these changes are reversible, indicating clearly that the electronic structure of iron in this lattice is determined, among other things, by the stoichiometry of the lattice.

It may be of interest to note that Wertheim and Remeika observed both Fe^{2+} and Fe^{3+} spectra in Al_2O_3:Co^{57} sources, whereas Al_2O_3:Fe^{57} absorbers give normally only Fe^{3+} spectra. The quadrupole interaction reported by Wertheim and Remeika for both Fe^{2+} and Fe^{3+} states in the Al_2O_3:Co^{57} system agrees very well with that reported here for the vacuum-annealed Al_2O_3:Fe^{57} system. The observation of Fe^{2+} spectra in addition to the normally expected Fe^{3+} spectra in the Al_2O_3:Co^{57} system was explained as, among other things, arising out of the Auger cascades that follow the electron-capture decay of Co^{57}. Similar spectra attributable to nonequilibrium charge states were reported by several authors.[6–9] However, the ingenious delayed-coincidence experiments of Triftshauser and Craig[10] set this controversy to rest. They showed conclusively that the observation of Fe^{3+} spectra in both NiO and CoO was not time-dependent within the time range accessible to the experiment and may arise out of the deviation from stoichiometry. Indeed, recently only Fe^{2+} spectra have been observed in nearly stoichiometric NiO [25] and CoO [26,27] sources. The observation of only the Fe^{3+} state in the air-fired Al_2O_3:Fe^{57} system and of the Fe^{2+} and Fe^{3+} states in the vacuum-annealed Al_2O_3:Fe^{57} system suggests that the Fe^{2+} spectrum in the Al_2O_3:Co^{57} system is at least in part due to the deviation from stoichiometry, besides being due to the fact that Co enters the Al_2O_3 lattice interstially and substitutionally in the divalent as well as the trivalent state.

ACKNOWLEDGMENTS

We wish to thank our colleagues for helpful discussions and, in particular, R. G. Sharma, Y. S. Reddy, H. C. Bhasin, and R. K. Jain for experimental assistance during the course of this investigation. Many stimulating discussions with Dr. G. K. Shenoy are gratefully acknowledged. One of us (S.K.D.) wishes to thank the Council of Scientific and Industrial Research for the award of a Senior Research Fellowship.

[25] J. D. Siegwarth, Phys. Rev. **155**, 285 (1967).
[26] C. J. Coston, R. Ingalls, and H. G. Drickamer, Phys. Rev. **145**, 409 (1966).
[27] J. G. Mullen and Hang Nam Ok, Phys. Rev. **168**, 550 (1968); **168**, 563 (1968).

PHYSICAL REVIEW VOLUME 172, NUMBER 2 10 AUGUST 1968

Quantum Noise. XI. Multitime Correspondence between Quantum and Classical Stochastic Processes

MELVIN LAX
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received 15 January 1968)

A correspondence $a_j \leftrightarrow \alpha_j$ between operators $\mathbf{a}=[a_1, a_2, \cdots a_f]$ and c numbers $\boldsymbol{\alpha}=[\alpha_1, \alpha_2, \cdots \alpha_f]$ together with an arbitrary ordering rule $\mathcal{C}$ (e.g., in sequence from 1 to f) permit an association $M(\mathbf{a})=\mathcal{C}M^{(c)}(\boldsymbol{\alpha})$ between a general operator $M(\mathbf{a})$ and an associated c number function $M^{(c)}(\boldsymbol{\alpha})$. A quasiprobability $P(\boldsymbol{\alpha}, t)$ is then defined so that a general ensemble average can be written as an ordinary integration: $\langle M(\mathbf{a}(t))\rangle = \int M^{(c)}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} P(\boldsymbol{\alpha}, t)$. The equation for $\partial P(\boldsymbol{\alpha}, t)/\partial t$ suggests that the $\boldsymbol{\alpha}$ obeys a classical Markoff process. If this classical Markoff process is taken literally, multitime classical averages can be computed. Do these correspond to appropriate quantum averages? For the case of field operators such that $[b, b^\dagger]=1$, important in discussing laser statistics, we show that with $a_1=b^\dagger$ and $a_f=b$, the classical multitime average is equivalent to the average of the corresponding quantum operators written in time-ordered, normal-ordered sequence. For the atomic operators in a laser problem, we obtain the desired correspondence, but find that the more complicated commutation rules necessarily lead to derivative correction terms when multitime averages are taken. Our derivation of multitime averages is based on the quantum regression theorem. We show that this theorem is equivalent to assuming the quantum system to be Markoffian, by showing that it leads to an appropriate factorization of a multitime density matrix and to a Chapman-Kolmogoroff-like condition on the conditional density matrix.

1. INTRODUCTION

IN the two preceding papers in this series,[1] QIX and QX, we set up a correspondence between the creation and destruction operators $b^\dagger$ and b of a normal mode of the electromagnetic field and a pair of c numbers β^* and β. This correspondence was based on the

[1] QIX refers to the ninth paper in the series on quantum noise by M. Lax. The series is as follows: (QI) Phys. Rev. **109**, 1921 (1958); (QII) *ibid.* **129**, 2342 (1963); (QIII) J. Phys. Chem. Solids **25**, 487 (1964); (QIV) Phys. Rev. **145**, 110 (1966); (QV) M. Lax, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Co., New York, 1966), p. 735; (QVI) *Moment Treatment of Maser Noise*, with D. R. Fredkin (unpublished); (QVII) J. Quantum Electron. **QE-3**, 37 (1967); (QVIII) H. Cheng and M. Lax, in *Quantum Theory of the Solid State*, edited by P.-O. Löwdin (Academic Press Inc., New York, 1966); (QIX) J. Quantum Electron. **QE-3**, 47 (1967) (with W. H. Louisell); (QX) Phys. Rev. **157**, 213 (1967); (QXI) the present paper; (QXII) *Density Operator Treatment of Field and Population Fluctuations*, with H. W. Louisell (to be published); (QXIII) *Six Classical Variable Description of Quantum Laser Fluctuations*, with H. Yuen, Phys. Rev. **172**, 362 (1968); Portions of QXI and QXII are contained in M. Lax's *1966 Tokyo Summer Lectures in Theoretical Physics, Part I*, edited by R. Kubo and H. Kamimura (W. A. Benjamin, Inc., New York, 1967) and M. Lax, *1966 Brandeis Summer Institute Lectures* (Gordon and Breach Science Publishers, Inc., New York, to be published).

use of normally ordered operators or antinormally ordered operators and it led to a correspondence between the density matrix $\rho(t)$ and a classical distribution function $P(\beta, \beta^*, t)$. We also established that with one choice of our ordering rule this correspondence was identical to that obtained by Glauber,[2] Sudarshan,[3] and Klauder[4] by the use of the coherent representation.

In order to deal realistically with lasers of a general kind, it is desirable to extend the discussion beyond the field variables to include atomic population variables such as $a_1^\dagger a_1$ and $a_2^\dagger a_2$ and in addition off-diagonal elements related to polarizations such as $a_1^\dagger a_2$ and $a_2^\dagger a_1$ for the two active levels 1 and 2 of a laser. Gordon[5] has recently found a way to generalize the coherent representation procedure to include these new variables. The coherent representation makes explicit use of the harmonic-oscillator character of the field variables, and Gordon's generalization,[5] in addition, makes explicit use of the nature of the atomic variables. Our own procedure does not make explicit use of the nature of the variables but merely introduces an ordering rule in order to obtain an appropriate quantum classical correspondence. Our procedure is then somewhat more general, because it can be applied to operators that may not have the special characteristics of the field and atomic variables mentioned above.

Our work in papers QX and QXII represented a trivial kind of generalization. In addition to the field variables we introduced the upper state population in QX and both upper- and lower-state populations in QXII. This generalization was trivial in that the population variables commuted with each other and with the field variables. Our calculation with Yuen,[6] in QXIII, makes use of all six variables however, and thus must squarely face the problem of many noncommuting variables. The calculation discussed in QXIII is based on our understanding of associated classical distribution functions and their meanings established in the present paper.

The basic idea of our generalized correspondence procedure is quite simple. The only distinction between operators and c numbers is that the former do not commute. If, however, an arbitrary function of our set of operators is always written in such a way that these operators appear in a chosen or selected order, then this leads to a unique form of the quantum-mechanical operator, and a classical function can then be associated with this particular form. This is the natural generalization of the correspondence between an operator function of b and $b^\dagger$ in normally ordered form being associated with a corresponding c number function as discussed in QIX. This correspondence between an operator function and a c number function valid for a set of noncommuting variables at one time is spelled out in Sec. 2. We find there that it is again possible to introduce a c number distribution function such that the mean of a general quantum-mechanical operator is obtainable by an ordinary integration of the c-number function corresponding to the operator against the distribution function.

