nu}=\omega_\nu+\left(\frac{\tilde{\omega}_{s\nu}}{2\tau_d'}+\frac{\tilde{\omega}_{d\nu}}{2\tau_{sd}'}\right)(\tilde{\Delta}_{sd\nu}^2+\tilde{\omega}_{sd\nu}^2)^{-1/2}, \quad (105)$$

$$\tilde{\Delta}_{sd\nu}=\bar{\Delta}_{sd}+(1/2\tau_{sd}''-1/2\tau_{sd}')\frac{\tilde{\Delta}_{sd\nu}}{(\tilde{\Delta}_{sd\nu}^2+\tilde{\omega}_{sd\nu}^2)^{1/2}}, \quad (106)$$

and thus

$$\tilde{\omega}_{sd\nu}=\tfrac{1}{2}[(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]\omega_\nu+\{(\tilde{\omega}_{sd\nu}/2\tau_{sd}')+\tfrac{1}{2}(\tilde{\omega}_{d\nu}/2\tau_s+\tilde{\omega}_{s\nu}/2\tau_d)\}(\tilde{\Delta}_{sd\nu}^2+\tilde{\omega}_{sd\nu}^2)^{-1/2}, \quad (107)$$

where

$$1/2\tau_s'=n_i\pi N_{sd}(0)\langle V_s^2(\mathbf{p})\rangle_\Omega, \quad (108)$$

$$1/2\tau_d'=n_i\pi N_{sd}(0)\langle V_d^2(\mathbf{p})\rangle_\Omega, \quad (109)$$

$$1/2\tau_{sd}'=n_i\pi N_{sd}(0)\langle V_{sd}^2(\mathbf{p})\rangle_\Omega, \quad (110)$$

$$1/2\tau_{sd}''=n_i\pi N_{sd}(0)\langle V_s(\mathbf{p})V_d(\mathbf{p})\rangle_\Omega. \quad (111)$$

Equation (64) should now be replaced by

$$\ln(2\gamma\omega_D/\pi k_BT)=k_BT\tfrac{1}{2}[(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]\times\sum_{\nu=-\infty}^{\infty}\int_{-\omega_D}^{\omega_D}\frac{dx}{[\omega_\nu+i(m_d/m_s)^{1/2}x][\omega_\nu-i(m_s/m_d)^{1/2}x]}. \quad (112)$$

The equation for the critical temperature of the pure superconductor in the strong interband phonon-coupling limit is

$$1/g_{sd}N_{sd}(0)=\ln(2\gamma\omega_D/\pi T_{csd}^{(0)}). \quad (113)$$

Combining Eq. (101) with Eqs. (112) and (113), we have an equation similar to Eq. (67). We let $\omega_D\to\infty$ and carry out the integral to obtain the following:

$$\ln\frac{T}{T_{csd}^{(0)}}=\frac{2\pi k_BT}{\bar{\Delta}_{sd}}\tfrac{1}{2}[(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]\times\sum_{\nu=0}^{\infty}\left(\frac{\tilde{\Delta}_{sd\nu}}{(\tilde{\omega}_{sd\nu}^2+\tilde{\Delta}_{sd\nu}^2)^{1/2}}-\frac{\bar{\Delta}_{sd}}{\omega_\nu'}\right), \quad (114)$$

where

$$\omega_\nu'=[\pi(2\nu+1)/2\beta][(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]. \quad (115)$$

We define

$$u_{sd\nu}=\tilde{\omega}_{sd\nu}/\tilde{\Delta}_{sd\nu}, \tag{116}$$

$$v_{sd\nu}=\omega_\nu'/\bar{\Delta}_{sd}$$
$$=[\pi(2\nu+1)/2\beta\bar{\Delta}_{sd}][(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]. \tag{117}$$

