# Intrinsic Resistive Transition in Thin Superconducting Wires Driven from Current Sources

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The Langer-Ambegaokar theory of dissipative fluctuations in narrow superconducting channels for $T$ near $T_c$ is extended to systems driven from current sources, with results identical to those previously obtained by Langer and Ambegaokar for voltage sources.

## I. INTRODUCTION

A STATISTICAL criterion describing the onset of dissipation in thin superconducting wires near $T_c$ has been proposed by Langer and Ambegaokar.[1] Their analysis is based upon a Helmholtz free-energy function of the Landau-Ginsburg form and is directly applicable to samples driven from a voltage source. Recent experiments by Webb and Warburton[2] on tin whisker crystals[3] confirm the principal theoretical predictions. However, because the samples in these experiments were driven from a current rather than a voltage source, it has been privately suggested that the Langer-Ambegaokar (LA) theory should not apply. In this paper we describe the modifications of the LA theory appropriate to a current source. We find that their final formulas remain unchanged and are therefore applicable to general sources.

The essential element in our treatment is the replacement of the Helmholtz free-energy function $F(\Delta\phi)$ by the Gibbs function

$$G(I)=F(\Delta\phi)-\hbar I\Delta\phi/2e, \tag{1}$$

where $I$ is the current through the superconductor and $\Delta\phi$ is the total change along the sample of the gauge-invariant phase of the Landau-Ginsburg order parameter. In this function, $I$ is the independent variable and $\Delta\phi$ the dependent variable. In the LA treatment based upon $F(\Delta\phi)$, $\Delta\phi$ is the independent variable and $I$ the dependent variable.

The phase difference $\Delta\phi$ changes reversibly with sample voltage $V$ according to the well-known Josephson equation

$$d(\Delta\phi)/dt=2eV/\hbar, \tag{2}$$

and it changes irreversibly through phase-slipping fluctuations. The work done on a system in a reversible isothermal process equals the change in its Helmholtz free energy.[4] It follows directly from Eq. (2) that the infinitesimal change in free energy appropriate to a reversible infinitesimal and isothermal change in phase difference $\Delta\phi$ is

$$dF=IVdt=(\hbar I/2e)\,d(\Delta\phi), \tag{3}$$

which verifies that $F$ is the thermodynamic potential appropriate to a specified phase difference $\Delta\phi$.[4] Differentiating (1) and using (3) to eliminate $dF$, we find

$$dG=-(\hbar\Delta\phi/2e)\,dI, \tag{4}$$

which verifies that $G$ is the thermodynamic potential appropriate to fixed current $I$.

The fluctuations responsible for dissipation are limited in the LA theory by free-energy barriers. For the voltage-controlled system ($V=0$, $\Delta\phi$ fixed) these describe the minimum changes in the Helmholtz free energy consistent with continuous deformations of the Landau-Ginsburg order parameter from one to another of the different nominally equilibrium configurations. For the current-controlled system they describe corresponding minimum changes in the Gibbs function $G(I)$. In what follows we compute and compare the free-energy barriers for the two cases. We find that the Gibbs-function barriers are identical to the *approximate* Helmholtz-function barriers used by LA to derive their final results. The differences between the exact and approximate Helmholtz-function barriers are small, generally negligible, corrections of relative order $\xi(T)/L\ll1$, where $\xi(T)$ is the coherence length and $L$ the effective sample length.

That the Helmholtz-function and Gibbs-function barriers are equal in the large-$L$ limit is a special case of a more general thermodynamic equality for large statistical systems.[5] If there exists a set of parameters $\lambda_i$ which together with $\Delta\phi$ or $I$ define the state of the system, variations in the $\lambda_i$ produce equal changes in the Gibbs function at constant current and in the Helmholtz function at constant phase difference when the variations $\delta\lambda_i$ are sufficiently small that

$$\delta I/\delta(\Delta\phi)=dI/d(\Delta\phi), \tag{5}$$

where $\delta I$ is the change in current at constant $\Delta\phi$ and $\delta(\Delta\phi)$ is the change in phase difference at constant $I$. The Gibbs-function and Helmholtz-function barriers are different only to the extent that Eq. (5) is violated. If the $\delta\lambda_i$ produce localized perturbations within a large system, Eq. (5) will always be valid in the limit

[1] J. S. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967).
[2] W. W. Webb and R. J. Warburton, Phys. Rev. Letters **20**, 461 (1968).
[3] J. Franks, Acta Met. **6**, 103 (1958).
[4] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958), pp. 45–48, 68–71.
[5] See Ref. 4, especially Eqs. (15.12) and (24.16).

that the system size is much larger than the extent of the perturbation.