In our specific calculations pertaining to lasers, we find the equation of motion for this c-number distribution. This equation of motion takes the form of a generalized Fokker-Planck equation. Thus it is natural to assume that the classical random process which would give rise to this Fokker-Planck equation for the purpose of calculating one-time averages can also be used to calculate multitime averages. To what extent is it legitimate to make this literal interpretation of the associated classical random process? That is one of the main questions to which we address ourselves in this paper.

One of our principal conclusions relates to the case of the field variables b, $b^\dagger$ (either for the single-mode or multimode case). In this case, we show that the assumption that the quantum-mechanical system is a Markoffian one establishes that the associated classical random process is Markoffian. In this case, as is well known for classical Markoffian systems, all multitime averages can be computed in terms of the two-time conditional probability, and the latter is the Green's-function solution of the one-time equation. Thus the single-time correspondence serves to determine all of the stochastic properties by the process of taking the single-time equation as the literal description of a classical Markoff process. These conclusions justify the work of Mandel and Wolf[7] and of Glauber[8] on photodetection fluctuations. These authors avoid the question of the proper determination of multitime averages by simply assuming that the field variables obey free-field equations. We have referred to our quantum classical correspondence as a dynamical one in order to imply that this correspondence is valid when our system variables interact with reservoirs, and not merely when they are free-field variables.

For the case in which atomic variables are present in addition to field variables, we establish that multitime averages over the *field variables* can still be computed by using the one-time equation to define a Markoff process. We also show that certain atomic multitime averages can also be computed under the same assumption. However, not all multitime averages can be computed by taking the classical process literally as a Markoffian process. Indeed, we find that to calculate certain atomic correlation functions it is desirable to use a classical distribution function based on a certain

[2] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[3] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[4] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).
[5] J. P. Gordon, Phys. Rev. **161**, 367 (1967).
[6] M. Lax and H. Yuen, following paper Phys. Rev. **172**, 362 (1968). This paper will be QXIII in the series of Ref. 1.
[7] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).
[8] R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. Dewitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, New York, 1965), p. 65ff. For a detailed review of quantum optics including photocounting problems, see also J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, to be published).

order of the atomic variables, whereas to calculate other averages it would be more convenient to use a different selected order. In summary then, one cannot take the associated classical process literally as a Markoffian process, but must investigate specifically how multitime averages are to be taken in each case.

The properties of our associated classical random process are derived under the assumption that the quantum system is describable in a Markoffian manner. This raises the question as to what is the most appropriate way of defining a Markoffian system in quantum mechanics. In papers QIV and QX, we found it convenient to define a Markoff process by the requirement that a Langevin force at time t be uncorrelated to any information at earlier times. From this assumption, we were able in QIV and QX to give a short derivation of the quantum regression theorem, which we had formulated earlier in paper QII.

For readers who do not like a Langevin description, it might be more appropriate to use the quantum regression theorem as a definition of a quantum Markoffian process. The appropriateness of this definition is verified in Sec. 5 for the field-variable case, because it leads to an associated classical description that is Markoffian. For a completely general case, we introduce multitime density matrices to permit the calculation of multitime averages. We then show in Sec. 7 that the quantum regression theorem forces such multitime density operators to factorize into two-time averages, in analogy to the classical Markoffian case. Moreover, we find that the conditional two-time average obeys a consistency condition which is the precise analog of the Chapman-Kolmogoroff equation. Indeed, this latter equation has recently been chosen by Bausch and Stahl[9] as their definition of a quantum Markoffian process.

2. ONE-TIME MULTIVARIABLE CORRESPONDENCE

Let us consider a set of system operators $\mathbf{a}\equiv[a_1, a_2, \cdots a_f]$ and an associated set of c-number variables $\boldsymbol{\alpha}\equiv[\alpha_1, \alpha_2, \cdots \alpha_f]$. A general operator function $M(\mathbf{a})$ can be set into correspondence with an associated classical function by means of the relation

$$M(\mathbf{a}) = \mathcal{C}M^{(c)}(\boldsymbol{\alpha}) = \mathcal{C}\bar{M}(\boldsymbol{\alpha}). \tag{2.1}$$

The operator $\mathcal{C}$ implies that each c number α_j should be replaced by the corresponding operator a_j and the operators should be placed in a chosen order. As a convention, we shall assume the order is from 1 to f going from left to right. We use the superscript c when we wish to emphasize the chosen-order nature of this relationship. If we merely wish to emphasize that the associated function is classical, we shall use a bar as a superscript. Using the abbreviation $d\boldsymbol{\alpha}\equiv d\alpha_1 d\alpha_2 \cdots d\alpha_f$, Eq. (2.1) can be rewritten in the equivalent form

$$M(\mathbf{a}) = \int d\boldsymbol{\alpha}\, M^{(c)}(\boldsymbol{\alpha})\, \delta(\alpha_1 - a_1)\, \delta(\alpha_2 - a_2) \cdots \delta(\alpha_f - a_f), \tag{2.2}$$

since the order of the δ functions specifies the appearance of the order of the operators in $M(\mathbf{a})$. The ensemble average of our general operator M at the time t can be defined by

$$\langle M(\mathbf{a}(t))\rangle = \mathrm{Tr}[M(\mathbf{a})\rho(t)], \tag{2.3}$$

the trace of M against the density operator at the time t. Taking the ensemble average of Eq. (2.2) we then obtain

$$\langle M(\mathbf{a}(t))\rangle = \int d\boldsymbol{\alpha}\, M^{(c)}(\boldsymbol{\alpha})\, P(\boldsymbol{\alpha}, t), \tag{2.4}$$

where the associated distribution function $P(\boldsymbol{\alpha}, t)$ is defined by

$$P(\boldsymbol{\alpha}, t) \equiv \langle \delta(\alpha_1 - a_1(t))\, \delta(\alpha_2 - a_2(t)) \cdots \delta(\alpha_f - a_f(t))\rangle. \tag{2.5}$$

Thus we have established that the mean of a general operator can be obtained by a c-number integration over the associated classical distribution function. We understand that the time dependence of the operator $a_j(t)$ in (2.5) includes not only interactions with the system but interactions with reservoirs. An alternative definition that avoids the use of δ functions by using their integral representation is

$$P(\boldsymbol{\alpha}, t) = (2\pi)^{-f} \int d\boldsymbol{\xi}\, e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\alpha}} \times \langle e^{i\xi_1 a_1(t)} e^{i\xi_2 a_2(t)} \cdots e^{i\xi_f a_f(t)}\rangle, \tag{2.6}$$

where

$$\boldsymbol{\xi}\cdot\boldsymbol{\alpha} \equiv \sum \xi_j \alpha_j.$$

The use of the exponential operator in (2.6) is a natural one for us, in that we obtained the equation of motion of our distribution function in papers QIX and QXII by first obtaining the motion of a general operator in exponential form.

Let us emphasize that the definition (2.1) above can be more general than the definition (2.2), in that the rule for placing the operators in some selected order can be a more complicated one than merely writing them in the order 1 to f, for example, we can introduce an ordering procedure which can be associated with the name of Wigner[10]:

$$M(\mathbf{a}) = WM^{(w)}(\boldsymbol{\alpha}) = \int d\boldsymbol{\alpha}\, M^{(w)}(\boldsymbol{\alpha})\, (2\pi)^{-f} \int d\boldsymbol{\xi}\, e^{-i\boldsymbol{\xi}\cdot\mathbf{a}} e^{i\boldsymbol{\xi}\cdot\boldsymbol{\alpha}}. \tag{2.7}$$

The Wigner-Moyal[11] association between operators

[9] R. Bausch and A. Stahl, Z. Physik **204**, 32 (1967).

[10] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[11] J. E. Moyal, Proc. Camb. Phil. Soc. **45**, 99 (1949).

and c numbers makes use of a completely symmetric arrangement of the operators. The ensemble average of Eq. (2.7) leads to

$$\langle M(\mathbf{a}(t))\rangle = \int M^{(w)}(\boldsymbol{\alpha}, t)\, d\boldsymbol{\alpha} P_w(\boldsymbol{\alpha}, t), \qquad (2.8)$$

a c-number integration as before, but over a different distribution function, the Wigner distribution function:

$$P_w(\boldsymbol{\alpha}, t) = (2\pi)^{-f} \int d\boldsymbol{\xi}\, e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\alpha}} \langle e^{i\boldsymbol{\xi}\cdot\mathbf{a}}\rangle. \qquad (2.9)$$

Haken, Risken, and Weidlich[12] have made use of ordered exponential operators of the form (2.6) and Gordon[5] has calculated the equation of motion of the Wigner distribution function for a laser.