If the impurity density is low enough, we obtain an approximate relation between $u_{sd\nu}$ and $v_{sd\nu}$:

$$v_{sd\nu}\cong u_{sd\nu}\{1-\gamma_{sd}/(1+u_{sd\nu}^2)^{1/2}\}, \tag{118}$$

where

$$\gamma_{sd}=(1/2\bar{\Delta}_{sd})[\tfrac{1}{2}(1/\tau_s+1/\tau_d)-1/\tau_{sd}'']. \tag{119}$$

Equation (114) can be written as

$$\ln\frac{T}{T_{csd}^{(0)}}=2v_{sd0}\sum_{\nu=0}^{\infty}[(1+u_{sd\nu}^2)^{-1/2}-v_{sd\nu}^{-1}] \tag{120}$$

or

$$\ln\frac{T}{T_{csd}^{(0)}}=2v_{sd0}\sum_{\nu=0}^{\infty}[(1+u_{sd\nu}^2)^{-1/2}-u_{sd\nu}^{-1}]$$
$$-2v_{sd0}\sum_{\nu=0}^{\infty}[v_{sd\nu}^{-1}-u_{sd\nu}^{-1}].$$

Notice that when $T\lesssim T_{csd}$, $\bar{\Delta}_{sd}\sim 0$, then $u_{sd\nu}\gg 1$. The first summation is negligible as compared with the second one. Therefore, for $T\lesssim T_{csd}$,

$$\ln\frac{T}{T_{csd}^{(0)}}\cong -2v_{sd0}\sum_{\nu=0}^{\infty}[v_{sd\nu}^{-1}-u_{sd\nu}^{-1}]. \tag{121}$$

From Eq. (118), for $T\lesssim T_{csd}$, we get

$$u_{sd\nu}\cong v_{sd\nu}+\gamma_{sd}. \tag{122}$$

Then, we have

$$\ln\frac{T_{csd}^{(0)}}{T}\cong 2\sum_{\nu=0}^{\infty}[(2\nu+1)^{-1}-(2\nu+1+\rho_{sd})^{-1}]$$
$$=\psi[\tfrac{1}{2}(1+\rho_{sd})]-\psi(\tfrac{1}{2}), \tag{123}$$

where $\psi(x)$ is the digamma function, and

$$\rho_{sd}=(\beta/\pi)[\tfrac{1}{2}(1/\tau_s+1/\tau_d)-1/\tau_{sd}'']$$
$$\times[(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]^{-1}. \tag{124}$$

Particularly, at T_{csd}, we have

$$\ln(T_{csd}^{(0)}/T_{csd})=\psi[\tfrac{1}{2}(1+\rho_{sd,c})]-\psi(\tfrac{1}{2}), \tag{125}$$

with

$$\rho_{sd,c}=(\pi k_B T_{csd})^{-1}[\tfrac{1}{2}(1/\tau_s+1/\tau_d)-1/\tau_{sd}'']$$
$$\times[(m_s/m_d)^{1/2}+(m_d/m_s)^{1/2}]^{-1}. \tag{126}$$

It is interesting to notice that T_{csd} can be either larger or smaller than $T_{csd}^{(0)}$, depending on whether $\rho_{sd,c}<0$ or >0, and thus on whether $1/\tau_{sd}''>\tfrac{1}{2}(1/\tau_s+1/\tau_d)$ or $<\tfrac{1}{2}(1/\tau_s+1/\tau_d)$. Physically, what we have shown is that stronger *s-d* interband impurity scattering would enhance the *s-d* pair formation and thus increase the critical temperature; on the other hand, stronger intraband impurity scattering would handicap the *s-d* pair formation and thus decrease the critical temperature.

V. CONCLUSION

So far we have investigated the two limiting cases of the general problem about the influence of the nonmagnetic impurities on the transition temperature. The general case, in which both the intraband and the interband phonon coupling are nonvanishing, is yet to be investigated. Even the investigation of the two limiting cases is not exhausted. One can look further into other thermodynamic properties and the electromagnetic properties. The present paper has set a background for their investigation. It is hoped that some real superconductors with overlapping bands might happen to satisfy the limiting cases.

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