## II. FREE-ENERGY BARRIERS

The longitudinal fluctuations appropriate to a long thin superconducting wire and the nature of their associated free-energy barriers have been described by Langer and Ambegaokar.[1] We shall not repeat their description but shall turn directly to the determination of the barriers, using only the necessary details of their paper.

If the transverse sample dimensions are small compared to a coherence length, it is accurate and convenient to write the Landau-Ginsburg Helmholtz free energy in the form[6]

$$F(\Delta\phi)=(\sigma H_c^2(T)\xi(T)/4\pi)\int_{-l/2}^{l/2}dx\left[-f^2(x)+\tfrac{1}{2}f^4(x)+\left(\frac{df(x)}{dx}\right)^2+\left(f(x)\,\frac{d\phi(x)}{dx}\right)^2\right],\qquad(6)$$

where $l\equiv L/\xi(T)$ is the sample length in dimensionless units (not to be confused with the electron mean free path), where $\sigma\ll\xi^2(T)$ is its constant cross-sectional area, and where $f(x)\,\exp[i\phi(x)]$ is proportional to the gauge-invariant Landau-Ginsburg order parameter. The phase difference $\Delta\phi$ is

$$\Delta\phi=\phi(\tfrac{1}{2}l)-\phi(-\tfrac{1}{2}l),\qquad\text{modulo }2\pi.\qquad(7)$$

In the limit $\Delta T\equiv T_c-T\to0^+$, the prefactor

$$\sigma H_c^2(T)\xi(T)/4\pi$$

in (6) vanishes as $(\Delta T)^{3/2}$.

For the voltage-controlled and current-controlled cases we must determine the extrema with respect to admissible variations of $f(x)$ and $\phi(x)$ of the Helmholtz function $F(\Delta\phi)$ and the Gibbs function $G(I)$, respectively. In both cases at these extrema, the functions $f(x)$ and $\phi(x)$ satisfy the Landau-Ginsburg equations, which in our notation are

$$d^2f(x)/dx^2=-f(x)+f^3(x)+[J^2/f^3(x)],\qquad(8a)$$

$$J=f^2(x)[d\phi(x)/dx],\qquad(8b)$$

where $J$ is independent of $x$ and related to the supercurrent $I$ in the sample by

$$I=J\sigma H_c^2(T)\xi(T)/\Phi_0.\qquad(9)$$

In the limit $\Delta T\to0$, the ratio $I/J$ vanishes as $(\Delta T)^{3/2}$, which is characteristic of the critical current ($J=J_c=2/3\sqrt{3}$) in the absence of fluctuations. In Eq. (9), $\Phi_0$ is the flux quantum $hc/2e$.

It is convenient to express $J$, $0\le J\le J_c$, in terms of a parameter $\kappa$ such that

$$J(\kappa)=\kappa(1-\kappa^2),\qquad\kappa^2\le\tfrac{1}{3}.\qquad(10)$$

That solution of Eqs. (8) which corresponds to a local free-energy minimum consistent with this current is

$$f^2(x)=1-\kappa^2,$$
$$\phi(x)=\kappa x+\phi_0,\qquad(11)$$

for which $\Delta\phi=\kappa l$. The free energies are, for the Helmholtz function,

$$F_0(\kappa)=-(H_c^2\xi/8\pi)l\sigma(1-\kappa^2)^2,\qquad(12a)$$

and, for the Gibbs function,

$$G_0(\kappa)=-(H_c^2\xi/8\pi)l\sigma(1-\kappa^2)(1+3\kappa^2).\qquad(12b)$$

In the absence of fluctuations these results describe thermal equilibrium.