3. QUANTUM REGRESSION THEOREM

The basis of our treatment of multitime averages will be the use of the quantum regression theorem as the simplest definition of a quantum Markoffian system. This theorem describes the motion of system operators in interaction with a reservoir. If M is a member (or a linear combination) of a *complete* set of system *Markoffian* operators M_μ, then the time evolution of such an operator can be written in the form

$$\langle M(t)\rangle = \sum O_\mu(t, t')\langle M_\mu(t')\rangle, \qquad (3.1)$$

where $t > t'$. If the numerical coefficients $O_\mu(t, t')$ are known for the single-time motion described by Eq. (3.1), then the mean of a two-time operator is given by

$$\langle Q(t')M(t)N(t')\rangle = \sum O_\mu(t, t')\langle Q(t')M_\mu(t')N(t')\rangle, \qquad (3.2)$$

and the mean of a three-time operator is given by

$$\langle Q(t', t'')M(t)N(t', t'')\rangle = \sum O_\mu(t, t') \times \langle Q(t', t'')M_\mu(t')N(t', t'')\rangle. \qquad (3.3)$$

Here N and Q are understood to be any *system* operators. In general, N and Q can be functions of system operators at an arbitrary number of times as long as these are times that precede the time t. These equations tell us that the time dependence in the variable t of these multitime averages is the same as the time dependence in the variable t displayed in the single-time equation (3.1). The only distinction between the values of these averages resides in the initial conditions when for example in (3.2) the time t is set equal to the earlier time t'.

In papers QIV and QX we started from a Langevin description of the form

$$dM_\mu/dt = A_\mu + F_\mu(t) \qquad (3.4)$$

and adopted the assumptions,

$$\langle Q(t')F_\mu(t)N(t')\rangle = 0, \qquad t > t'$$
$$\langle Q(t', t'')F_\mu(t)N(t', t'')\rangle = 0, \qquad t > t' > t'' \qquad (3.5)$$

the lack of correlation of the Langevin force $F_\mu(t)$ with information at earlier times, as our definition of a Markoff process. We then found that the quantum regression theorem was an immediate consequence of this assumption. The steps of this proof could also be taken in reverse order. The quantum regression theorem could be used as the definition of a Markoff process and the properties Eqs. (3.5) and (3.6) of the Langevin force would then be a necessary consequence of this assumption.

4. CLASSICAL EQUIVALENCE BETWEEN THE MARKOFF PROPERTY AND THE REGRESSION THEOREM

In this section, we shall establish the complete equivalence between the classical regression theorem and the usual definition of a Markoffian system. This proof is a slight generalization of one sketched in Sec. 5 of our Japanese lecture notes.[13]

We can take as our definition of a classical Markoffian process the factorization of the multitime probability density:

$$P(\boldsymbol{\alpha}_n, t_n; \boldsymbol{\alpha}_{n-1}, t_{n-1}; \cdots \boldsymbol{\alpha}_1, t_1) = P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}_{n-1}, t_{n-1}) \times P(\boldsymbol{\alpha}_{n-1}, t_{n-1}; \cdots \boldsymbol{\alpha}_1, t_1). \qquad (4.1)$$

where the $\boldsymbol{\alpha}_n$ are values taken by the random variable $\boldsymbol{\alpha}(t_n)$. For a non-Markoff process the first factor, the conditional probability, would depend on all earlier times. We rewrite Eq. (4.1) in the completely equivalent form

$$P(\boldsymbol{\alpha}_n, t_n; \cdots \boldsymbol{\alpha}_1, t_1) = \int P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}', t_{n-1})\, d\boldsymbol{\alpha}' \delta(\boldsymbol{\alpha}' - \boldsymbol{\alpha}_{n-1}) \times P(\boldsymbol{\alpha}_{n-1}, t_{n-1}; \cdots \boldsymbol{\alpha}_1, t_1), \qquad (4.2)$$

multiply both sides by $M(\boldsymbol{\alpha}_n)N(\boldsymbol{\alpha}_{n-1}, \boldsymbol{\alpha}_{n-2}, \cdots \boldsymbol{\alpha}_1)$, and integrate to obtain

$$\langle M(\boldsymbol{\alpha}(t_n))N(\boldsymbol{\alpha}(t_{n-1}), \cdots \boldsymbol{\alpha}(t_1))\rangle = \iint M(\boldsymbol{\alpha}_n)\, d\boldsymbol{\alpha}_n P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}', t_{n-1})\, d\boldsymbol{\alpha}' \times \langle \delta(\boldsymbol{\alpha}' - \boldsymbol{\alpha}(t_{n-1}))N(\boldsymbol{\alpha}(t_{n-1}), \cdots \boldsymbol{\alpha}(t_1))\rangle. \qquad (4.3)$$

For any process, Markoff or not, the two-time correlation function can be written

$$P(\boldsymbol{\alpha}_n, t_n; \boldsymbol{\alpha}_{n-1}, t_{n-1}) = P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}_{n-1}, t_{n-1})\, P(\boldsymbol{\alpha}_{n-1}, t_{n-1}) = P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}_{n-1}, t_{n-1}) \times \langle \delta(\boldsymbol{\alpha}_{n-1} - \boldsymbol{\alpha}(t_{n-1}))\rangle, \qquad (4.4)$$

[12] H. Haken, H. Risken, and W. Weidlich, Z. Physik **206**, 355 (1967).

[13] M. Lax, in *1966 Tokyo Summer Lectures in Theoretical Physics, Part I*, edited by R. Kubo and H. Kamimura (W. A. Benjamin, Inc., New York, 1967).

since this is merely the definition of the conditional probability. Let us multiply by $M(\boldsymbol{\alpha}_n)$ and integrate over both variables $\boldsymbol{\alpha}_n$ and $\boldsymbol{\alpha}_{n-1}$. If we change the name of the integration variable $\boldsymbol{\alpha}_{n-1}$ to $\boldsymbol{\alpha}'$ we obtain

$$\langle M(\boldsymbol{\alpha}(t_n))\rangle = \iint M(\boldsymbol{\alpha}_n)\, d\boldsymbol{\alpha}_n P(\boldsymbol{\alpha}_n, t_n \mid \boldsymbol{\alpha}', t_{n-1})\, d\boldsymbol{\alpha}' \times \langle \delta(\boldsymbol{\alpha}' - \boldsymbol{\alpha}(t_{n-1}))\rangle. \quad (4.5)$$

The set of functions $\delta(\boldsymbol{\alpha}' - \boldsymbol{\alpha}(t_{n-1}))$ for all $\boldsymbol{\alpha}'$ is the complete set of variables M_μ referred to in the regression theorem. The comparison between Eqs. (4.3) and (4.5) is essentially the statement of the regression theorem, since Eq. (4.3) may be obtained from (4.5) merely by adding the factor N under the ensemble average on each side of Eq. (4.5).

Since the steps in the proof presented in this section can be stated in reverse order, the regression theorem implies the Markoff property as well as the converse. It was this full equivalence between the regression theorem and the Markoff property that suggested to us the natural use of the regression theorem as the simplest way to generalize the Markoff property from classical systems to quantum-mechanical systems.

5. REGRESSION THEOREM AND MARKOFF PROPERTY IN THE COHERENT REPRESENTATION

We consider now the important special case in which we have only two variables taken in the normal order, in other words, $a_1 = b^\dagger$ and $a_2 = b$, where $[b, b^\dagger] = 1$. Our associated distribution function for this special case is then defined by

$$P(\beta, \beta^*, t) = \langle \delta(\beta^* - b^\dagger(t))\delta(\beta - b(t))\rangle, \quad (5.1)$$

and the mean of a general operator can be written as an integral:

$$\langle M(b, b\dagger)\rangle = \int M^{(n)}(\beta, \beta^*)\, d^2\beta\; P(\beta, \beta^*, t), \quad (5.2)$$

where $d^2\beta \equiv d\mathrm{Re}\beta\, d\mathrm{Im}\beta$. We use the superscript n on M to emphasize that the chosen order in this particular case is the normal order. Since Eq. (5.2) is true for a completely general operator M and since Glauber[2] and Sudarshan[3] have established this equation by the methods of the coherent representation it is clear that their P function is identical to ours. We can now generalize Eq. (5.1) to define an n-time distribution function by means of

$$P_n \equiv P(\beta_n, \beta_n^*, t_n; \beta_{n-1}, \beta_{n-1}^*, t_{n-1}; \cdots \beta_1, \beta_1^*, t_1) \equiv \langle \delta(\beta_1^* - b^\dagger(t_1))\delta(\beta_2^* - b^\dagger(t_2))\cdots\delta(\beta_n^* - b^\dagger(t_n)) \times \delta(\beta_n - b(t_n))\cdots\delta(\beta_2 - b(t_2))\delta(\beta_1 - b(t_1))\rangle. \quad (5.3)$$

Let us now define an operator T_N which places operators in normal order and in appropriate time sequence. If we use A_j as an abbreviation for $A(b^\dagger(t_j))$ and $B_j = B(b(t_j))$ then our operator T_N can be defined by

$$M(b_1^\dagger, \cdots b_n^\dagger; b_n \cdots b_1) \equiv T_N(A_1B_1A_2B_2A_3B_3\cdots A_nB_n) = A_1A_2A_3\cdots A_nB_nB_{n-1}\cdots B_2B_1. \quad (5.4)$$

With this normal-ordered time-ordered sequence understood as part of the definition of operator $\mathcal{C}$ we can set up a multitime association

$$M(b_1^\dagger \cdots b_n; b_n \cdots b_1) = \mathcal{C}A_1(\beta_1^*)A_2(\beta_2^*)\cdots A_n(\beta_n^*)B_n(\beta_n)\cdots B_1(\beta_1) = \mathcal{C}\bar{M}(\beta_1, \beta_1^*; \cdots \beta_n, \beta_n^*), \quad (5.5)$$

in which the c numbers β_j, β_j^* are replaced by the operator b_j, $b_j^\dagger$ and the results are placed in the time-ordered normal ordered sequence defined by Eq. (5.4). In precise analogy to Eq. (2.4) the ensemble average of this multitime operator can be written as a c-number integration over the associated multitime distribution function:

$$\langle M(b_1^\dagger, \cdots b_n^\dagger; b_n, \cdots b_1)\rangle = \int \bar{M}(\beta_1^*, \cdots \beta_n^*; \beta_n, \cdots \beta_1)\, d^2\beta_1 \cdots d^2\beta_n \times P(\beta_n, \beta_n^*, t_n; \cdots \beta_1, \beta_1^*, t_1). \quad (5.6)$$

Our single-time distribution function obeys

$$\langle \delta(\beta_n^* - b^\dagger(t_n))\delta(\beta_n - b(t_n))\rangle = \int P(\beta_n, \beta_n^*, t_n \mid \alpha, \alpha^*, t_{n-1})\, d^2\alpha \times \langle \delta(\alpha^* - b^\dagger(t_{n-1}))\delta(\alpha - b(t_{n-1}))\rangle, \quad (5.7)$$

since this is merely the definition of the conditional probability distribution. We shall now apply the quantum regression theorem using this single-time motion to simplify the n-time distribution function P_n of (5.3). The regression theorem then yields

$$P_n = \int P(\beta_n, \beta_n^*, t_n \mid \alpha, \alpha^*, t_{n-1})\, d^2\alpha \times \langle \delta(\beta_1^* - b_1^\dagger)\cdots\delta(\beta_{n-1}^* - b_{n-1}^\dagger)\,\delta(\alpha^* - b_{n-1}^\dagger)\delta(\alpha - b_{n-1}) \times \delta(\beta_{n-1} - b_{n-1})\cdots\delta(\beta_1 - b_1)\rangle. \quad (5.8)$$

Inside the ensemble average in Eq. (5.8) one can make the replacements

$$\delta(\alpha - b_{n-1})\,\delta(\beta_{n-1} - b_{n-1}) \to \delta(\alpha - \beta_{n-1})\,\delta(\beta_{n-1} - b_{n-1}),$$
$$\delta(\beta_{n-1}^* - b_{n-1}^\dagger)\,\delta(\alpha^* - b_{n-1}^\dagger) \to \delta(\alpha^* - \beta_{n-1}^*)\,\delta(\beta_{n-1}^* - b_{n-1}^\dagger), \quad (5.9)$$

since the operator underneath the $\langle\ \rangle$ is in the chosen order and could be evaluated with the help of Eq.

(5.6). The result,

$$P(\beta_n, \beta_n^*, t_n; \cdots \beta_1, \beta_1^*, t_1) = P(\beta_n, \beta_n^*, t_n \mid \beta_{n-1}, \beta_{n-1}^*, t_{n-1}) \times P(\beta_{n-1}, \beta_{n-1}^*, t_{n-1}; \cdots \beta_1, \beta_1^*, t_1), \quad (5.10)$$

is simply the definition (4.1) of a classical Markoff process.

In summary, the time-ordered normal-ordered sequence used in our definition by Eq. (5.3) is just that appropriate to evaluating the means of time-ordered normal-ordered quantum-mechanical operators by means of c-number integration. Moreover, such multitime averages can be performed by assuming that the process is a Markoff process whose conditional probabilities are obtainable from the Green's-function solution of the single-time equation. Also, as we have previously emphasized, it is the time-ordered normal-ordered operators that are appropriate to measurement processes in which photons are destroyed.

In order to show that not all results are as simple as the above, let us consider the case in which we wish to obtain a photon number correlation:

$$\langle b^\dagger(t_2) b(t_2) b^\dagger(t_1) b(t_1) \rangle.$$

For convenience in evaluating this average, it is useful to introduce the two-time distribution function

$$P(\beta_2, \beta_2^*, t_2; \beta_1, \beta_1^*, t_1) = \langle \delta(\beta_2^* - b_2^\dagger) \delta(\beta_2 - b_2) \delta(\beta_1^* - b_1^\dagger) \delta(\beta_1 - b_1) \rangle. \quad (5.11)$$

With the help of the relation (5.7) and the use of the quantum regression theorem, this two time average can be rewritten in the form

$$P_2 = \int P(\beta_2^*, \beta_2, t_2 \mid \alpha^*, \alpha, t_1) d^2\alpha \times \langle \delta(\alpha^* - b_1^\dagger) \delta(\alpha - b_1) \delta(\beta_1^* - b_1^\dagger) \delta(\beta_1 - b_1) \rangle, \quad (5.12)$$

where the first factor, the conditional probability, is the same as that obtained from the Green's-function solution of the one-time distribution function. Unfortunately, the operators at the single time t_1 underneath the ensemble average in Eq. (5.12) are not in normal order. Thus the replacement used in Eq. (5.9) is no longer valid and the ensemble average is not simply the single-time probability distribution $P(\alpha, \alpha^*, t_1)$. The time dependence on the variable t_2 is, of course, the same as before but the initial conditions have been changed. In terms of the decay eigenvalues of the single-time equation these will remain unchanged, but the extent to which each decay eigenvalue enters the two-time average is different for photon number fluctuations than it is for the intensity fluctuations which are measured by time-ordered normal-ordered operators. Even worse difficulties will occur if we ask questions involving photon numbers at three times. Fortunately, for this particular example there is a way of avoiding these difficulties. If one starts with an initial distribution that is independent of the phases of the β's then all subsequent distributions will remain independent of such phases. Thus it is possible to introduce the variable $b^\dagger b$ itself as a fundamental variable and discard all phase information. A multitime probability distribution function can then be defined by means of

$$P(\pi_1, t_1; \cdots \pi_n, t_n) = \langle \delta(\pi_n - b_n^\dagger b_n) \cdots \delta(\pi_2 - b_2^\dagger b_2) \delta(\pi_1 - b_1^\dagger b_1) \rangle, \quad (5.13)$$

where

$$b_j \equiv b(t_j).$$

The regression theorem can be used to establish the Markoff character of this process so that the random process is completely defined by the one-time equation of motion for generalized Fokker-Planck equation. Such an equation was developed in paper QVII for the probability distribution of the number of photons and the population difference. The present discussion justifies the literal use of this classical equation in determining photon number fluctuations.

6. PHOTODETECTION FLUCTUATIONS

Photodetection fluctuations are not a primary subject of the present paper. We mention, however, the formula of Mandel,[14] Glauber,[15] and Kelley and Kleiner,[16] for the probability $p(m, T)$ for obtaining m photocounts in a time T:

$$p(m, T) = \langle T_N (\Omega^m / m!) \exp(-\Omega) \rangle, \quad (6.1)$$

where

$$\Omega = \epsilon \int_t^{t+T} b^\dagger(s) b(s) ds z. \quad (6.2)$$

This formula is a complicated one involving time-ordered normal-ordered operators at, in general, an infinite number of times. The previous section has established, however, that such an average can be taken for the associated classical process. Namely, if we define $\bar{\Omega}$ by

$$\bar{\Omega} = \epsilon \int_t^{t+T} |\beta(s)|^2 ds, \quad (6.3)$$

then the probability distribution is given by the classical average

$$p(m, T) = \langle [(\bar{\Omega})^m / m!] \exp(-\bar{\Omega}) \rangle_{\mathrm{cl}}. \quad (6.4)$$

It is often convenient to rewrite the preceding result in the form

$$p(m, T) = [(-1)^m / m!] (d/d\lambda)^m \langle \exp(-\lambda \bar{\Omega}) \rangle_{\mathrm{cl}} \mid_{\lambda=1}, \quad (6.5)$$

[14] L. Mandel, Proc. Phys. Soc. (London) **74**, 233 (1959); and Ref. 7.

[15] See Ref. 8.