To compute the fluctuation-produced resistance, we require for the voltage-source case the Helmholtz free energy at the barrier saddle points appropriate to $\kappa\to\kappa\pm2\pi/l$ transitions. The saddle-point solutions of Eqs. (8) have been described in detail by Langer and Ambegaokar.[1] If for $\kappa>0$ we define new parameters $\kappa_\pm$ such that

$$\kappa=\kappa_+-(2\pi/l)+(2/l)\,\arctan[(1-3\kappa_+^2)/2\kappa_+^2]^{1/2}\qquad(13a)$$

$$=\kappa_-+(2/l)\,\arctan[(1-3\kappa_-^2)/2\kappa_-^2]^{1/2},\qquad(13b)$$

the saddle-point free energies are

$$F_\pm(\kappa)=-(H_c^2\xi/8\pi)l\sigma[(1-\kappa_\pm^2)^2-(8\sqrt{2}/3l)(1-3\kappa_\pm^2)^{1/2}].\qquad(14)$$

Subtracting (12a) from (14), we obtain the barriers

$$\delta F_\pm(\kappa)\equiv F_\pm(\kappa)-F_0(\kappa)\qquad(15a)$$

$$\approx(H_c^2\xi\sigma/8\pi)\tfrac{8}{3}\sqrt{2}(1-3\kappa^2)^{1/2}+[\partial F_0(\kappa)/\partial\kappa]\delta\kappa_\pm,\qquad(15b)$$

where

$$\delta\kappa_\pm\equiv\kappa_\pm-\kappa\qquad(16a)$$

$$\approx\binom{2\pi/l}{0}-\frac{2}{l}\arctan\left(\frac{1-3\kappa^2}{2\kappa^2}\right)^{1/2}.\qquad(16b)$$

The LA result follows from Eqs. (15b) and (16b). It is approximate in that it differs from the exact expression by small, generally negligible, terms of order $l^{-1}$.

For the constant-current case the parameter $\kappa$ defined

[6] Our notation differs slightly from Ref. 1. This disadvantage is compensated by final expressions which are somewhat more transparent.

by Eqs. (9) and (10) remains unchanged, but fluctuations produce $\Delta\phi \rightarrow \Delta\phi \pm 2\pi$ transitions. The saddle-point solutions of (8) appropriate to the Gibbs-energy barrier are similar to the Helmholtz-function solutions described by Langer and Ambegaokar but are different in that the same functions $f(x)$ and $\phi(x)$ are used at both saddle points. The phase difference $\Delta\phi$ differs at those two points by $2\pi$, which is consistent with the modulo-$2\pi$ uncertainty in Eq. (7).

The saddle-point Gibbs energies are

$$G_{\pm}(\kappa) = -(H_c^2\xi/8\pi) l\sigma\{(1-\kappa^2)(1+3\kappa^2) - (8\sqrt{2}/3l)(1-3\kappa^2)^{1/2} + 4\kappa(1-\kappa^2)\delta(\Delta\phi)_{\pm}/l\}, \quad (17)$$

where for $\kappa > 0$,

$$\delta(\Delta\phi)_+ = \delta(\Delta\phi)_- + 2\pi = 2\arctan[(1-3\kappa^2)/2\kappa^2]^{1/2}. \quad (18)$$

These $\delta(\Delta\phi)_{\pm}$ are the changes which must be induced in $\Delta\phi$ by an external voltage [via the Josephson equation (2)] in order to maintain the current constant in going from the initial free-energy minimum to the saddle point. The Gibbs-energy barriers are

$$\delta G_+(\kappa) = (H_c^2\xi\sigma/8\pi)\{\tfrac{8}{3}\sqrt{2}(1-3\kappa^2)^{1/2} - 8\kappa(1-\kappa^2)\arctan[(1-3\kappa^2)/2\kappa^2]^{1/2}\},$$

$$\delta G_-(\kappa) = (H_c^2\xi\sigma/8\pi)(\tfrac{8}{3}\sqrt{2}(1-3\kappa^2)^{1/2} + 8\kappa(1-\kappa^2) \times\{\pi - \arctan[(1-3\kappa^2)/2\kappa^2]^{1/2}\}), \quad (19)$$

which are identical to the Helmholtz free-energy barriers computed by Langer and Ambegaokar with the approximations (15b) and (16b).