[16] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

because general methods for evaluating means of exponentials of time integrals have been developed.[17] A discussion of these methods has been given in III.[18] Moreover, evaluations of the exponential average have been given for a simple linear model of a laser below threshold by Bedard[19] and Glauber.[8] Our own results in the region below threshold are summarized in Chap. 18 of our Brandeis lectures. In addition an approximate technique based on fitting the first two moments has been introduced by Rice[20] and applied by Mandel[21] to a laser below threshold. An extension of this method to the region above threshold is also given in our Brandeis lectures.[22] We also evaluate the mean of this exponential for a simple relaxation model of a maser that properly takes into account the steady state distribution of the number of photons in the laser.

7. QUANTUM REGRESSION THEOREM AND THE CHAPMAN-KOLMOGOROFF CONDITION

If we write a set of operators $MNQ\cdots$ in the second quantized form

$$M=\sum M_{ij}a_i^\dagger a_j,$$
$$N=\sum N_{kl}a_k^\dagger a_l,$$
$$Q=\sum Q_{mn}a_m^\dagger a_n,$$

then the mean of a multitime product can be written in the form

$$\langle M(t_n)N(t_{n-1})Q(t_{n-2})\cdots\rangle$$
$$=\sum M_{ij}N_{kl}Q_{mn}\cdots\rho(ijt_n;\,klt_{n-1};\,mnt_{n-2};\,\cdots),\quad(7.1)$$

in which we have introduced a kind of multitime density matrix defined by

$$\rho(ijt_n,\,klt_{n-1},\,mnt_{n-2},\,\cdots)$$
$$=\langle a_i^\dagger(t_n)a_j(t_n)a_k^\dagger(t_{n-1})a_l(t_{n-1})$$
$$\times a_m^\dagger(t_{n-2})a_n(t_{n-2})\cdots\rangle.\quad(7.2)$$

As established in paper QII, any function of the operators $a_i^\dagger a_j$ is necessarily reducible to a bilinear form in those operators by means of

$$a_\mu^\dagger a_\lambda a_k^\dagger a_l=\delta_{\lambda k}a_\mu^\dagger a_l.\quad(7.3)$$

Thus the bilinear operators $a_i^\dagger a_j$ constitute a complete set and any operator is linearly expressible in terms of them. Hence, if the operator $a_i^\dagger(t)a_j(t)$ is expressed in terms of similar operators at an earlier time t' the equation necessarily takes the linear form

$$\langle a_i^\dagger(t)a_j(t)\rangle=\sum\rho_{i\mu}{}^{j\lambda}(t\mid t')\langle a_\mu^\dagger(t')a_\lambda(t')\rangle,\quad(7.4)$$

which is the precise transcription of Eq. (5.13) of QII with the numerical function $O_{qp}{}^{ji}$ of that paper replaced by a numerical function $\rho_{i\mu}{}^{j\lambda}$ in this paper. If now Eq. (7.2) is simplified by using the quantum regression theorem in conjunction with Eq. (7.4), we immediately obtain the result

$$\rho(ijt_n,\,klt_{n-1},\,mnt_{n-2},\,\cdots)$$
$$=\sum_\mu\rho_{i\mu}{}^{jk}(t_n\mid t_{n-1})\rho(\mu lt_{n-1};\,mnt_{n-2},\,\cdots),\quad(7.5)$$

which displays a factorization similar to that of (4.1) characteristic of classical Markoff processes. If this factorization is iterated once we obtain

$$\rho(ijt_n,\,klt_{n-1},\,mnt_{n-2},\,\cdots)$$
$$=\sum_{\mu\nu}\rho_{i\mu}{}^{jk}(t_n\mid t_{n-1})\rho_{\mu\nu}{}^{lm}(t_{n-1}\mid t_{n-2})\rho(\nu nt_{n-2};\,\cdots),\quad(7.6)$$

and if it is iterated twice we obtain

$$\rho(ijt_n,\,klt_{n-1},\,mnt_{n-2},\,pqt_{n-3},\,\cdots)$$
$$=\sum_{\mu,\nu,\lambda}\rho_{i\mu}{}^{jk}(n\mid n-1)\rho_{\mu\nu}{}^{lm}(n-1\mid n-2)$$
$$\times\rho_{\nu\lambda}{}^{np}(n-2\mid n-3)\rho(\lambda qt_{n-3};\,\cdots).\quad(7.7)$$

In general, we can iterate enough times to express the results entirely in terms of a product of conditional density matrices and density matrix at the earliest time. Applying Eq. (7.6) directly to the three-time case we obtain

$$\rho(ij3,\,kl2,\,mn1)=\sum_\mu\rho_{i\mu}{}^{jk}(3\mid 2)\rho_{\mu\nu}{}^{lm}(2\mid 1)\rho_{\nu n}(1).\quad(7.8)$$

If we set $k=l$ and sum on this variable and use the condition

$$\sum a_k^\dagger a_k=1,\quad(7.9)$$

we obtain

$$\rho(ij3,\,mn1)=\sum_{\mu k\nu}\rho_{i\mu}{}^{jk}(3\mid 2)\rho_{\mu\nu}{}^{km}(2\mid 1)\rho_{\nu n}(1).\quad(7.10)$$

On the other hand, Eq. (7.5) used for two times implies

$$\rho(ij3,\,mn1)=\sum\rho_{i\nu}{}^{jm}(3\mid 1)\rho_{\nu n}(1).\quad(7.11)$$

If we compare Eq. (7.10) and (7.11), we obtain a condition

$$\rho_{i\nu}{}^{jm}(3\mid 1)=\sum_{k,\mu}\rho_{i\mu}{}^{jk}(3\mid 2)\rho_{\mu\nu}{}^{km}(2\mid 1),\quad(7.12)$$

which is the precise quantum analog of the classical Chapman-Kolmogoroff condition. This condition, indeed, was assumed by Bausch and Stahl[9] as the appropriate definition for a quantum Markoff process.

---

[17] D. Slepian, Bell System Tech. J. **37**, 163 (1958); U. Grenander, H. J. Pollack, and D. Slepian, J. Soc. Ind. and Appl. Math. **7**, 374 (1959).

[18] III refers to the third paper in the series on classical noise by M. Lax. The complete series is as follows: (I) Rev. Mod. Phys. **32**, 25 (1960); (II) J. Phys. Chem. Solids **14**, 248 (1960); (III) Rev. Mod. Phys. **38**, 359 (1966); (IV) *ibid.* **38**, 541 (1966); (V) Phys. Rev. **160**, 290 (1967); (VI) *ibid.* **161**, 350 (1967) (with R. D. Hempstead).

[19] G. Bedard, Phys. Rev. **151**, 1038 (1966).

[20] S. O. Rice, Bell System Tech. J. **24**, 46 (1965).

[21] L. Mandel, Ref. 14; G. Bedard, J. C. Chang, and L. Mandel, Phys. Rev. **160**, 1496 (1967).

[22] M. Lax, in *1966 Brandeis Summer Institute Lectures* (Gordon and Breach Science Publishers, Inc., New York, to be published).

If we combine Eqs. (7.1) and (7.8), we find that the mean of a three-time operator can be written in the form

$$\langle M(3)N(2)Q(1)\rangle = \sum M_{ij}N_{kl}Q_{mn} \times \rho_{i\mu}{}^{jk}(3\mid 2)\rho_{\mu\nu}{}^{lm}(2\mid 1)\rho_{\nu n}(1). \quad (7.13)$$

If we wish to avoid all subscripts, the same result can be written in purely operator form:

$$\langle M(3)N(2)Q(1)\rangle = \mathrm{Tr}[M(3)U(3\mid 2)N(2)U(2\mid 1)Q(1)\rho(1)], \quad (7.14)$$

where the density matrix $\rho(1)$ can be written in the form

$$\rho(1) = \sum a_n{}^\dagger(1)a_\nu(1)\rho_{\nu n}(1), \quad (7.15)$$

to conform with the notation introduced in QII. Our general operator at the time t_3 can be written in the form

$$M(3) = \sum M_{ij}a_i{}^\dagger(3)a_j(3), \quad (7.16)$$

and the propagator $U(3\mid 2)$ can be written

$$U(3\mid 2) \equiv \sum_{r,s,\mu,\lambda} \mid a_s{}^\dagger(3)a_r(3)\rangle\rho_{r\mu}{}^{s\lambda}(3\mid 2)a_\mu{}^\dagger(2)a_\lambda(2), \quad (7.17)$$

where the symbol $\mid\ \rangle$ is defined by

$$A\mid B\rangle \equiv \mathrm{Tr}AB. \quad (7.18)$$

We emphasize that Eq. (7.14) is identical in content to (7.13). Indeed, it was so constructed that when all of the necessary traces are taken Eq. (7.13) is obtained identically.

We can also obtain Eq. (7.14) by a direct application of the quantum regression theorem. For example, we can rewrite Eq. (7.14) in the form,

$$\begin{aligned}\langle a_i{}^\dagger(3)a_j(3)\rangle &= \sum_{\mu\lambda}\rho_{i\mu}{}^{j\lambda}(3\mid 2)\langle a_\mu{}^\dagger(2)a_\lambda(2)\rangle \\ &= a_i{}^\dagger(3)a_j(3)\sum_{r,s,\mu,\lambda}\mid a_s{}^\dagger(3)a_r(3)\rangle \\ &\quad\times\rho_{r\mu}{}^{s\lambda}(3\mid 2)\langle a_\mu{}^\dagger(2)a_\lambda(2)\rangle \\ &= \langle a_i{}^\dagger(3)a_j(3)U(3\mid 2)\rangle. \end{aligned} \quad (7.19)$$

Thus, for purposes of taking averages, we make the replacement

$$M(3)\rightarrow M(3)U(3\mid 2). \quad (7.20)$$

It is to be emphasized that the operator on the left-hand side of Eq. (7.20) is an operator at the time three, whereas, after the traces implied by U are taken, the operator on the right-hand side of (7.20) is an operator that acts at the time two. The quantum regression theorem then tells us that a relationship such as (7.20) can be bordered by operators on the right-hand side so that, for the purpose of taking a two-time average, one may make the replacement

$$M(3)U(3\mid 2)N(2)\rightarrow M(3)U(3\mid 2)N(2)U(2\mid 1). \quad (7.21)$$

Indeed, if we repeat this process and add one more propagator and the Q operator we obtain an operator that acts only at the time one so that, when the average is taken, we immediately obtain Eq. (7.14).

The significance of the propagator U is defined by the last line of Eq. (7.19). Alternatively, we can use Eq. (7.19) to write

$$\langle M(3)\rangle = \mathrm{Tr}M(3)\rho(3) = \mathrm{Tr}[M(3)U(3\mid 2)\rho(2)]. \quad (7.22)$$

Since the operator M is arbitrary, we learn from Eq. (7.20) that

$$\rho(3) = U(3\mid 2)\rho(2), \quad (7.23)$$

in other words, that the propagator U also describes the propagation in time of the density matrix.

These results can be immediately generalized to take an average of a product of n distinct operators taken at n different times:

$$\begin{aligned}&\langle B_n(n)B_{n-1}(n-1)\cdots B_2(2)B_1(1)\rangle \\ &= \mathrm{Tr}[B_n(n)U(n\mid n-1)B_{n-1}(n-1)U(n-1\mid n-2)\cdots \\ &\quad\times B_2(2)U(2\mid 1)B_1(1)\rho(1)]. \end{aligned} \quad (7.24)$$

If we wish to take an average of the set of operators in the reverse time order, it is convenient to rewrite Eq. (7.19) in the form

$$\begin{aligned}a_i{}^\dagger(3)a_j(3) &\rightarrow \sum_{\mu\lambda} a_\mu{}^\dagger(2)a_\lambda(2)\rho_{i\mu}{}^{j\lambda}(3\mid 2) \\ &\rightarrow V(2\mid 3)a_i{}^\dagger(3)a_j(3), \end{aligned} \quad (7.25)$$

where the new propagator V is defined by

$$\begin{aligned}V(2\mid 3) &= \sum a_\mu{}^\dagger(2)a_\lambda(2)\rho_{r\mu}{}^{s\lambda}(3\mid 2)\langle a_s{}^\dagger(3)a_r(3)\mid \\ &= U(3\mid 2)^\dagger. \end{aligned} \quad (7.26)$$

An average of a product of n operators in the reverse time order is then given by

$$\begin{aligned}\langle A_1(1)A_2(2)\cdots A_n(n)\rangle &= \mathrm{Tr}[\rho(1)A_1(1)V(1\mid 2)A_2(2) \\ &\quad\times V(2\mid 3)\cdots V(n-1\mid n)A_n(n)]. \end{aligned} \quad (7.27)$$

A more complicated kind of time average, of the sort that previously appeared in connection with electromagnetic field operators, can by the same arguments be written in the form

$$\begin{aligned}&\langle A_1(1)\cdots A_n(n)B_n(n)\cdots B_1(1)\rangle \\ &\quad = \mathrm{Tr}[\rho(1)A_1(1)V(1\mid 2) \\ &\quad\times\cdots V(n-1\mid n)A_n(n)B_n(n)U(n\mid n-1) \\ &\quad\times\cdots B_2(2)U(2\mid 1)B_1(1)]. \end{aligned} \quad (7.28)$$

More complicated time orderings can be handled by the same methods. We leave to the reader the working out of any new special cases.

In this section, we have developed a completely

general method for taking the averages of products of operators at several times. This method, however, requires a knowledge of the propagation of the full density matrix of the system. Often, however, we are concerned with a description that is far less complete, e.g., in terms of a smaller number of variables. Indeed, it was just this simpler problem which led us to introduce the correspondence between quantum and classical variables. The problem in the remaining sections of this paper is to show that we can exploit this quantum classical correspondence to simplify the calculation of multitime averages without the full knowledge of the evolution of the density matrix of the complete system.

Our result (7.24) is similar to a result derived recently by Haken and Weidlich.[23] Their result (35) has the same form as (7.24) with a set of projection operators inserted after each operator B_j. Their result is correct if the multitime average consists of a sequence of measurements at each separate time. Our exact result (7.24) contains no projection operators and thus should be interpreted as a *single measurement* of the correlation product. If n measurements are understood, the right-hand side should contain the Haken-Weidlich projection operators—but so should the left-hand side, to indicate that a different correlation function has been measured. The differences should be slight when the reaction of the measuring apparatus on the system is weak, e.g., when light fluctuations are measured far from a laser source.

8. MULTIVARIABLE, MULTITIME CASE

In the present section, we investigate the extent to which, in the multivariable case, the literal use of the Markoff random process implied by the one-time equation of motion, can be used to calculate multitime averages. [This procedure was found successful for the variables b and $b^\dagger$ in Sec. 5.] We start by writing the Green's-function solution of the single-time equation in the form

$$P(\alpha, t)=\int P(\alpha, t \mid \alpha', t')\, d\alpha P(\alpha', t'). \quad (8.1)$$

We then take the average of a general operator $M(t)\equiv M(\mathbf{a}(t))$ by integrating the associated chosen-ordered function against Eq. (8.1). After replacing the initial distribution $P(\alpha', t')$ by its definition, we obtain

$$\langle M(t)\rangle=\iint d\alpha\, M^{(c)}(\alpha) P(\alpha, t \mid \alpha', t')\, d\alpha' \times\langle\prod_1^f \delta(\alpha_j'-a_j(t'))\rangle. \quad (8.2)$$

The quantum regression theorem of Eq. (3.2) can now be applied immediately to yield

$$\langle Q(t')M(t)N(t')\rangle=\iint d\alpha\, \bar{M}(\alpha) P(\alpha, t \mid \alpha', t')\, d\alpha' \times\langle Q(t')\prod_1^f \delta(\alpha_j'-a_j(t'))N(t')\rangle. \quad (8.3)$$

If the operator inside the ensemble average on the right-hand side of Eq. (8.3) requires no rearrangement to be placed in chosen order, then this ensemble average can also be written as an integration. For example, if the operators Q and N are functions, respectively, only of a_1 and a_f, then no permutation is needed to obtain a chosen order, and we can write

$$\langle Q(a_1(t'))M(t)N(a_f(t'))\rangle =\iint d\alpha\, \bar{M}(\alpha) P(\alpha, t \mid \alpha', t')\, d\alpha' \times Q(\alpha_1') P(\alpha', t') N(\alpha_f'). \quad (8.4)$$

The right-hand side of (8.4), however, is precisely the way the corresponding average would have been calculated in the associated classical random process, so that we can rewrite (8.4) in the form

$$\langle Q(a_1(t'))M(t)N(a_f(t'))\rangle =\langle \bar{M}(\alpha(t))\bar{Q}(\alpha_1(t'))\bar{N}(\alpha_f(t'))\rangle_{\mathrm{cl}}. \quad (8.5)$$

It is clear, however, that if Q were a function of a_2, and a_2 did not commute with a_1, then the operator beneath the ensemble average on the right-hand side of (8.3) would not be a normal-order, and the result could not be written as a c-number integration without first producing some commutator corrections. Similar remarks occur if the operator N depends on the variable a_{f-1}. Thus we see that, even for quantum Markoff processes, it is not legitimate to compute all averages by taking the associated classical random process literally.