## III. DISCUSSION

These results and those of Langer and Ambegaokar[1] have all been derived under the assumption that the sample length $L$ is large compared to the coherence length $\xi(t)$—that is, $l \equiv L/\xi(T) \gg 1$. If $l \lesssim 1$, the mathematical expressions are considerably more complex [because the "energy" $E$ of LA, Fig. 2 and Eq. (3.8), is unequal to $U_{\text{eff}}(f_0)$], but our general conclusion remains. The constant-voltage and constant-current formulations are equivalent whenever

$$J(\kappa_{\pm}) \approx J(\kappa) + (\partial J/\partial\kappa)\delta\kappa_{\pm}, \quad (20)$$

which is equivalent to (5) and sufficient to validate the approximations (15b) and (16b).

One might object that it is incorrect to use the LA analysis when $l \lesssim 1$ because neglected boundary corrections to the free energy could dominate the bulk contribution. This objection is less damaging than it appears because it is possible, in some cases with relatively little effort, to extend the "system" to include its boundaries.

An instructive one-dimensional example is that of a long thin sample of total length $l_T \equiv L_T/\xi(T) \gg 1$ whose transverse dimensions are variable but everywhere small compared to the coherence length $\xi(T)$. If the dimensional changes are slow over distances $\xi_0 \ll \xi(T)$, the free energy (6) obtains as before, except that the integrand must be weighted by the factor $\sigma(x)/\sigma$, where $\sigma(x)$ is the cross-sectional area at $x$ and $\sigma$ is a constant average or reference cross section. The Landau-Ginsburg equations for this system are [cf. Eqs. (8)]

$$\frac{d}{dx}\left[\frac{\sigma(x)}{\sigma}\frac{df}{dx}(x)\right] = \frac{\sigma(x)}{\sigma}\left[-f(x) + f^3(x) + \frac{\sigma^2 J^2}{\sigma^2(x) f^3(x)}\right], \quad (21a)$$

$$J = [\sigma(x)/\sigma] f^3(x)(d\phi/dx). \quad (21b)$$

A simple representation of a short length-$L$ "sample" and that to which our above remarks refer has

$$\sigma(x) = \sigma \quad \text{for} \quad |x| \le \tfrac{1}{2}l$$
$$= \sigma_e \quad \text{for} \quad \tfrac{1}{2}l < |x| \le \tfrac{1}{2}l_T, \quad (22)$$

where $\sigma \ll \sigma_e \ll \xi^2(T)$.

A second important "one-dimensional" example, whose properties are formally very similar to those of simply connected systems with non-negligible self-inductance, is that of a closed loop of thin superconducting wire in an external magnetic field. In this example, the sample is of finite length but has no ends. Its analysis is slightly more complicated than those considered above in that the external magnetic field rather than the phase difference $\Delta\phi$ or the current $I$ is the independent thermodynamic variable. (The current is the complementary dependent variable.) The total free energy includes that stored in the wire [Eq. (6)] plus that stored in the field. Although the phase of the complex order parameter is quantized by the structure periodicity, the *gauge-invariant* phase $\phi(x)$ and phase difference $\Delta\phi$ are not, so that the mathematical details are not very different from Langer and Ambegaokar.[1] The phase difference $\Delta\phi$ is related to the total flux $\Phi$ linking the loop by

$$\Delta\phi = 2\pi\Phi/\Phi_0, \qquad \text{modulo } 2\pi, \quad (23)$$

where $\Phi_0 = hc/2e$ is the flux quantum. The condition of fixed external field or equivalently of fixed external flux

$$\Phi_e = \Phi + L_s I, \quad (24)$$

where $L_s$ is the self-inductance of the loop, reduces for $2\pi L_s dI/d(\Delta\phi) \gg \Phi_0$ to that of constant current and for $2\pi L_s dI/d(\Delta\phi) \ll \Phi_0$ to that of constant phase difference. For intermediate values the behavior is mixed.