If we make use of Eq. (8.1) in the form

$$P(\alpha', t')=\int P(\alpha', t' \mid \alpha'', t'')\, d\alpha'' P(\alpha'', t''), \quad (8.6)$$

then we can apply the quantum regression theorem to Eq. (8.4) to obtain the three-time result

$$\langle S(t'')Q(t')M(t)N(t')R(t')\rangle =\iiint d\alpha d\alpha' d\alpha'' \bar{M}(\alpha) P(\alpha, t \mid \alpha', t') \times Q(\alpha_1') N(\alpha_f') P(\alpha', t \mid \alpha'', t'') \times\langle S(t'')\prod_1^f \delta(\alpha_j''-a_j(t''))R(t'')\rangle. \quad (8.7)$$

Again the last ensemble average in (8.7) can be reduced to an integration when no rearrangements are

[23] H. Haken and W. Weidlich, Z. Physik **205**, 96 (1967).

required. For example, we obtain

$$\langle S(a_1(t''))Q(a_1(t'))M(t)N(a_f(t'))R(a_f(t''))\rangle$$
$$=\langle \bar{S}(\alpha_1(t''))\bar{Q}(\alpha_1(t'))\bar{M}(\boldsymbol{\alpha}(t))\bar{N}(\alpha_f(t'))\bar{R}(\alpha_f(t''))\rangle_{\mathrm{cl}}. \tag{8.8}$$

It is clear from the above results that the first and last variables in our sequence have an especially favored status. All multitime averages involving only these two operators written in the order in which a_1 appears before a_f can all be evaluated by using the usual literal Markoffian interpretation of the associated classical random process. It is for this reason that in our work with Yuen[6] on the six-variable description of the laser we have placed the variable $b^\dagger$ to the extreme left and b to the extreme right.

9. SIX-VARIABLE DESCRIPTION OF A LASER

If we introduce the population operators by means of a sum over atoms m,

$$N_i=\sum_m (a_i^\dagger a_i)_m, \qquad i=1,2 \tag{9.1}$$

and the polarization operator by means of

$$M=\sum_m (a_1^\dagger a_2)_m, \tag{9.2}$$

we obtain the commutation rules

$$[b,N_1]=[b,N_2]=[b,M]=[b,M^\dagger]=[N_1,N_2]=0, \tag{9.3}$$

$$[N_1,M]=M, \qquad [N_2,M]=-M,$$
$$[M^\dagger,M]=N_2-N_1. \tag{9.4}$$

In accord with the discussion at the end of Sec. 8, a convenient choice of order for our six operators is given by

$$a_1=b^\dagger, \quad a_2=M^\dagger, \quad a_3=N_1, \quad a_4=N_2, \quad a_5=M, \quad a_6=b. \tag{9.5}$$

The associated classical variables can be denoted by

$$\alpha_1=\beta, \quad \alpha_2=\mathfrak{M}^*, \quad \alpha_3=\mathfrak{N}, \quad \alpha_4=\mathfrak{N}_2, \quad \alpha_5=\mathfrak{M}, \quad \alpha_6=\beta. \tag{9.6}$$

In view of the remarks at the end of Sec. 8 all time-ordered normal-ordered averages involving only photon operators are given by the associated classical averages. In particular, the most important of these averages are

$$\langle b^\dagger(t)b(0)\rangle=\langle \beta^*(t)\beta(0)\rangle, \tag{9.7}$$

$$\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle=\langle |\beta(t)|^2\,|\beta(0)|^2\rangle, \tag{9.8}$$

$$\left\langle T_N \exp\left[-\lambda\int_t^{t+T} b^\dagger(s)b(s)\,ds\right]\right\rangle$$
$$=\left\langle \exp\left[-\lambda\int_t^{t+T} |\beta(s)|^2 ds\right]\right\rangle. \tag{9.9}$$

We have omitted the subscript cl on the right-hand side, since the averages in question are clearly classical. Since at any one time the polarization variables commute with the field variables, averages involving only the polarization and field variables in normal and time order are clearly reducible to the corresponding averages of the associated classical problem. For example we have

$$\langle M^\dagger(t)M(0)\rangle=\langle \mathfrak{M}^*(t)\mathfrak{M}(0)\rangle, \tag{9.10}$$

$$\langle b^\dagger(t)M(0)\rangle=\langle \beta^*(t)\mathfrak{M}(0)\rangle, \tag{9.11}$$

$$\langle b^\dagger(0)M(t)^\dagger b(t)M(0)\rangle=\langle \beta^*(0)\beta(t)\mathfrak{M}^*(t)\mathfrak{M}(0)\rangle. \tag{9.12}$$

On the other hand, if we wished to compute moments involving population fluctuations these would be difficult to obtain since the populations do not commute through the polarizations, so that

$$\langle N_1(t)N_2(0)\rangle \neq \langle \mathfrak{N}_1(t)\mathfrak{N}_2(0)\rangle. \tag{9.13}$$

Since a computation of the commutators leads to considerable algebra, if we were interested in population correlations it would be simpler to introduce a new classical distribution function based on the new order of variables:

$$a_1=b^\dagger, \quad a_2=N_1, \quad a_3=M^\dagger, \quad a_4=M, \quad a_5=N_2, \quad a_6=b. \tag{9.14}$$

For this new choice of variables, we would indeed have

$$\langle N_1(t)N_2(0)\rangle=\langle \mathfrak{N}_1(t)\mathfrak{N}_2(0)\rangle. \tag{9.15}$$

It may be that, because of other averages that we wish to compute, we do not wish to adopt a change in the order of variables given in Eq. (9.14), but prefer instead to retain the original choice of ordering given in Eq. (9.5). How then are we to compute the average of Eq. (9.13)? From Eqs. (8.3) we can write

$$\langle N_2(t)N_1(0)\rangle=\iint d\boldsymbol{\alpha}\,\mathfrak{N}_2 P(\boldsymbol{\alpha},t\,|\,\boldsymbol{\alpha}',0)\,d\boldsymbol{\alpha}'$$
$$\times\langle\prod_1^6 \delta(\alpha_j'-a_j(0))N_1(0)\rangle. \tag{9.16}$$

The difficulty now is that the quantities underneath the averaging bracket are not in the chosen order. We must therefore somehow commute N_1 through $\delta(\mathfrak{M}'-M(0))$. To accomplish this, we make use of the commutation rule

$$e^{\xi M}N_1=N_1e^{\xi M}-\xi Me^{\xi M}$$
$$=(N_1-M\partial/\partial M)e^{\xi M}. \tag{9.17}$$

The second form of Eq. (9.17) is sufficiently general that the exponential operator involving M can be replaced by a completely general function of M. In particular, if we make use of the correspondence

$$Q(b^\dagger, M^\dagger, N_1, N_2, M, b) = \mathcal{C}Q^{(c)}(\beta^*, \mathfrak{M}^*, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}, \beta), \tag{9.18}$$

then Eq. (9.17) permits us to write the new correspondence

$$Q(b^\dagger, M^\dagger, N_1, N_2, M, b)N_1 = \mathcal{C}(\mathfrak{N}_1 - \mathfrak{M}\partial/\partial\mathfrak{M})Q^{(c)}. \tag{9.19}$$

In accord with the general procedure derived in Eq. (2.4), the average of any operator at one time can be written as an integral over the product of the associated classical function and the distribution function. For the present case, we obtain

$$\begin{aligned}\langle Q(b^\dagger, M^\dagger, N_1, N_2, M, b)N_1\rangle &= \int (\mathfrak{N}_1 - \mathfrak{M}(\partial/\partial\mathfrak{M}))Q^c(\beta^*, \mathfrak{M}^*, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}, \beta) \\ &\quad \times d\boldsymbol{\alpha}\, P(\beta^\dagger, \mathfrak{M}^*, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}, \beta) \\ &= \int Q^c(\boldsymbol{\alpha})d\boldsymbol{\alpha}\,(\mathfrak{N}_1 + (\partial/\partial\mathfrak{M})\mathfrak{M})P(\boldsymbol{\alpha}, t). \end{aligned} \tag{9.20}$$

The second form of (9.20) is obtained after an integration by parts. When Eq. (9.20) is inserted into (9.16) with Q standing for the product of δ functions, we can rewrite (9.16) in the form

$$\begin{aligned}\langle N_2(t)N_1(0)\rangle &= \iint d\boldsymbol{\alpha}\,\mathfrak{N}_2 P(\alpha, t \mid \boldsymbol{\alpha}', 0)\,d\alpha' \\ &\quad \times (\mathfrak{N}_1' + (\partial/\partial\mathfrak{M}')\mathfrak{M}')P(\boldsymbol{\alpha}', 0). \end{aligned} \tag{9.21}$$

The result has almost the form expected for a classical two-time average. Instead of replacing the operator N_1 by $\mathfrak{N}_1$, we obtain the slightly different replacement

$$(N_1)_r \rightarrow \mathfrak{N}_1 + (\partial/\partial\mathfrak{M})\mathfrak{M}. \tag{9.22}$$

We have written a subscript r on the operator N_1 to indicate that this is the appropriate replacement when the operator in question appears as a right-hand multiplication factor. If it had appeared as a left-hand multiplication factor, it would have had to commute through a different δ function, and a different replacement would have been necessary.

A complete set of commutation relations was derived in paper QXIII. These relations

$$N_1 e^{\xi M^\dagger} = e^{\xi M^\dagger}N_1 - \xi M^\dagger e^{\xi M^\dagger}, \tag{9.23a}$$

$$e^{\xi M}N_1 = N_1 e^{\xi M} - \xi M e^{\xi M}, \tag{9.23b}$$

$$N_2 e^{\xi M^\dagger} = e^{\xi M^\dagger} + \xi M^\dagger e^{\xi M^\dagger}, \tag{9.23c}$$

$$e^{\xi M}N_2 = N_2 e^{\xi M} + \xi M e^{\xi M}, \tag{9.23d}$$

$$Me^{\xi M^\dagger} = e^{\xi M^\dagger}M - \xi e^{\xi M^\dagger}N_2 + \xi e^{\xi M^\dagger}N_1 - \xi^2 M^\dagger e^{\xi M^\dagger}, \tag{9.23e}$$

$$e^{\xi M}M^\dagger = M^\dagger e^{\xi M} - \xi N_2 e^{\xi M} + \xi N_1 e^{\xi M} - \xi^2 M e^{\xi M}, \tag{9.23f}$$

$$Me^{\xi N_2} = e^{\xi}e^{\xi N_2}M, \tag{9.23g}$$

$$e^{\xi N_2}M^\dagger = e^{\xi}M^\dagger e^{\xi N_2}, \tag{9.23h}$$

$$Me^{\xi M_1} = e^{-\xi}e^{\xi N_1}M, \tag{9.23i}$$

$$e^{\xi N_2}M^\dagger = e^{-\xi}M^\dagger e^{\xi N_1}, \tag{9.23j}$$

$$e^{\xi b}b^\dagger = b^\dagger e^{\xi b} + \xi e^{\xi b}, \tag{9.23k}$$

$$be^{\xi b^\dagger} = e^{\xi b^\dagger} + \xi e^{\xi b^\dagger} \tag{9.23l}$$

are sufficiently complete to permit us to perform all of the necessary rearrangements to place a set of operators in the chosen order. With the help of these commutation relations, we can show in the manner above that our six operators, when added as factors to the right, can be treated by making the replacements

$$(b^\dagger)_r \rightarrow \beta^* - \partial/\partial\beta, \tag{9.24a}$$

$$(M^\dagger)_r \rightarrow \exp(\partial/\partial\mathfrak{N}_1 - \partial/\partial\mathfrak{N}_2)\mathfrak{M}^* + (\partial/\partial\mathfrak{M})\mathfrak{N}_2 - (\partial/\partial\mathfrak{M})\mathfrak{N}_1 + (\partial/\partial\mathfrak{M})^2\mathfrak{M}, \tag{9.24b}$$

$$(N_1)_r \rightarrow \mathfrak{N}_1 + (\partial/\partial\mathfrak{M})\mathfrak{M}, \tag{9.24c}$$

$$(N_2)_r \rightarrow \mathfrak{N}_2 - (\partial/\partial\mathfrak{M})\mathfrak{M}, \tag{9.24d}$$

$$M_r \rightarrow \mathfrak{M}, \tag{9.24e}$$

$$b_r \rightarrow \beta. \tag{9.24f}$$

Conversely, if we wish to add factors at the left, we must make the replacements

$$(b^\dagger)_l \rightarrow \beta^*, \tag{9.25a}$$

$$(M^\dagger)_l \rightarrow \mathfrak{M}^*, \tag{9.25b}$$

$$(N_1)_l \rightarrow \mathfrak{N}_1 + (\partial/\partial\mathfrak{M}^*)\mathfrak{M}^*, \tag{9.25c}$$

$$(N_2)_l \rightarrow \mathfrak{N}_2 - (\partial/\partial\mathfrak{M}^*)\mathfrak{M}^*, \tag{9.25d}$$

$$(M)_l \rightarrow \exp(\partial/\partial\mathfrak{N}_1 - \partial/\partial\mathfrak{N}_2)\,\mathfrak{M} + (\partial/\partial\mathfrak{M}^*)(\mathfrak{N}_2 - \mathfrak{N}_1) + (\partial/\partial\mathfrak{M}^*)^2\mathfrak{M}^*, \tag{9.25e}$$

$$(b)_l \rightarrow \beta - \partial/\partial\beta^*. \tag{9.25f}$$

For example, a four-time average can be written in the form

$$\langle b(3)N_1(2)N_2(1)b^\dagger(0)\rangle = \left\langle \beta^{(3)}\left(\mathfrak{N}_1+\frac{\partial}{\partial\mathfrak{M}}\mathfrak{M}\right)^{(2)}\left(\mathfrak{N}_2-\frac{\partial}{\partial\mathfrak{M}}\mathfrak{M}\right)^{(1)}\left(\beta^*-\frac{\partial}{\partial\beta}\right)^{(0)}\right\rangle_{cl}$$

$$= \iiiint d\alpha^3 d\alpha^2 d\alpha^1 d\alpha^0 \beta^{(3)} P(\alpha^3, 3 \mid \alpha^2, 2)\left(\mathfrak{N}_1+\frac{\partial}{\partial\mathfrak{M}}\mathfrak{M}\right)^{(2)} P(\alpha^2, 2 \mid \alpha^1, 1)$$

$$\times\left(\mathfrak{N}_2-\frac{\partial}{\partial\mathfrak{M}}\mathfrak{M}\right)^{(1)} P(\alpha^1, 1 \mid \alpha^0, 0)\left(\beta^*-\frac{\partial}{\partial\beta}\right)^{(0)} P(\alpha^0, 0). \quad (9.26)$$

We use superscripts to denote the time at which the variable acts. The first form on the right-hand side of Eq. (9.26) is symbolic, and the fully displayed version is given in the second form. The variables α are of course the six variables of our problem defined by Eq. (9.6), so that, for example, we can write

$$\left(\mathfrak{N}_1+\frac{\partial}{\partial\mathfrak{M}}\mathfrak{M}\right)^{(2)} P(\alpha^2, 2 \mid \alpha^1, 1)$$

$$= \left(\alpha_3^2+\frac{\partial}{\partial\alpha_5^2}\alpha_5^2\right) P(\alpha^2, 2 \mid \alpha^1, 1), \quad (9.27)$$

where the subscripts run from 1 to 6 in accord with Eq. (9.6).

In summary then, the average of any product of operators at several times can be computed if one knows the steady-state distribution function and the two-time conditional distribution function. The latter is the Green's-function solution of the equation obeyed by $\partial P/\partial t$. This equation of motion, which has a generalized Fokker-Planck form, is derived in QXIII.

10. SUMMARY

In Sec. 4 of this paper, we have established a complete equivalence between the Markoff property of a classical system and the regression theorem. We have therefore found that the simplest definition of a Markoffian quantum system is one that obeys the regression theorem. Indeed, in Sec. 9, by defining an appropriate multitime density matrix, we have shown that the regression theorem implies a factorization of this density matrix, which is analogous to the factorization of a multitime probability in classical stochastic processes. Moreover, we show that the two-time conditional density matrix obeys a Chapman-Kolmogoroff condition quite analogous to the one obeyed by probabilities in classical random processes. The factorization of this multitime density matrix permits us to calculate arbitrary multitime averages in terms of the conditional two-time density matrix in a manner that is somewhat analogous to that used in classical Markoffian processes.

The above procedure applies to the full density matrix of the system. Often, however, we are satisfied with less information. We are only concerned with a certain limited number of operators, and we only wish to answer questions concerning these operators. One of the main contributions of the present paper, then, was to set up a correspondence between a set of quantum-mechanical operators and an associated set of c-number variables. This correspondence, using ordering rules, was so devised as to permit the calculation of the average of any function of these quantum-mechanical operators to be written in terms of a c-number integration over an associated classical distribution function. In this way, to a given quantum-random process, an associated classical random process was set up. The problem that remains is the extent to which the associated classical random process can be taken literally for the purpose of computing multitime quantum averages. In Sec. 5, we showed indeed that for the field operators b and $b^\dagger$ this literal interpretation is in fact correct. The average of any multitime function for the associated classical random process yields an average of the corresponding quantum operators, providing that those operators are written in normal order and in the appropriate time sequence. For the atomic operators appropriate to a laser problem, we found that a similar classical correspondence could be set up. Single time averages, again, are given by ordinary integrations over the associated classical distribution function. Some multitime operators can be obtained by taking the associated classical random process literally. More generally, however, we found that the commutation rules, associated with the atomic operators, forced the calculation of multitime, averages to introduce a more complicated correspondence in which certain quantum-mechanical operators were replaced by c-number variables with extra terms involving derivatives of the c-number variables. With these corrections that arise from the computation rules, we have shown in Sec. 9 that it is indeed possible to calculate all of the desired multitime averages, in a relatively simple way, in terms of the distribution function and the conditional distribution function of the associated classical random process.